Abstract

In this paper the quantum version of the source coding theorem is obtained for a completely ergodic source. This results extends Shannon’s classical theorem as well as Schumacher’s quantum noiseless coding theorem for memoryless sources. The control of the memory effects requires some earlier results of Hiai and Petz on high probability subspaces.

1 Introduction

The objective of quantum information theory is the transmission and manipulation of information stored in systems obeying quantum mechanics. A quantum channel has a source that emits systems in quantum states to the channel. For example, the source could be a laser that emits individual monochromatic photons and the channel could be an optical fiber. The noisy signal output of the channel arrives at the receiver. In principle, there are two very different problems about quantum channels. The sender has a quantum systems in an unknown state and wants to have the receiver to end up with a similar system in the same state. In this case we speak of a pure quantum channel which has a quantum mechanical input and output. On the other hand, one might want to use quantum states to carry classical information, roughly speaking a sequence of zeros and ones. Now both the input and the output are classical, however there is a quantum mechanical section inbetween. The classical information is encoded into a quantum state and this is sent down the channel. The higher the channel noise is, the more redundant the encoding must be in order to restore the original signal at the reciever, where the quantum signal is converted into classical information. In this paper we do not deal with the problem how such a scheme can be realistically implemented; practical quantum encoding and decoding requires sophisticated ability to manipulating quantum states. However, we are interested in the amount of classical information getting through the channel which is will be asumed to be noiseless. It was emphasized already by Shannon that a computer memory is a communication channel. (Quantum or classical depends on the type of the computer.) In an optimal situation the computer memory is free of any noise and this is the case we are concentrating on in the present paper. We want to consider rather general noiseless quantum channels (with possibly memory effects but strong ergodic properties) and our aim is to discuss the quantum source coding theorem.

To each classical input message \( x_i \) there corresponds a signal state \( \varphi_i \) of the quantum communication system. The quantum states \( \varphi_i \) are functioning as codewords of the messages.
The signal states $\varphi_i$ could be pure and orthogonal in the sense of quantum mechanics but for example in quantum cryptography nonorthogonal states are used intentionally in order to avoid eavesdropping. At the moment we do not impose any condition on the signal states, they could be arbitrary pure or mixed states. In the stochastic model of communication, one assumes that each input message $x_i$ appears with certain probability. Let $p_{ji}$ be the probability that the message $x_i$ is sent and $y_j$ is received. The joint distribution $p_{ji}$ yields marginal probability distributions $p_i$ and $q_j$ on the set of input and output messages. According to Shannon the mutual information

$$I = \sum_{i,j} p_{ji} \log \frac{p_{ji}}{p_i q_j}$$

measures the amount of information going through the channel from Alice to Bob. Of course, the relation of $I$ to the quantum encoding and decoding should be made clear. This comes next.

The message $x_i$ has a priori probability $p_i$ and the mixed quantum state of the channel is

$$\varphi = \sum_i p_i \varphi_i.$$ 

This might be considered as the statistical operator of the message ensemble, for example when $\varphi_i$ is a pure state $|i\rangle\langle i|$, then $\varphi = \sum_i p_i |i\rangle\langle i|$ acts on the input Hilbert space $\mathcal{H}$. The distribution of the output is determined by a measurement, which is nothing else but a physical word for decoding. To each output message there corresponds an observable $A_j$ on the output Hilbert space $\mathcal{K}$. It is customary to assume that $0 \leq A_j; \sum_j A_j = \text{id}$ (id stands for the identity operator) and $p_{ji} = \varphi_i(A_j)$. The so-called Kholevo bound ([7]) provides an upper bound on the amount of information accessible to Bob in terms of von Neumann entropies:

$$I \leq S(\varphi) - \sum_i p_i S(\varphi_i)$$

(When $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the statistical operator of a quantum state $\psi$, then $S(\psi) = -\sum_k \lambda_k \log(\lambda_k)$.) In particular, if all signal states $\varphi_i$ are pure, then $S(\varphi_i) = 0$ and we have $I \leq S(\varphi)$. In this way the von Neumann entropy gets an information theoretical interpretation. Kholevo’s bound is actually not very strong, it is attained only in trivial situations ([12]).

The basic problem of communication theory is to maximize the amount of information received by Bob from Alice. However, up to now this problem is not well-posed in our discussion yet. Let us deal with messages of length $n$, they are $n$-term-sequences of 0 and 1. (So the size of this message set is $2^n$.) For each message length $n$ we carry out the above procedure of coding and decoding and the amount of information going through the channel is $I_n$. Since $I_n$ is presumably proportional to $n$, the good information quantity is $I_n/n$, that is, the transmitted information per letter. Since Shannon’s theory is not only stochastic but asymptotic as well, we are going to let $n$ to $\infty$. In this way we need to repeat the above information transmission scheme for each $n$. The message set, the input Hilbert space $\mathcal{H}(n)$, our coding, the channel state $\varphi^{(n)}$, the output Hilbert space $\mathcal{K}(n)$ and the observables applied in the measurement are all depending on the parameter $n$.

The subject of the present paper is faithful signal transmission, which bears the name noiseless channel. In place of faithful transmission, one can think of information storage. In this case the aim is to use the least possible number of Hilbert space dimension per signal for coding. The new feature of the noiseless channel we are studying is the memory effect. Mathematically this means that the channel state (of the $n$-fold channel) is not of product type
but we assume stationarity and good ergodic properties. In Section 2 we use the standard formalism of statistical mechanics to describe such a channel. It turns out that the mean von Neumann entropy, familiar also from statistical mechanics, gives the optimal coding rate. The proof of our main result, Theorems 3.1 and 3.2, is similar to Shannon’s original proof as well as to the proof presented in [9] for Schumacher’s coding theorem, however instead of typical sequences we use the high probability subspace of strongly ergodic stationary states, a subject studied by Hiai and Petz in [4]. We note for the interested reader that most of the concepts used in the present paper are treated in details in the monograph [11].

2 An infinite system setting of the source

Let $X^n$ denote the set of all messages of length $n$. If $x^n \in X^n$ is a message then a quantum state $\varphi(x^n)$ of the $n$-fold quantum system is corresponded with it. The Hilbert space of the $n$-fold system is the $n$-fold tensor product $\mathcal{H}^{\otimes n}$ and $\varphi(x^n)$ has a statistical operator $D(x^n)$. If messages of length $n$ are to be transmitted then our quantum source should be put in the mixed state $\varphi_n = \sum_{x^n} p(x^n)\varphi(x^n)$ with statistical operator $D_n = \sum_{x^n} p(x^n)D(x^n)$, where $p(x^n)$ is the probability of the message $x^n$. Since we want to let $n \to \infty$, it is reasonable to view all the $n$-fold systems as subsystem of an infinite one. In this way we can conveniently use a formalism standard in statistical physics, see Chap. 15 of [11].

Let an infinitely extended system be considered over the lattice $\mathbb{Z}$ of integers. The observables confined to a lattice site $k \in \mathbb{Z}$ form the selfadjoint part of a finite dimensional matrix algebra $\mathcal{A}_k$, that is the set of all operators acting on the finite dimensional space $\mathcal{H}$. It is assumed that the local observables in any finite subset $\Lambda \subset \mathbb{Z}$ are those of the finite quantum system

$$\mathcal{A}_\Lambda = \bigotimes_{k \in \Lambda} \mathcal{A}_k.$$ 

The quasilocal algebra $\mathcal{A}$ is the norm completion of the normed algebra $\mathcal{A}_\infty = \bigcup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with finite intervals $\Lambda \subset \mathbb{Z}^\nu$.

A state $\varphi$ of the infinite system is a positive normalized functional $\mathcal{A} \to \mathbb{C}$. It does not make sense to associate a statistical operator to a state of the infinite system in general. However, $\varphi$ restricted to a finite dimensional local algebra $\mathcal{A}_\Lambda$ admits a density matrix $D_\Lambda$. We regard the algebra $\mathcal{A}_{[1,\infty]}$ as the set of all operators acting on the $n$-fold tensor product space $\mathcal{H}^{\otimes n}$. Moreover, we assume that the density $D_n$ from the first part of this section is identical with $D_{[1,\infty]}$. Under this assumptions we call the state $\varphi$ the state of the (infinite) channel. Roughly speaking, all the states used in the transmission of messages of length $n$ are marginals of this $\varphi$. Coding, transmission and decoding could be well formulated using the states $\varphi_n \equiv \varphi_{[1,\infty]}$. However, it is more convenient to formulate our setting in the form of an infinite system, in particular because we do not want to assume that the channel state $\varphi$ is a product type. This corresponds to the possibility that our quantum source has a memory effect.

The right shift on the set $\mathbb{Z}$ induces a transformation $\gamma$ on $\mathcal{A}$. A state $\varphi$ is called stationary if $\varphi \circ \gamma = \varphi$. The state $\varphi$ is called ergodic if it is an extremal point in the set of stationary states. Moreover, $\varphi$ is completely ergodic when it is an extreme point for every $m \in \mathbb{N}$ in the convex set of all states $\psi$ such that $\psi \circ \gamma^m = \psi$. By a completely ergodic stationary quantum source we simply mean a completely ergodic stationary state $\varphi$ of the infinite system $\mathcal{A}$. Of course, a stationary product state, corresponding to a memoryless channel, is completely ergodic. The emphesis is put to other states here.
Below we show an example of a completely ergodic stationary quantum source from the context of algebraic states. For the details see the original paper [5].

**Example 2.1** Let $\mathcal{A} := M_3(\mathbb{C})$, $\mathcal{B} := M_2(\mathbb{C})$, moreover let $\{E_{ij}\}_{i,j=1}^3$ be the usual matrix units of $M_3(\mathbb{C})$. Set

$$
V_1 := \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & 0 
\end{bmatrix}, \quad V_2 := \begin{bmatrix}
0 & 0 \\
\frac{1}{\sqrt{2}} & 0 
\end{bmatrix}, \quad V_3 := \begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}.
$$

Then $\sum_{i=1}^3 V_i^* V_i = I_B$.

Let $\rho$ be a state on $\mathcal{B}$ with density matrix

$$
\begin{bmatrix}
\frac{2}{3} & 0 \\
0 & \frac{1}{3} 
\end{bmatrix}.
$$

Define $\Sigma : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}$ by $\Sigma(E_{ij} \otimes x) := V_i^* x V_j$. It is easy to check that $\Sigma$ is a completely positive unital map and $\rho(\Sigma(I_{\mathcal{A}} \otimes x)) = \rho(x), x \in \mathcal{B}$.

Then the algebraic state $\varphi$ generated by $(\mathcal{B}, \Sigma, \rho)$ is given by

$$
\varphi(E_{i_1j_1} \otimes \ldots E_{i_nj_n}) = \rho(V_{i_1}^* \ldots V_n^* V_{j_n} \ldots V_{j_1}).
$$

It is shown in [5] that $\varphi$ is completely ergodic. Of course, it is not a product state.

It is well-known in quantum statistical mechanics that due to the subadditivity of the von Neumann entropy (proven first in [10] by Lieb and Ruskai) the limit

$$
\lim_{n \to +\infty} \frac{1}{n} S(\varphi_n) =: h = \inf \frac{1}{n} S(\varphi_n)
$$

exists for any stationary state and this quantity is called the mean entropy of $\varphi$. (See [11] for a textbook treatment of the subject or [13] for some related properties of the mean entropy.)

## 3 Source coding

For a while we fix a message length $n$ and we denote by $d$ the dimension of the Hilbert space $\mathcal{H}$. Assume that our $n$-fold composite quantum system is operating as a quantum source and emits the quantum states $D^{(1)}, D^{(2)}, \ldots, D^{(m)}$ with a-priory probabilities $p_1, p_2, \ldots, p_m$. (Therefore the state of the system is $D_n = \sum_i p_i D^{(i)}$.) By source coding we mean an association

$$
D^{(i)} \mapsto \tilde{D}^{(i)},
$$

where $\tilde{D}^{(i)}$ is some other statistical operator on the Hilbert space $\mathcal{H}^\otimes n$. We denote by $\mathcal{K}_n$ the subspace spanned by the eigenvectors corresponding to all nonzero eigenvalues of all statistical operators $\tilde{D}^{(i)}, 1 \leq i \leq m$. The goal of source coding is to keep the dimension of $\mathcal{K}_n$ to be small and to fulfill some fidelity criterion. The source coding rate

$$
\limsup_{n \to \infty} \frac{\log \dim(\mathcal{K}_n)}{n}
$$

expresses the resolution of the encoder in qubits per input symbol. (It is actually more precise to speak about “qunats” per input symbol, but the difference is only a constant factor.)
The distortion measure is a number which allows us to compare the goodness or badness of communication systems. The *fidelity* of the coding scheme was introduced by Schumacher ([14]):

\[ F := \sum p_i \text{Tr} D^{(i)} \tilde{D}^{(i)}, \]

where \( p_i \) is a probability distribution on the input and \( \tilde{D}^{(i)} \) is the density used to encode the density \( D^{(i)} \). Note that \( 0 \leq F \leq 1 \) and \( F = 1 \) if and only if \( D^{(i)} = \tilde{D}^{(i)} \) are pure states.

First we present our positive source coding theorem for a completely ergodic source. The result says that the source coding rate may approach the mean entropy while we can keep the fidelity arbitrarily good.

**Theorem 3.1** Let \( \mathcal{H} \) be a finite dimensional Hilbert space, and \( \varphi \) be a completely ergodic state on \( B(\mathcal{H})^{\otimes \infty} \). Then for every \( \varepsilon, \delta > 0 \) there exists \( n_{\varepsilon, \delta} \in \mathbb{N} \) such that for \( n \geq n_{\varepsilon, \delta} \) there is a subspace \( \mathcal{K}_n(\varepsilon, \delta) \) of \( \mathcal{H}^{\otimes n} \) such that

(i) \( \log \dim \mathcal{K}_n(\varepsilon, \delta) < n(s + \delta) \) and

(ii) for every decomposition \( D_n = \sum_{i=1}^m p_i D^{(i)} \) one can find an encoding \( D^{(i)} \mapsto \tilde{D}^{(i)} \) with density matrices \( \tilde{D}^{(i)} \) supported in \( \mathcal{K}_n(\varepsilon, \delta) \) such that the fidelity \( F := \sum_{i=1}^m p_i \text{Tr} D^{(i)} \tilde{D}^{(i)} \) exceeds \( 1 - \varepsilon \).

The negative part of the coding theorem tells that the source coding rate cannot exceed the mean entropy when the fidelity is good.

**Theorem 3.2** Let \( \mathcal{H} \) be a finite dimensional Hilbert space, and \( \varphi \) be a completely ergodic state on \( B(\mathcal{H})^{\otimes \infty} \). Then for every \( \varepsilon, \delta > 0 \) there exists \( n_{\varepsilon, \delta} \in \mathbb{N} \) such that for \( n \geq n_{\varepsilon, \delta} \)

(i) for all subspaces \( \mathcal{K}_n(\varepsilon, \delta) \) of \( \mathcal{H}^{\otimes n} \) with the property \( \log \dim \mathcal{K}_n(\varepsilon, \delta) < n(s - \delta) \) and

(ii) for every decomposition \( D_n = \sum_{i=1}^m p_i D^{(i)} \) and for every encoding \( D^{(i)} \mapsto \tilde{D}^{(i)} \) with density matrices \( \tilde{D}^{(i)} \) supported in \( \mathcal{K}_n(\varepsilon, \delta) \), the fidelity \( F := \sum_{i=1}^m p_i \text{Tr} D^{(i)} \tilde{D}^{(i)} \) is smaller than \( \varepsilon \).

The detailed proofs are given in the next section of the paper.

## 4 High probability subspace

The proof of Shannon’s original source coding theorem is based on the typical sequences ([1], Chap. 1). The quantum extension of this result obtained by Schumacher still benefits from the classical result. When the channel state is a product, the densities \( D_n \) commute and simultaneous diagonalization is possible. If the memory effects are present, then these densities do not commute and in some sense we are in a really quantum mechanical non-commutative situation. Nevertheless, the high probability subspace can be used but new techniques are required.

Let \( \mathcal{K} \) be a Hilbert space and \( D \) be a density matrix on \( \mathcal{K} \). \( D \) has a Schatten decomposition \( D = \sum_i \lambda_i |f_i\rangle\langle f_i| \), where \( |f_i\rangle \)’s are eigenvectors and the eigenvalues \( \lambda_i \) are numbered...

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decreasingly: $\lambda_1 \geq \lambda_2 \geq \ldots$. Choose and fix $0 < \varepsilon < 1$. Let $n(\varepsilon)$ be the smallest integer such that
\[
\sum_{i=1}^{n(\varepsilon)} \lambda_i \geq 1 - \varepsilon.
\]

The subspace $HP(D, \varepsilon)$ spanned by the eigenvectors $|f_1\rangle, \ldots, |f_{n(\varepsilon)}\rangle$ is called the high probability subspace corresponding to the level $\varepsilon$. Note that $HP(D, \varepsilon)$ is not completely well-defined, if there are multiplicities in the spectrum of $D$, then the Schatten decomposition is not unique. However, the dimension $n(\varepsilon)$ of $HP(D, \varepsilon)$ is determined. The term “high probability subspace” is borrowed from the monographs [3] and its role in macroscopic uniformity was discussed in [6].

In the following, $\varphi$ will be a completely ergodic state on $A^{\otimes \infty}$. For $\varepsilon \in (0, 1)$ let
\[
\beta_{\varepsilon,n} := \inf \{ \log \text{Tr}_n(q) : q \in P(A^{\otimes n}), \varphi_n(q) \geq 1 - \varepsilon \},
\]
where $P(A^{\otimes n})$ denotes the set of projections of $A^{\otimes n}$. ($\exp \beta_{\varepsilon,n}$ is the dimension of the high probability subspace.)

It was shown in [4] that
\[
\limsup_{n \to +\infty} \frac{1}{n} \beta_{\varepsilon,n} \leq s \quad (1)
\]
\[
\liminf_{n \to +\infty} \frac{1}{n} \beta_{\varepsilon,n} \geq \frac{1}{1 - \varepsilon} s - \frac{\varepsilon}{1 - \varepsilon} \log d, \quad (2)
\]
so we have the following

**Theorem 4.1** For every positive $\varepsilon$ and $\delta$ there exists $n_{\varepsilon,\delta} \in \mathbb{N}$ such that

(i) for every $n > n_{\varepsilon,\delta}$ there exists a projection $q$ in $A^{\otimes n}$ such that

\[
\log(\text{Tr}_n(q)) < n(s + \delta) \quad \text{and} \quad \varphi_n(q) \geq 1 - \varepsilon,
\]

(ii) for every $n > n_{\varepsilon,\delta}$ and for every projection $q$ in $A^{\otimes n}$

\[
\log(\text{Tr}_n(q)) < n(s - \delta),
\]

implies $\varphi_n(q) \leq \varepsilon$.

Next we prove the source coding theorem.

**Proof of Theorem 3.1**: Use part (i) of Theorem 4.1 and set $K_n(\varepsilon, \delta) := \text{Ran} q_n$. Given $D_n = \sum_{i=1}^k p_i D^{(i)}$ we construct the coding densities $\tilde{D}^{(i)}$. Let
\[
\tilde{D}^{(i)} := q_n D^{(i)} q_n + c_n E_i,
\]
where $E_i \leq q_n$ is an arbitrary projection and the constant $c_n$ is chosen such a way that $\text{Tr} \tilde{D}^{(i)} = 1$ should be. Then
\[
\text{Tr} D^{(i)} \tilde{D}^{(i)} \geq \text{Tr} D^{(i)} q_n D^{(i)} q_n \geq 2 \text{Tr} q_n D^{(i)} - 1.
\]

We need to sum over $i$:
\[
\sum_i p_i \text{Tr} D^{(i)} \tilde{D}^{(i)} \geq \sum_i p_i (2 \text{Tr} q_n D^{(i)} - 1) = 2 \text{Tr} D_n q_n - 1 \geq 1 - 2 \varepsilon.
\]
It is worth to note that if the densities $D^{(i)}$ are describing pure states, then we can choose $\tilde{D}^{(i)}$ to be a pure state as well.

Proof of Theorem 3.2: For the given $\varepsilon$ and $\delta$ we choose $n(\varepsilon, \delta)$ according to Theorem 4.1. Let $q$ be the projection onto the subspace $\mathcal{K}(\varepsilon, \delta)$. Then $D^{(i)} = qD^{(i)}q$ from the hypothesis and we have

$$F := \sum_{i=1}^{m} p_i \text{Tr} D^{(i)} \tilde{D}^{(i)} = \sum_{i=1}^{m} p_i \text{Tr} D^{(i)} q \tilde{D}^{(i)} q \leq \sum_{i=1}^{m} p_i \text{Tr} D^{(i)} q = \varphi_n(q) \leq \varepsilon.$$

5 Discussion

In this paper a theory of quantum source coding subject to a fidelity criterion or quantum data compression is presented. The maximum of the source coding rate is studied under the conditions that Schumacher’s fidelity must exceeds $1 - \varepsilon$ and the quantum mechanical state of the channel has a strong ergodic property. This later condition allows many states with memory effect. For the mathematical model and in the proof of the main result techniques of quantum statistical mechanics are used. We prove that the maximal source coding rate is the mean entropy of the channel state, and, to some extent, it is independent of the message ensemble.

References


