Bounds on transverse momentum dependent distribution and fragmentation functions.

A. Bacchetta, M. Boglione, A. Henneman and P.J. Mulders

Department of Theoretical Physics, Faculty of Science, Vrije Universiteit
De Boelelaan 1081, NL-1081 HV Amsterdam, the Netherlands

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We give bounds on the distribution and fragmentation functions that appear at leading order in deep inelastic 1-particle inclusive leptoproduction or in Drell-Yan processes. These bounds simply follow from positivity of the defining matrix elements and are an important guidance in estimating the magnitude of the azimuthal and spin asymmetries in these processes.

In deep-inelastic processes the transition from hadrons to quarks and gluons is described in terms of distribution and fragmentation functions. For instance, at leading order in the inverse hard scale \(1/Q\), the cross section for inclusive electroproduction \(e^- H \rightarrow e^- X\) is given as a charge squared weighted sum over quark distribution functions, which describe the probability of finding quarks in hadron \(H\). In electron-positron annihilation, the 1-particle inclusive cross section for \(e^+ e^- \rightarrow hX\) is given as a charge squared weighted sum over quark and antiquark fragmentation functions, describing the decay of the produced (anti)quarks into hadron \(h\).

The distribution functions for a quark can be obtained from the lightcone correlation function [1–3]

\[
\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi} e^{ip\xi} \langle P, S | \overline{\psi}_j(0) \psi_i(\xi) | P, S \rangle \bigg|_{\xi^+ = \xi_T = 0},
\]

depending on the lightcone fraction \(x = p^+ / P^+\). To be precise, the lightlike directions \(n_+\) and \(n_-\), satisfying \(n_+^2 = n_-^2 = 0\) and \(n_+ \cdot n_- = 1\), define the lightcone coordinates \(a^\pm = a \cdot n^\pm\). The hadron momentum \(P\) is chosen so that it has no components orthogonal to \(n_+\) or \(n_-\) thus the transverse hadron momentum \(P_T = 0\). The correlator contains the soft parts appearing in hard scattering processes, and is related to the forward amplitude for antiquark-hadron scattering (see Fig. 1). The relevant part is \(\Phi_{\lambda_-} = \Phi_{\gamma^+}\). Inserting a complete set of intermediate states and generalizing to off-diagonal spin, one obtains

\[
(\Phi_{\gamma^+})_{ij, s's} = \int \frac{d\xi^-}{2\pi \sqrt{2}} e^{ip\xi} \langle P, s'| \overline{\psi}_{s'}(0) \psi_i(\xi) | P, s \rangle \bigg|_{\xi^+ = \xi_T = 0} = \frac{1}{\sqrt{2}} \sum_n \langle \overline{P}_n | \overline{\psi}_{s'}(0) | P, s' \rangle^* \langle P_n | \psi_i(0) | P, s \rangle \delta \left( P_n^+ - (1 - x) P^+ \right),
\]

where \(\psi_i \equiv P_s \psi = \frac{1}{2} \gamma^- \gamma^+ \psi\) is the good component of the quark field [4]. This representation shows that the correlation functions have a natural interpretation as lightcone momentum densities.

In order to study the correlation function in a spin 1/2 target one introduces a spin vector \(S\) that parametrizes the spin density matrix \(\rho(P, S)\). It satisfies \(P \cdot S = 0\) and \(S^2 = -1\) (spacelike) for a pure state, \(-1 < S^2 < 0\) for a mixed state. Using \(\lambda \equiv MS^+/P^+\) and the transverse spin vector \(S_T\), the condition becomes \(\lambda^2 + S_T^2 \leq 1\), as can be seen from the rest-frame expression \(S = (0, S_T, \lambda)\). The precise equivalence of a \(2 \times 2\) matrix \(\tilde{M}_{s's'}\) in the target spin space and the \(S\)-dependent function \(M(S)\) is \(M(S) = \text{Tr} \left[ \rho(P) \tilde{M} \right]\). Explicitly, the \(S\)-dependent function \(M(S) = M_O + \lambda M_L + S_T^1 M_L^1 + S_T^2 M_L^2\), corresponds to a matrix, which in the target rest-frame with as basis the spin 1/2 states with \(\lambda = +1\) and \(\lambda = -1\) becomes

\[
\tilde{M}_{s's'} = \begin{pmatrix}
M_O + M_L & M_L^2 - i M_L^3 \\
M_L^1 + i M_L^2 & M_O - M_L
\end{pmatrix}
\]

From Eq. 2 follows that after transposing in Dirac space, and subsequently extending the matrix \(M(S) = (\Phi_{\gamma^+})^T\) to the target spin space gives a matrix in the combined Dirac \(\otimes\) target spin space which satisfies \(v^\dagger M v \geq 0\) for any vector \(v\) in that combined space.
The most general form for the quantity $\Phi \gamma^+$ for a spin 1/2 target in terms of the spin vector is

$$\Phi(x) \gamma^+ = \left\{ f_1(x) + \lambda g_1(x) \gamma_5 + h_1(x) \gamma_5 \right\} P_+,$$

where the functions $f_1$, $g_1$ and $h_1$ are the leading order quark distribution functions [5]. By tracing over the Dirac indices one projects out $f_1$, which is the quark momentum density (see Eq. 2). By writing $\gamma_5$ as the difference of the chirality projectors $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ it follows that in a longitudinally polarized target ($\lambda \neq 0$) $g_1$ is the difference of densities for right-handed and left-handed quarks. By writing $\gamma^+ \gamma_5$ as the difference of the transverse spin projectors $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5 \gamma_5)$, it follows that in a transversely polarized target ($S_T \neq 0$) $h_1$ is the difference of quarks with transverse spin along and opposite the target spin [6–8]. Since $f_1(x)$ is the sum of the densities it is positive and gives bounds $|g_1(x)| \leq f_1(x)$ and $|h_1(x)| \leq f_1(x)$.

By considering the combined Dirac ⊗ target spin space stricter bounds can be found. As mentioned above, we need to consider the function $M(S) = \left(\Phi \gamma^+\right)^T$ in Dirac space. For this we use a chiral representation. In that representation the good projector $P_+$ only leaves two (independent) dirac spinors, one right-handed (R), one left-handed (L). On this (2-dimensional) basis of good R and L spinors the matrix $M = \left(\Phi(x) \gamma^+\right)^T$ obtained from Eq. 4 is given by

$$M_{ij} = \begin{pmatrix} f_1(x) + \lambda g_1(x) & (S_1^+ + i S_2^2) h_1(x) \\ (S_1^+ - i S_2^2) h_1(x) & f_1(x) - \lambda g_1(x) \end{pmatrix},$$

Next we make the spin-structure of the target explicit as outlined in Eq. 3, yielding on the basis $+R$, $-R$, $+L$ and $-L$

$$\tilde{M} = \begin{pmatrix} f_1 + g_1 & 0 & 0 & 2h_1 \\ 0 & f_1 - g_1 & 0 & 0 \\ 0 & 0 & f_1 - g_1 & 0 \\ 2h_1 & 0 & 0 & f_1 + g_1 \end{pmatrix}.$$

From the positivity of the diagonal elements one recovers the trivial bounds $f_1(x) \geq 0$ and $|g_1(x)| \leq f_1(x)$, but requiring the eigenvalues of the matrix to be positive gives the stricter Soffer bound [9],

$$|h_1(x)| \leq \frac{1}{2} \left( f_1(x) + g_1(x) \right).$$

Analogously bounds can be obtained for transverse momentum dependent distribution and fragmentation functions. Transverse momenta of partons play an important role in hard processes with more than one hadron [10]. Examples are 1-particle inclusive deep-inelastic electroproduction, $e^{-} H \rightarrow e^{-} hX$ [11], or Drell-Yan scattering, $H_1 H_2 \rightarrow \mu^+ \mu^- X$ [12].

The soft parts involving the distribution functions are contained in the lightfront correlation function

$$\Phi_{ij}(x, p_T) = \int \frac{d\xi d^2\xi_T}{(2\pi)^2} e^{ip \xi} \langle P, S \mid \overline{\psi}_j(0) \psi_i(\xi) \mid P, S \rangle \bigg|_{\xi^+ = 0},$$

depending on $x = p^+/P^+$ and the quark transverse momentum $p_T$ in a target with $P_+ = 0$. For the description of quark fragmentation one needs [13]
\[ \Delta_{ij}(z, k_r) = \sum_X \int \frac{d^2 \xi}{(2\pi)^3} e^{ik \cdot \xi} T_r(0|\psi_i(\xi)|P_h, X)\langle P_h, X|\psi_j(0)\rangle \bigg|_{\xi^+ = 0} , \]  
(9)

(see Fig. 2) depending on \( z = P^+_h/k^+ \) and the quark transverse momentum \( k_r \), leading to a hadron with \( P_{h\perp} = 0 \). A simple boost shows that this is equivalent to a quark producing a hadron with transverse momentum \( P_{h\perp} = -z k_r \) with respect to the quark. Notice that the expressions given here are in a lightcone gauge \( A^+ = 0 \).

In a general gauge, a gauge link running along \( n \) needs to be included. In the presence of transverse momentum dependence [14] and hence separation in \( x, \xi \), the links run to lightcone infinity \( \xi^- = \pm \infty \).

Separating the terms corresponding to unpolarized (O), longitudinally polarized (L) and transversely polarized targets (T), the most general parametrizations with \( p_r \)-dependence, relevant at leading order, are

\[ \Phi_O(x, p_r) \gamma^+ = \left\{ f_1(x, p_r) + i h^+_{1L}(x, p_r) \frac{\not{p}_r}{M} \right\} P_+ \]  
(10)

\[ \Phi_L(x, p_r) \gamma^+ = \left\{ \lambda g_{1L}(x, p_r) \gamma_5 + \lambda h^+_{1L}(x, p_r) \gamma_5 \frac{\not{p}_r}{M} \right\} P_+ \]  
(11)

\[ \Phi_T(x, p_r) \gamma^+ = \left\{ f^+_{1T}(x, p_r) \frac{\epsilon_{\tau \rho \sigma} S_\rho p^\sigma_{T}}{M} + g_{1T}(x, p_r) \frac{p_{\tau r} S_T}{M} \gamma_5 h^+_{1T} \right\} P_+ + h^+_{1T}(x, p_r) \gamma_5 S_T + \frac{h^+_{1T}(x, p_r)}{M} \gamma_5 \frac{\not{p}_r}{M} \]  
(12)

As before, \( f_\ldots, g_\ldots \) and \( h_\ldots \) indicate unpolarized, chirality and transverse spin distributions. The subscripts \( L \) and \( T \) indicate the target polarization, and the superscript \( \perp \) signals explicit presence of transverse momentum of partons. Using the notation \( f^{(1)}(x, p_r) \equiv (p^2_r/2M^2) f(x, p_r) \), one sees that \( f_1, g_1 = g_{1L} \) and \( h_1 = h_{1T} + h_{1T}^{(1)} \) are the functions surviving \( p_r \)-integration.

Analogously, \( \Delta \) is parametrized in terms of unpolarized, chirality and transverse-spin fragmentation functions [11], denoted by capital letters \( D_\ldots, G_\ldots \), and \( H_\ldots \), respectively. So far, time reversal invariance has not been used. For distribution functions it leads to \( f^{(1)}_{1T} = 0 \) and \( h^{(1)}_1 = 0 \) [15]. For fragmentation functions it cannot be used [16–18], allowing for non-vanishing fragmentation functions \( D^{(1)}_{1T} \) [11] and \( H^{(1)}_1 \) [19].

To put bounds on the transverse momentum dependent functions, we again make the matrix structure explicit. One finds for \( M = (\Phi(x, p_r) \gamma^+)^T \) the full spin matrix \( M \) to be

\[
\begin{bmatrix}
    f_1 + g_{1L} & |p_r|^2 e^{i\phi} (g_{1T} + i f^+_{1T}) & |p_r|^2 e^{-i\phi} (h^+_{1L} + i h^+_{1T}) & 2h_1 \\
    |p_r|^2 e^{-i\phi} (g_{1T} - i f^+_{1T}) & f_1 - g_{1L} & |p_r|^2 e^{-2i\phi} h^+_{1T} & -|p_r|^2 e^{-i\phi} (h^+_{1L} - i h^+_{1T}) \\
    |p_r|^2 e^{i\phi} (h^+_{1L} - i h^+_{1T}) & |p_r|^2 e^{2i\phi} h^+_{1T} & f_1 - g_{1L} & -|p_r|^2 e^{i\phi} (g_{1T} - i f^+_{1T}) \\
    2h_1 & -|p_r|^2 e^{i\phi} (h^+_{1L} + i h^+_{1T}) & -|p_r|^2 e^{-i\phi} (g_{1T} + i f^+_{1T}) & f_1 + g_{1L}
\end{bmatrix}.
\]

First of all, this matrix is illustrative as it shows the full quark helicity structure accessible in a polarized nucleon [20], which is equivalent to the full helicity structure of the forward antiquark-nucleon scattering amplitude. Bounds to assure positivity of any matrix element can for instance be obtained by looking at the 1-dimensional subspaces, giving the the trivial bounds \( f_1 \geq 0 \) and \( \langle g_{1L} \rangle \leq f_1 \). From the 2-dimensional subspace one finds

\[ |h_1| \leq \frac{1}{2} (f_1 + g_{1L}) \leq f_1 , \]  
(13)

\[ |h_{1T}^{(1)}| \leq \frac{1}{2} (f_1 - g_{1L}) \leq f_1 , \]  
(14)

\[ \left( g^{(1)}_{1T} \right)^2 + \left( f^{(1)}_{1T} \right)^2 \leq \frac{p^2_T}{4M^2} (f_1 + g_{1L}) (f_1 - g_{1L}) \leq \frac{p^2_T}{4M^2} f^2_1 , \]  
(15)

\[ \left( h^{(1)}_{1L} \right)^2 + \left( h^{(1)}_{1T} \right)^2 \leq \frac{p^2_T}{4M^2} (f_1 + g_{1L}) (f_1 - g_{1L}) \leq \frac{p^2_T}{4M^2} f^2_1 . \]  
(16)
Besides the Soffer bound, new bounds for the distribution functions are found. In particular, one sees that functions like $g_{1T}$ and $h_{1L}^{(1)}$ appearing in azimuthal asymmetries in lepton production are real-valued but $f$ functions. Actually, one sees that the T-odd functions go for fragmentation functions, where for instance the magnitude of $\alpha_{1}$ can serve as important guidance to estimate evolution [26], these future experiments may provide us with the knowledge of the full helicity structure of quarks in a nucleon. The elementary bounds derived in this paper can serve as important guidance to estimate the magnitudes of asymmetries expected in the various processes.

Requiring them to be positive can be converted into the conditions

$$
F + G \geq 0.
$$

$$
|\alpha F - \beta G| \leq F + G, \quad \text{i.e. } |h_{1T}| \leq f_{1}.
$$

$$
|\gamma + \delta|^{2} \leq (1 - \alpha)(1 + \beta),
$$

$$
|\gamma - \delta|^{2} \leq (1 + \alpha)(1 - \beta).
$$

It is interesting for the phenomenology of deep inelastic processes that a bound for the transverse spin distribution $h_{1}$ is provided not only by the inclusively measured functions $f_{1}$ and $g_{1}$, but also by the functions $g_{1T}$ and $h_{1L}^{(1)}$, responsible for specific azimuthal asymmetries [11,21]. This is illustrated in Fig. 3. The same goes for fragmentation functions, where for instance the magnitude of $H_{1}$ constrains the magnitude of $H_{1}$ [22]. Recently SMC [23], HERMES [24] and LEP [25] have reported preliminary results for azimuthal asymmetries. More results are likely to come in the next few years from HERMES, HERA, RHIC and COMPASS experiments. Although much theoretical work is needed, for instance on factorization and the stability of the bounds under evolution [26], these future experiments may provide us with the knowledge of the full helicity structure of quarks in a nucleon. The elementary bounds derived in this paper can serve as important guidance to estimate the magnitudes of asymmetries expected in the various processes.
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[5] Other common notations for a quark flavor \( q \) are \( f^q(x) = q(x), g^q(x) = \Delta q(x) \) and \( h^q(x) = \Delta_q q(x) \).
[22] R.L. Jaffe, Phys. Rev. D 54 (1996) 6581. Here the notation \( \hat{f}, \hat{g}, \hat{h} \) is used instead of \( D, G, H \).