It is easy to construct classical 2-state systems illustrating the behavior of the short-lived and long-lived neutral K mesons in the limit of CP conservation. The emulation of CP violation is more tricky, but is provided by the two-dimensional motion of a Foucault pendulum. Analogies are drawn between the pendulum and observables in neutral kaon decays. An emulation of CP- and CPT-violation using electric circuits is also discussed.

I. INTRODUCTION

Two-state systems abound in quantum mechanics, and have the virtue of permitting exact solutions [1]. One such system is that of the neutral kaons $K^0$ and $\bar{K}^0$ [2], which mix with one another [3] to form short-lived and long-lived mass eigenstates. Previously [4] we have shown that one can emulate this system using coupled resonant circuits, with the violation of CP invariance [5] exhibited in kaon decays corresponding to asymmetric coupling between the circuits.

We sought to implement the suggestion of Ref. [4] in a physical device, such as a pair of coupled pendula which may be used to illustrate the CP-conserving limit [6]. In the course of this activity, we came upon a familiar system which has many of the features of the neutral kaon system, including the asymmetric coupling between two oscillation modes which leads to the phenomenon of CP violation. This system is the Foucault pendulum. In the present article we explore the parallels between the Foucault pendulum and the neutral kaon system, showing how one might construct a pendulum with the closest possible relation to the actual short-lived and long-lived neutral kaon states, and discussing not only CP, but T and CPT violation as well.

\[^1\text{To be submitted to Am. J. Phys.}\]
This article is arranged as follows. We first recall the motion of the pendulum in Section II, review the neutral kaons in Section III (partly a recapitulation of results from Ref. [4]), and explore the parallels in Section IV. Some suggestions for modifying the basic pendulum to make it emulate the actual kaon system are made in Section V. Section VI contains some remarks on emulation of CP- and CPT-violation using coupled electric circuits, while Section VII concludes.

II. FOUCAULT PENDULUM

A. Equations of motion in a rotating frame

The motion of a body subject to a force \( \mathbf{F} \) in a system rotating with constant angular velocity \( \mathbf{\Omega}_E \) is described by [7]

\[
m\ddot{\mathbf{r}} = \mathbf{F} - 2m(\mathbf{\Omega}_E \times \dot{\mathbf{r}}) - m\mathbf{\Omega}_E \times (\mathbf{\Omega}_E \times \mathbf{r}) - 2m\beta \dot{\mathbf{r}} ,
\]

where the last term has been inserted to describe damping due to air resistance. For the case in question, \( \mathbf{\Omega}_E \) will be a vector pointing toward the Earth’s North Pole, with magnitude \( 2\pi d^{-1} \).

Now consider a pendulum with a spherically symmetric support point so that it is free to move in two directions. Denote the corresponding axes by \( \hat{x} \) (East), \( \hat{y} \) (North), and \( \hat{z} \) (up, i.e., perpendicular to the surface of the Earth). The components of \( \mathbf{\Omega}_E \) are

\[
\Omega_{E_x} = 0 , \quad \Omega_{E_y} = \Omega_E \cos \theta , \quad \Omega_{E_z} = \Omega_E \sin \theta , \quad (2)
\]

where \( \theta \) is the latitude (positive for North latitude). The components of \( \mathbf{\Omega}_E \times \dot{\mathbf{r}} \) are

\[
(\mathbf{\Omega}_E \times \dot{\mathbf{r}})_x = -\Omega_E \sin \theta \dot{y} , \quad (\mathbf{\Omega}_E \times \dot{\mathbf{r}})_y = \Omega_E \sin \theta \dot{x} , \quad (\mathbf{\Omega}_E \times \dot{\mathbf{r}})_z = -\Omega_E \cos \theta \dot{x} , \quad (3)
\]

where we have neglected \( \dot{z} \) for small oscillations of the pendulum.

The centripetal acceleration term \( -m\mathbf{\Omega}_E \times (\mathbf{\Omega}_E \times \mathbf{r}) \) in Eq. (1) has magnitude \( m\Omega_E^2a \cos \theta \) and acts in the direction \( -\hat{z} \cos \theta + \hat{y} \sin \theta \), where \( a \) is the radius of the Earth and \( \Omega_E^2a = 3.38 \text{ cm s}^{-2} [7] \). It thus changes the local acceleration of gravity slightly, \( g \to g_{\text{eff}} \), and displaces the equilibrium position of the pendulum. We shall take account of these effects by redefining \( g \equiv g_{\text{eff}} \) and setting \( x = y = 0 \) to be the equilibrium position.

We then define \( \omega_0^2 \equiv g/l \) and \( \Omega \equiv \Omega_E \sin \theta \), write Eq. (1) in component form, and cancel a factor of \( m \). The result is

\[
\ddot{x} = -\omega_0^2x + 2\Omega \dot{y} - \beta \dot{x} , \quad \ddot{y} = -\omega_0^2y - 2\Omega \dot{x} - \beta \dot{y} . \quad (4)
\]

The coupled equations (4) can be solved for periodic motion by substituting \( x = x_0 \exp(-i\omega t) \), \( y = y_0 \exp(-i\omega t) \), leading to an eigenvalue equation for \( \omega^2 \). Expanding around \( \omega = \omega_0 \) and dividing by \( 2\omega_0 \), the result is

\[
\mathcal{M} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \omega \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} , \quad (5)
\]
where
\[ M \equiv \begin{bmatrix} \omega_0 - i\beta & -i\Omega \\ i\Omega & \omega_0 - i\beta \end{bmatrix}. \] (6)

Eq. (6) is very close to the result which one would obtain for mixing of a neutral-kaon system in which the \( K^0 \) is represented by the basis vector \( \hat{x} \) while the \( \bar{K}^0 \) is represented by the basis vector \( \hat{y} \). We shall explore this parallel presently. First, however, we show that Eq. (6) leads to the familiar behavior of the Foucault pendulum in which the plane of linear oscillations precesses by a daily amount which depends on the latitude.

**B. Solution of equations of motion**

The normalized eigenmodes of the system (6) are
\[ |R\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad |L\rangle \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \] (7)
with eigenvalues
\[ \mu_R = \omega_0 - i\beta + \Omega, \quad \mu_L = \omega_0 - i\beta - \Omega. \] (8)

An arbitrary two-component solution \( \mathbf{x}(t) \) describing motion in the \( x - y \) plane can then be written as a linear combination of \( |R\rangle \) and \( |L\rangle \) as
\[ \mathbf{x}(t) = c_R |R\rangle e^{-i\mu_R t} + c_L |L\rangle e^{-i\mu_L t}. \] (9)

The initial conditions on \( \mathbf{x}(0) \) and \( \dot{\mathbf{x}}(0) \) then permit us to specify the complex quantities \( c_R \) and \( c_L \). Imposing the condition
\[ \text{Re} (c_R + c_L)/\sqrt{2} = x(0), \quad \text{Im} (c_L - c_R)/\sqrt{2} = y(0), \] (10)
on the solution (9) at \( t = 0 \), we find
\[ \text{Re} \ c_R = \frac{x_0 \omega_0 - \Omega}{\sqrt{2}} \frac{\omega_0}{\omega_0}, \quad \text{Re} \ c_L = \frac{x_0 \omega_0 + \Omega}{\sqrt{2}} \frac{\omega_0}{\omega_0}, \]
(12)
\[ \text{Im} \ c_R = \text{Im} \ c_L = \frac{\beta x_0}{\sqrt{2} \omega_0}. \] (13)
The solution for the motion of the pendulum as a function of time is then

\[
x(t) = x_0 \text{Re} \left\{ \frac{\omega_0 - \Omega + i\beta}{2\omega_0} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i(\omega_0 + \Omega - i\beta)t} \\
+ \frac{\omega_0 + \Omega + i\beta}{2\omega_0} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-i(\omega_0 - \Omega - i\beta)t} \right\},
\]

(14)

so that

\[
x(t) = x_0 e^{-\beta t} \left[ \cos(\omega_0 t + \beta \omega_0 \sin \omega_0 t) \cos \Omega t + \frac{\Omega}{\omega_0} \sin \omega_0 t \sin \Omega t \right],
\]

\[
y(t) = x_0 e^{-\beta t} \left[ \cos(\omega_0 t + \beta \omega_0 \sin \omega_0 t) \sin \Omega t - \frac{\Omega}{\omega_0} \sin \omega_0 t \cos \Omega t \right].
\]

(15)

For \( \Omega \ll \omega_0 \), one sees by making the replacement \( \Omega t \to \Omega t + \frac{\pi}{2} \) and comparing the terms \( \sim \sin \Omega t \) and \( \cos \Omega t \) in Eq. (15) that the plane of oscillation rotates with angular frequency \( -\Omega = -\Omega_E \sin \theta \). For \( \theta = 0 \) there is no precession of the plane since at the Equator the only effect of the Earth’s rotation to lowest order is a change in the effective value of \( g \). At the North or South pole the pendulum oscillates in a fixed plane in an inertial frame, and thus its plane of oscillation rotates with respect to the frame fixed with respect to the Earth with angular velocity \( \mp \Omega_E \).

III. THE TWO-STATE KAON SYSTEM

We review the discussion of Ref. [4]. In the limit of CP conservation, the states \( K^0 \) and \( \bar{K}^0 \) mix with one another through shared intermediate states such as \( \pi \pi \) and through short-distance second-order weak processes to form the states of definite mass and lifetime

\[
K_1 = \frac{K^0 + \bar{K}^0}{\sqrt{2}}, \quad K_2 = \frac{K^0 - \bar{K}^0}{\sqrt{2}}.
\]

(16)

When CP is violated [5], the states of definite mass and lifetime \( K_S \) (for “short”) and \( K_L \) (for “long”) can be parametrized approximately as

\[
|S\rangle \simeq |K_1\rangle + \epsilon|K_2\rangle, \quad |L\rangle \simeq |K_2\rangle + \epsilon|K_1\rangle,
\]

(17)

where we shall use \( S, L \) to denote \( K_S, K_L \). Here \( |\epsilon| \simeq (2.28 \pm 0.02) \times 10^{-3} \) and \( \text{Arg} \epsilon \simeq 43^\circ \) [8]. The states \( |S\rangle \) and \( |L\rangle \) have a scalar product \( \langle L|S\rangle \approx 2 \text{ Re } \epsilon \).

The \( K_S \) and \( K_L \) are eigenstates of a \( 2 \times 2 \) matrix \( M \) [9, 10], with the basis states \( K^0 \) and \( \bar{K}^0 \) evolving in proper time [11] as

\[
i \frac{\partial}{\partial t} \begin{bmatrix} K^0 \\ \bar{K}^0 \end{bmatrix} = M \begin{bmatrix} K^0 \\ \bar{K}^0 \end{bmatrix} ; \quad M = M - i\Gamma/2,
\]

(18)

where \( M \) and \( \Gamma \) are Hermitian. CPT invariance implies \( M_{11} = M_{22} \) and hence \( M_{11} = M_{22}, \Gamma_{11} = \Gamma_{22} \). In that case the eigenstates of \( M \) can be written

\[
|S\rangle = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left[ (1 + \epsilon)|K^0\rangle + (1 - \epsilon)|\bar{K}^0\rangle \right],
\]

(19)
\[ |L\rangle = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left[ (1 + \epsilon)|K^0\rangle - (1 - \epsilon)|\bar{K}^0\rangle \right] , \quad (20) \]

implying the approximate relation (17).

The eigenvalues of \( \mathcal{M} \) are related to its elements by

\[ \mu_S = \mathcal{M}_{11} + \sqrt{\mathcal{M}_{12}\mathcal{M}_{21}} \quad ; \quad \mu_L = \mathcal{M}_{11} - \sqrt{\mathcal{M}_{12}\mathcal{M}_{21}} . \quad (21) \]

The parameter \( \epsilon \) is related to \( \mathcal{M} \) by

\[ \epsilon = \frac{\sqrt{\mathcal{M}_{12}} - \sqrt{\mathcal{M}_{21}}}{\sqrt{\mathcal{M}_{12}} + \sqrt{\mathcal{M}_{21}}} \simeq \frac{\mathcal{M}_{12} - \mathcal{M}_{21}}{4\sqrt{\mathcal{M}_{12}\mathcal{M}_{21}}} \simeq \frac{\text{Im}(\Gamma_{12}/2) + i \text{Im}M_{12}}{\mu_S - \mu_L} . \quad (22) \]

One can show [12] that \( |\text{Im}\Gamma_{12}/2| \ll |\text{Im} M_{12}| \). This result then implies that

\[ \text{Arg} \epsilon \approx \left\{ \begin{array}{c} \frac{90^\circ}{2700} \\
\end{array} \right\} - \text{Arg}(\mu_S - \mu_L) \quad \text{for} \quad \left\{ \begin{array}{c} \text{Im}M_{12} > 0 \\
\text{Im}M_{12} < 0 
\end{array} \right\} \quad (23) \]

Given the measurements [8, 13] \( m_S - m_L = -0.474 \Gamma_S, \Gamma_S - \Gamma_L = 0.998 \Gamma_S \), we have

\[ \mu_S - \mu_L = -(0.474 + 0.499i)\Gamma_S, \quad \text{or Arg} (\mu_S - \mu_L) = (3\pi/2) - \text{arctan} (0.474/0.499) = (3\pi/2) - 43.5^\circ. \] Thus \( \text{Arg} \epsilon = (43.5 \pm 0.1)^\circ \) for \( \text{Im} M_{12} < 0 \), and \( \text{Arg} \epsilon = \pi + (43.5 \pm 0.1)^\circ \) for \( \text{Im} M_{12} > 0 \).

Equation (22) implies that \( \epsilon \neq 0 \) arises from \( \mathcal{M}_{12} \neq \mathcal{M}_{21} \). As we shall see, the Foucault pendulum provides a mechanical illustration of this behavior. Additional CP-violating effects have now been seen in the kaon system which cannot be parametrized by \( \epsilon \): specifically, the fact that the amplitude ratios \( A(L \to \pi\pi)/A(S \to \pi\pi) \) differ for charged and neutral pions [14, 15]. We shall not attempt to emulate such effects in the present work, though they are an interesting subject for future study.

One quantity sensitive to \( \epsilon \) is the asymmetry in the semileptonic decay of a neutral kaon beam to those final states which can arise from \( K^0 \) or \( \bar{K}^0 \). We shall assume the \( \Delta S = \Delta Q \) rule, where \( S \) is strangeness and \( Q \) is hadron charge. Thus the allowed processes are \( K^0 \to \pi^-\ell^+\nu_\ell \) and \( \bar{K}^0 \to \pi^+\ell^-\bar{\nu}_\ell \), where \( \ell = e \) or \( \mu \). This rule is expected to hold if the basic subprocesses governing the decays at the quark level are \( s \to u\ell^-\bar{\nu}_\ell \) and \( \bar{s} \to \bar{u}\ell^+\nu_\ell \). Suppose, for example, that all the \( K_S \) in a neutral kaon beam have decayed away, leaving pure \( K_L \). The leptonic asymmetry

\[ A_\ell \equiv \frac{\Gamma(K^0 \to \pi^-\ell^+\nu_\ell) - \Gamma(\bar{K}^0 \to \pi^+\ell^-\bar{\nu}_\ell)}{\Gamma(K^0 \to \pi^-\ell^+\nu_\ell) + \Gamma(\bar{K}^0 \to \pi^+\ell^-\bar{\nu}_\ell)} \quad (24) \]

is then just \( A_\ell = 2\text{Re}\epsilon \). Measurements of this quantity are consistent with the measured magnitude and phase of \( \epsilon \).

**IV. PARALLELS: BASIC PENDULUM**

The eigenstates \( R \) and \( L \) of the Foucault pendulum problem, which are eigenvectors of (6), can be put into correspondence with neutral-kaon eigenstates \( S \) and \( L \) by the correspondence \( R \leftrightarrow S, \ L \leftrightarrow L \). Aside from overall phases assigned to
the states, one then sees from Eq. (22) that $\epsilon = -i \text{sgn}(\Omega)$. The eigenstates have different oscillation frequencies, $\mu_S - \mu_L = 2\Omega$, but their lifetimes are the same. Since $\epsilon$ is imaginary, the eigenstates $S$ and $L$ are orthogonal to one another, just as in the CP-conserving case.

The $\pi\pi$ mode is excited by the decay of $K_1$ in (16), which corresponds in the Foucault pendulum case to an oscillation along the line $x = y$. Both the eigenmodes (7) have a component along this direction. Since the plane of oscillation of the Foucault pendulum is always rotating (as long as $\Omega \neq 0$), the excitation of the $\pi\pi$ mode will undergo variations in time as this plane rotates with angular frequency $\Omega$.

The analogue of the leptonic asymmetry $A_\ell$ discussed at the end of the previous Section is the asymmetry of the projection of oscillations of the $L$ eigenmode onto the $\hat{x} \leftrightarrow K^0$ and $\hat{y} \leftrightarrow \bar{K}^0$ directions. As one sees from (7), there is no asymmetry, only a phase difference, with respect to oscillations in the $\hat{x}$ and $\hat{y}$ directions. The same conclusion can be drawn from the fact that $\text{Re} \epsilon = 0$ in this example.

V. MODIFICATIONS TO ILLUSTRATE ACTUAL KAON SYSTEM

One needs eigenmodes with vastly different lifetimes in order to emulate the true neutral-kaon system, since $\Gamma_S/\Gamma_L = \tau_L/\tau_S \approx 579$. It is not difficult to simulate such eigenmodes in a two-state system [4, 6]. For example, two pendula of the same natural frequency can be coupled to one another through a dissipative spring [6], leading to a difference in both frequency and lifetime between the eigenmodes in which the pendula oscillate in or out of phase with respect to each other.

For the spherical pendulum, one can simply introduce damping in one of the two directions, for example by using a permanent magnet as the pendulum bob, and placing a flat coil with windings oriented along one direction just beneath the pendulum. The coil should be connected to a dissipative load (e.g., a resistor). A coil with its windings along the $\hat{x}$ direction will damp oscillations in the $\hat{y}$ direction. However, if we were to treat the $\hat{x}$ and $\hat{y}$ damping differently, we would be violating CPT invariance, since then $M_{22} \neq M_{11}$.

In the present Section we wish to emulate a CPT-preserving system, which is most convenient in a new basis. In the $(K^0, \bar{K}^0)$ basis a CPT-preserving mass matrix has the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{11} \end{bmatrix}.$$  \hfill (25)

Thus one must have equal natural frequencies and damping terms for oscillations in the $\hat{x}$ and $\hat{y}$ directions. However, one can transform [16] to the basis $(K_1, K_2)$ corresponding to oscillations in the $(K^0 \pm \bar{K}^0)/\sqrt{2} = (\hat{x} \pm \hat{y})/\sqrt{2}$ directions. In this basis the mass matrix is $N = UMU^\dagger$, where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U^\dagger.$$  \hfill (26)

One finds

$$N = \begin{bmatrix} N_{11} & N_{12} \\ -N_{12} & N_{22} \end{bmatrix}.$$  \hfill (27)
where $N_{11} = \mathcal{M}_{11} + (\mathcal{M}_{12} + \mathcal{M}_{21})/2$, $N_{22} = \mathcal{M}_{11} - (\mathcal{M}_{12} + \mathcal{M}_{21})/2$, and $N_{12} = (\mathcal{M}_{21} - \mathcal{M}_{12})/2$. Thus, in this basis, one can have a mass matrix of the form

$$
\mathcal{N} = \begin{bmatrix}
\omega_1 - i\gamma_1 & i\Omega' \\
-i\Omega' & \omega_2 - i\gamma_2
\end{bmatrix},
$$

(28)

where $\Omega'$ is not necessarily real. The CPT-invariance is expressed in the $K_1-K_2$ basis by the condition $N_{21} = -N_{12}$.

Physically one can emulate the system described by the matrix $\mathcal{N}$ by having a Foucault pendulum with different damping in the $K_1$ and $K_2$ directions. Such damping could be implemented, for example, by the inductive setup noted above. Natural frequencies in two orthogonal directions can be made to differ using a hinged set of supports. One could also introduce different damping constants in two perpendicular directions through properties of the hinged joints themselves.

The Foucault pendulum case corresponds to real $\Omega'$ for the new basis. Note that $\Omega'$ in $\mathcal{N}$ is then the same as $\Omega$ in $\mathcal{M}$; aside from a sign flip in the off-diagonal elements, $\mathcal{N}$ and $\mathcal{M}$ are the same matrix.

For $\omega_1 - i\gamma_1 \neq \omega_2 - i\gamma_2$ and $\Omega \ll \omega_{1,2}$, the eigenvectors and their corresponding eigenvalues are approximately

$$
|S\rangle = \begin{bmatrix} 1 \\ \epsilon_S \end{bmatrix}, \quad \mu_S = \omega_1 - i\gamma_1 + \delta_S ,
$$

(29)

$$
\epsilon_S = \frac{-i\Omega}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} , \quad \delta_S = \frac{\Omega^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} ,
$$

(30)

$$
|L\rangle = \begin{bmatrix} \epsilon_L \\ 1 \end{bmatrix}, \quad \mu_L = \omega_2 - i\gamma_2 + \delta_L ,
$$

(31)

$$
\epsilon_L = \frac{-i\Omega}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} = \epsilon_S \equiv \epsilon , \quad \delta_L = -i\Omega\epsilon = -\delta_S .
$$

(32)

Let us now discuss the time-evolution of states which are initially $|S\rangle$ and $|L\rangle$:

$$
|S\rangle = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} e^{-i\mu_S t} ,
$$

(33)

$$
|L\rangle = \begin{bmatrix} \epsilon \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \epsilon \\ 1 \end{bmatrix} e^{-i\mu_L t} .
$$

(34)

Defining $\phi \equiv \text{Arg}\epsilon$, these lead to the time-dependences

$$
\text{Re } x_S(t) = x_0 \text{Re } \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} e^{-i\mu_S t} \approx e^{-\gamma_1 t} \begin{bmatrix} \cos \omega_1 t \\ |\epsilon| \cos (\omega_1 t - \phi) \end{bmatrix} ,
$$

(35)

$$
\text{Re } x_L(t) = x_0 \text{Re } \begin{bmatrix} \epsilon \\ 1 \end{bmatrix} e^{-i\mu_L t} \approx e^{-\gamma_2 t} \begin{bmatrix} |\epsilon| \cos (\omega_2 t - \phi) \\ \cos \omega_2 t \end{bmatrix} ,
$$

(36)
where contributions of order $\delta S, L$ have been omitted. Thus both $S$ and $L$ eigenstates correspond to orbits which are slightly rotated to favor an orientation in the direction of $K^0$-like decays by an amount proportional to $\epsilon$. The Coriolis force in this example induces an effect which is akin to the leptonic asymmetry parameter $A_\ell$ discussed at the end of Sec. III. Moreover, the decay to $\pi\pi$ in the $K_1-K_2$ basis is like the $x$-projection of the eigenstate. Because of the Coriolis force, although the dominant damping is along the $\hat{x}$-direction (we assume $\gamma_1 \gg \gamma_2$), the eigenstate $L$ continues to have a projection proportional to $\epsilon$ along $\hat{x}$, i.e., the $K_L$ does decay to $\pi\pi$.

**VI. CPT-VIOLATING CASE**

The matrices $M$ and $N$ are arbitrary when CPT is violated. The relation between them is

$$
N_{11} = (M_{11} + M_{12} + M_{21} + M_{22})/2, \quad N_{12} = (M_{11} - M_{12} + M_{21} - M_{22})/2, \\
N_{21} = (M_{11} + M_{12} - M_{21} - M_{22})/2, \quad N_{22} = (M_{11} - M_{12} - M_{21} + M_{22})/2.
$$

A simple example of a CPT-violating mass matrix $N$ involves coupling between two resonant circuits, as discussed in Ref. [4]. If the circuits are taken to have different frequencies, the matrix takes the form

$$
N = \begin{bmatrix}
\omega_1 - i\gamma_1 & \alpha \\
\alpha & \omega_2 - i\gamma_2
\end{bmatrix},
$$

in which the off-diagonal elements are equal (rather than equal and opposite as in the CPT-preserving case). An analysis parallel to that for the eigenstates $S$ and $L$ performed in the previous Section leads to the results (for $|\alpha| \ll \omega_{1,2}$)

$$
|S\rangle = \begin{bmatrix} 1 \\ \epsilon_S \end{bmatrix}, \quad |L\rangle = \begin{bmatrix} \epsilon_L \\ 1 \end{bmatrix},
$$

with

$$
\epsilon_S = \frac{\alpha}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)} = -\epsilon_L \equiv \tilde{\epsilon},
$$

$$
\mu_S = \omega_1 - i\gamma_1 - \frac{\alpha^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}, \quad \mu_L = \omega_2 - i\gamma_2 - \frac{\alpha^2}{\omega_1 - \omega_2 - i(\gamma_1 - \gamma_2)}.
$$

The eigenstates are thus

$$
|S\rangle \simeq |1\rangle + \tilde{\epsilon}|2\rangle, \quad |L\rangle \simeq |2\rangle - \tilde{\epsilon}|1\rangle.
$$

This case (see, e.g., Ref. [17]) corresponds to invariance with respect to time-reversal, so that CP and CPT are violated. Both (17) and (42) can be written in the more general form

$$
|S\rangle \simeq |1\rangle + \epsilon_S|2\rangle, \quad |L\rangle \simeq |2\rangle + \epsilon_L|1\rangle.
$$
The CPT-preserving, CP-violating case corresponds to $\epsilon_S = \epsilon_L = \epsilon$, while the case (42) corresponds to $\epsilon_S = -\epsilon_L = \bar{\epsilon}$. To lowest order in $\epsilon_{S,L}$, one finds

$$|K^0\rangle = \frac{1}{\sqrt{2}}(|S\rangle(1-\epsilon_L) + |L\rangle(1-\epsilon_S)), \quad |\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(|S\rangle(1+\epsilon_L) - |L\rangle(1+\epsilon_S)) \quad (44)$$

Since the states $|S, L\rangle$ evolve with proper time $t$ as $|S, L\rangle \rightarrow e^{-i\mu_{S,L}t}|S, L\rangle$, a short calculation shows that

$$|K^0\rangle \rightarrow |K^0\rangle[f_+(t) + (\epsilon_S - \epsilon_L)f_-(t)] + |\bar{K}^0\rangle[1 - (\epsilon_S + \epsilon_L)]f_-(t),$$

$$|\bar{K}^0\rangle \rightarrow |\bar{K}^0\rangle[f_+(t) + (\epsilon_L - \epsilon_S)f_-(t)] + |K^0\rangle[1 + (\epsilon_S + \epsilon_L)]f_-(t), \quad (45)$$

where $f_\pm(t) \equiv [\exp(-i\mu_{S,L}t) \pm \exp(-i\mu_{S,L}t)]/2$.

For $\epsilon_L = -\epsilon_S$, the evolution of $K^0$ into $\bar{K}^0$ is the same as that for $\bar{K}^0$ into $K^0$, corresponding to a time-reversal-invariant situation. However, the amplitudes for $K^0 \rightarrow K^0$ and $\bar{K}^0 \rightarrow \bar{K}^0$ differ from one another, corresponding to CPT violation.

For $\epsilon_L = \epsilon_S = \epsilon$, the amplitudes for $K^0 \rightarrow K^0$ and $\bar{K}^0 \rightarrow \bar{K}^0$ are the same, corresponding to $CPT$ invariance, while those for $K^0 \rightarrow \bar{K}^0$ and $\bar{K}^0 \rightarrow K^0$ differ from one another, corresponding to T violation. In this case the terms $|f_-(t)|^2$ cancel in the rate asymmetry

$$A_T \equiv \frac{\Gamma[K^0(0) \rightarrow K^0(t)] - \Gamma[\bar{K}^0(0) \rightarrow \bar{K}^0(t)]}{\Gamma[K^0(0) \rightarrow \bar{K}^0(t)] + \Gamma[\bar{K}^0(0) \rightarrow K^0(t)]}, \quad (46)$$

and to lowest order one finds $[18] A_T = 4 \text{Re} \, \epsilon$. This relation has recently passed an experimental test at CPLEAR [19].

**VII. CONCLUSIONS**

We have shown that the motion of a Foucault pendulum has many features in common with the CP-violating neutral kaon system. When the natural frequencies for oscillation in two perpendicular directions and the damping terms for these directions are equal, the parameter $\epsilon$ describing the eigenstates $|K_S\rangle \simeq |K_1\rangle + \epsilon|K_2\rangle$ and $|K_L\rangle \simeq |K_2\rangle + \epsilon|K_1\rangle$ takes on the special value $\epsilon = -i$. In order to simulate the physical situation in which $|\epsilon| = O(2 \times 10^{-3})$ and $\text{Arg}(\epsilon) \simeq \pi/4$, one must construct a Foucault pendulum with slightly different natural frequencies in two perpendicular directions, and with vastly different damping constants in these directions. While the practical realization of such a construction sounds challenging, it is interesting that, at least in principle, it seems feasible entirely within the realm of classical mechanics.

The phenomenon of CPT violation, with preservation of T and violation of CP, can be emulated by coupled resonant circuits, building upon the results of Ref. [4]. The program set forth in that work is still incomplete; we would be delighted to see an implementation of CP and T violation, with CPT conservation, through the asymmetric coupling of two resonant circuits with equal frequencies.

The classical emulation of CP violation via the Foucault pendulum leaves us with one big puzzle. In the classical system, the asymmetry in coupling between the two
modes is imposed, so to speak, from the outside, through the Earth’s rotation. In the neutral-kaon system, the corresponding asymmetry in $K^0 - \bar{K}^0$ mixing is thought to arise from a complex phase in the Cabibbo-Kobayashi-Maskawa matrix [20] describing the charge-changing weak transitions of quarks. Is that phase an indication of a new fundamental asymmetry arising from physics beyond the Standard Model?

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References

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