Analytical results for string propagation near a Kaluza-Klein black hole

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Abstract

This brief report presents analytical solutions to the equations of motion of a null string. The background spacetime is a magnetically charged Kaluza-Klein black hole. The string coordinates are expanded with the world-sheet velocity of light as an expansion parameter. It is shown that the zeroth order solutions can be expressed in terms of elementary functions in an appropriate large distance approximation. In addition, a class of exact solutions corresponding to the Pollard-Gross-Perry-Sorkin monopole case are also obtained.

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String propagation in curved spacetime has been a subject of study in a large number of papers (For a review, see [1]). To study string interaction with the background spacetime, the string equations of motion are obtained from the world-sheet action and are solved by simplifying them with suitable ansatze. It has been suggested by de Vega and Nicolaidis [2] that the world-sheet velocity of light $c$, the velocity of wave propagation along the string, can be an expansion parameter. The limit of small $c$ corresponds to the strong gravity limit [2–4], the length of the string being much larger than the curvature radius.

The formalism developed in the literature [1–4] is applicable to both cosmic strings and fundamental strings. However, in the latter case the theory is formulated in higher dimensions. Thus we expect a nontrivial contribution to the string dynamics from the extra compact dimensions. In other words, it is of interest to study in what way the propagation of a string probe is affected by these extra dimensions. An attempt in this direction was made in [5]. The propagation of a null string in a Kaluza-Klein black hole background was studied. It was shown that, even at the classical level, the unfolding of an extra dimension can be seen by the string probe. The Kaluza-Klein radius, i.e., the radius around which the extra dimension winds, decreases for a magnetically charged black hole, as viewed by a null string approaching it. For an electrically charged background, the Kaluza-Klein radius behaves in the opposite sense.

In this brief report we extend the work done already in ref. [5] and obtain analytical counterparts of the solutions presented there for the magnetically charged black hole case, for small values of the scalar charge. In addition, we show that the problem is solvable exactly for the case of the Pollard-Gross-Perry-Sorkin (PGPS) monopole, which has been shown recently to arise as a solution of a suitable dimensionally reduced string theory [6]. This report is more or less technical in content. We believe our results may be of some significance since analytical solutions are presented for physically interesting higher dimensional backgrounds.

We start with the string world-sheet action [7] given by

$$ S = -T_0 \int d\tau d\sigma \sqrt{\det g_{ab}} \tag{1} $$

where $g_{ab} = G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu$ is the two-dimensional world-sheet metric; $\sigma$ and $\tau$ are the world sheet coordinates and $T_0$ is the string tension.

The classical equations of motion in the gauge $g_{ab} = \rho(\sigma, \tau) \eta_{ab}$ ($\eta_{ab}$ is the two-dimensional Minkowskian metric) are given by

$$ \partial_\tau^2 X^\mu - c^2 \partial_\sigma^2 X^\mu + \Gamma^\mu_{\nu\rho} \left[ \partial_\tau X^\nu \partial_\tau X^\rho - c^2 \partial_\sigma X^\nu \partial_\sigma X^\rho \right] = 0. \tag{2} $$

and the constraints are given by

$$ \partial_\tau X^\mu \partial_\sigma X^\nu G_{\mu\nu} = 0 \tag{3} $$

$$ [\partial_\tau X^\mu \partial_\tau X^\nu + c^2 \partial_\sigma X^\mu \partial_\sigma X^\nu] G_{\mu\nu} = 0, \tag{4} $$

c being the velocity of wave propagation along the string.

The string coordinates are expanded perturbatively with the world-sheet velocity of light as the expansion parameter, viz.,
\[ X^\mu(\sigma, \tau) = X^\mu_0(\sigma, \tau) + c^2 X^\mu_1(\sigma, \tau) + c^4 X^\mu_2(\sigma, \tau) + \ldots, \] (5)

The zeroth order \( X^\mu_0(\sigma, \tau) \) satisfies the following set of equations

\[
\begin{align*}
\ddot{X}_0^\mu + \Gamma_{\nu\rho}^\mu \dot{X}_0^\nu \dot{X}_0^\rho &= 0, \\
X_0^\mu X_0^\nu G_{\mu\nu} &= 0, \\
\dot{X}_0^\mu X_0^\nu G_{\mu\nu} &= 0,
\end{align*}
\] (6)

where dot and prime denote differentiation w.r.t. \( \tau \) and \( \sigma \) respectively. These equations describe the motion of a null string [2]. The second equation constrains the motion to be perpendicular to the string.

We study string propagation in the following metric ansatz [8]

\[ ds^2 = -e^{4k \frac{\varphi}{\sqrt{3}}} (dx_5 + 2k A_\alpha dx^\alpha)^2 + e^{-2k \frac{\varphi}{\sqrt{3}}} g_{\alpha\beta} dx^\alpha dx^\beta, \] (7)

where \( k^2 = 4\pi G; \) \( x_5 \) is the extra dimension and winds around a circle, \( \varphi \) and \( A_\mu \) represent the scalar field and the gauge field respectively. Here \( g_{\alpha\beta} \) is the four-dimensional metric.

The mass \( M \) of the black hole, the electric charge \( Q \), the magnetic charge \( P \) and the scalar charge \( \Sigma \) are constrained by

\[ \frac{2}{3} \Sigma = \frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M}, \] (8)

where the scalar charge is defined by

\[ k\varphi \longrightarrow \frac{\Sigma}{r} + O\left(\frac{1}{r^2}\right) \text{ as } r \longrightarrow \infty. \]

The notation used here is the same as that used in ref. [9]. The black hole solutions are

\[ e^{4\varphi/\sqrt{3}} = \frac{B}{A}, A_\mu dx^\mu = \frac{Q}{B} (r - \Sigma) dt + P \cos \theta d\phi \] (9)

\[ g_{\mu\nu} dx^\mu dx^\nu = \frac{f^2}{\sqrt{AB}} dt^2 - \frac{\sqrt{AB}}{f^2} dr^2 - \sqrt{AB} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

with \( A, B \) and \( f \) given by

\[ A = \left( r - \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2P^2 \Sigma}{\Sigma - \sqrt{3}M} \] (10)

\[ B = \left( r + \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2Q^2 \Sigma}{\Sigma + \sqrt{3}M} \]

\[ f^2 = (r - M)^2 - (M^2 + \Sigma^2 - P^2 - Q^2) \]

The zeroth order equations of motion for the string coordinates in a purely magnetically charged background \( (Q = 0) \) are
\[ \frac{\partial^2 t}{\partial \tau^2} + 2 \left( \frac{f'}{f} - \frac{B'}{2B} \right) \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} = 0, \]  
(11)

Here we restrict ourselves to the exterior region \( r > M \) and the equatorial plane, i.e. \( \theta = \pi/2 \).

These equations can be reduced to quadratures

\[ \tau = \int \frac{dr}{\sqrt{\frac{B}{A} c^2 - \frac{f^3}{2AB^2} (B'f - 2f'B)}} \]  
(12)

\[ x_5 = \int \frac{c_3 A dr}{B \sqrt{\frac{B}{A} c^2 - \frac{f^3}{2AB^2}}} \]

\[ t = \int \frac{c_1 B dr}{f^2 \sqrt{\frac{B}{A} c^2 - \frac{f^3}{2AB^2}}} \]

up to constants of integration which depend on \( \sigma \) and can be solved to obtain \( t, r, \phi \) and \( x_5 \) as functions of \( \tau \). Here we have taken the case of a string falling in ‘head-on’.

The integrals (13) have been evaluated numerically [5] and inverted to obtain the coordinates as function of \( \tau \) in the limit \( r >> \Sigma \). However, the quadratures can be reduced to combinations of elliptical integrals, depending on the relative values of constants \( \Sigma_1 \) and \( M \) (see [10]). The solutions reduce to elementary functions in the region where \( r \) is very large compared to the scalar charge. We take for instance the case when \( c_1 = c_3 = 1 \). The solutions (up to integration constants which depend on \( \sigma \)), retaining terms up to the first order in \( \Sigma_1 \), are

\[ \tau = -\frac{2}{3\sqrt{2(M + 3\Sigma_1)}} \left\{ r + \frac{M\Sigma_1}{M + 3\Sigma_1} \right\}^{3/2} \]  
(13)

\[ x_5 = -\frac{2}{\sqrt{2(M + 3\Sigma_1)}} \left\{ \left( \frac{r}{3} + \alpha - \frac{9\Sigma_1}{2} \right) (r - \Sigma_1)^{1/2} + \left( \frac{1}{2} \Sigma_1 - \alpha \right)^{3/2} \arctan \left\{ \frac{\sqrt{2(r - 3\Sigma_1)}}{\sqrt{\Sigma_1 - \alpha}} \right\} \right\} \]

\[ t = -\frac{2}{\sqrt{2(M + 3\Sigma_1)}} \left\{ (r - \alpha) \left( \frac{r}{3} + 2M + \Sigma_1 \right) + \frac{(2M + \Sigma_1)^2}{\sqrt{\alpha - 2M - \Sigma_1}} \arctan \left\{ \frac{\sqrt{r - \alpha}}{\sqrt{\alpha - 2M - \Sigma_1}} \right\} \right\} \]

where \( \Sigma_1 = \Sigma/\sqrt{3} \) and \( \alpha = \frac{3M\Sigma_1}{\sqrt{3} \Sigma_1 + M} \). These solutions are valid in the region outside the horizon but not asymptotically far from the black hole. The negative sign comes because we consider an ‘in-falling’ string. The solutions match with the numerical solutions reported in [5] for the corresponding values of \( c_1 \) and \( c_3 \).
The integrals of motion can be solved analytically without resorting to the limit $r \gg \Sigma$, if $P = 2M$ and $Q = 0$, i.e., the Pollard-Gross-Perry-Sorkin monopole [11]. The constraint equation (8) implies that $\Sigma_1 = -M$ or $\Sigma_1 = 2M$. In the former case, the metric reduces to the form reported in ref. [11]. Here the time coordinate $t$ is the same as the proper time $\tau$ of the string. The solutions are

$$
\tau = t = \frac{1}{\sqrt{1 - c_3^2}} \left[ (r - \beta M)^{1/2} (r + 3M)^{1/2} + (3 + \beta) M \ln \left[ (r - \beta M)^{1/2} + (r + 3M)^{1/2} \right] \right] (14)
$$

$$
x_5 = \frac{1}{\sqrt{1 - c_3^2}} \left[ (r - \beta M)^{1/2} (r + 3M)^{1/2} + (11 + \beta) M \ln \left[ (r - \beta M)^{1/2} + (r + 3M)^{1/2} \right] \right] + \frac{16M}{\sqrt{\beta - 1} \sqrt{1 - c_3^2}} \left[ \arctan \left( \frac{2 \sqrt{r - \beta M}}{\sqrt{\beta - 1} \sqrt{r + 3M}} \right) \right]
$$

where $\beta = \frac{1 + 3c_3^2}{1 - c_3^2}$. We choose $c_1 = 1$; the condition of reality of the solutions then forces $c_3 < 1$ and consequently $\beta > 1$. The solutions, physically speaking, are the same as those in the previous case. In the limit when the magnitude of the scalar charge is the same as the mass of the black hole, the quadratures are solved without expanding in powers of $\Sigma_1/r$.

In addition to the above cases, there is another solution (which was not considered in [8,9]) corresponding to $\Sigma_1 = 2M$. The integrals can be solved in terms of elliptical functions, the solutions being

$$
\tau = \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M) \left( r - \frac{10M}{7} \right)} - \frac{32M}{21 \sqrt{7}} \left[ \left\{ E \left( g(r), \frac{3}{7} \right) - 4F \left( g(r), \frac{3}{7} \right) \right\} \right] (15)
$$

$$
x_5 = \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M) \left( r - \frac{10M}{7} \right)} - \frac{32M}{21 \sqrt{7}} \left[ \left\{ 22 E \left( g(r), \frac{3}{7} \right) - 4F \left( g(r), \frac{3}{7} \right) \right\} \right]
$$

$$
t = \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M) \left( r - \frac{10M}{7} \right)} - \frac{2M}{21 \sqrt{7}} \left[ 52 F \left( g(r), \frac{3}{7} \right) \right] + \frac{M}{21 \sqrt{7}} \left[ 59 \left\{ 8M E \left( g(r), \frac{3}{7} \right) - 6F \left( g(r), \frac{3}{7} \right) \right\} \right] + \frac{2M}{21 \sqrt{7}} \left[ 63 \left\{ 4 \pi \left( g(r), \frac{3}{7} \right) \right\} \right],
$$

where $g(r) = \arcsin \left[ \frac{7}{2} \sqrt{\frac{7}{6} \sqrt{1 + 2M}} \right]$. In these expressions $F$, $E$ and $\Pi$ are elliptical functions of first, second and third kind respectively (see [10]). We have chosen $c_1 = c_3 = 1$ in this case.

We have found out analytical solutions to the equations of motion to zeroth order in the expansion in powers of $c$. The solutions obtained in the cases when $r$ is large compared to the scalar charge match with those obtained numerically in [5], for corresponding values of the integration constants $c_1$ and $c_3$. Although the results presented for the large $r$ case are technically the same as the ones obtained in [5], the analytical solutions reveal two new cases where the problem is solved exactly.

The equations of motion for the electrically charged black hole case do not reduce to quadratures (see [5]). However, the analytical results obtained in the magnetically charged case give hope that some qualitative predictions can be made in that case as well. The optimism comes from the fact that the electrically charged black hole solutions are related
to the magnetically charged ones by the duality transformation $P \rightarrow Q$, $Q \rightarrow P$ and $\Sigma \rightarrow -\Sigma$ [8,9]. These analytical zeroth order results also form the basis for higher order calculations, work on which is in progress.

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