Global structure of exact cosmological solutions in the brane world

Shinji Mukohyama

*Canadian Institute for Theoretical Astrophysics, University of Toronto*
*Toronto, ON, M5S 3H8*
*and*
*Department of Physics and Astronomy, University of Victoria*
*Victoria, BC, Canada V8W 3P6*

Tetsuya Shiromizu

*DAMTP, University of Cambridge*
*Silver Street, Cambridge CB3 9EW, United Kingdom*
*Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan*
*and*
*Research Centre for the Early Universe (RESCEU), The University of Tokyo, Tokyo 113-0033, Japan*

Kei-ichi Maeda

*Department of Physics, Waseda University, Shinjuku, Tokyo 169-8555, Japan*
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We find the explicit coordinate transformation which links two exact cosmological solutions of the brane world which have been recently discovered. This means that both solutions are exactly the same with each other. One of two solutions is described by the motion of a domain wall in the well-known 5-dimensional Schwarzshild-AdS spacetime. Hence, we can easily understand the region covered by the coordinate used by another solution.

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The Randall-Sundrum brane world [1,2] may give drastic changes to the conventional gravity theory or cosmology. Since the 3-brane world is motivated by the relation between the 11-dimensional supergravity and \( E_8 \times E_8 \) superstring theory [3], the drastic changes might be realistic. Hence we have to seriously think of this scenario. In this respect, the conventional 4-dimensional theory should be recovered in low energy limits. This recovery is easily confirmed on a brane with positive tension [4,5]. Hence for simplicity, hereafter, we will consider the single brane case with the positive tension. For the brane world cosmology with negative tension, the issue is still in debate [4–9], because the fine-tuning and radius stabilization problem seem to be necessary to recover the conventional Einstein theory and the single brane is not apparently acceptable.

Recently several authors have found exact cosmological solutions in the brane world [10–14]. In one of them, the 3-brane is described as a ‘domain wall’ moving in 5-dimensional black-hole geometries [10,11]. It may be worth noting that the radiation dominated Friedmann universe is also expected by the AdS/CFT correspondence [24,25]. Another one was given by exactly solving the Einstein equations in the Gaussian normal coordinate [12–14]. These global solutions will be important to discuss its stability as the full spacetime, the gravitational force between two bodies on the brane and the cosmological perturbation on the brane.

Since each of the two solutions is general enough in each coordinate system, it is easily expected that both solutions represent the same spacetime in different coordinate systems. In this brief note, we give the explicit coordinate transformation from the coordinate used in Refs. [10,11] to the Gaussian normal coordinate adopted in Refs. [12–14]. Moreover, we identify the region which the Gaussian normal coordinate covers.

As stated above, some authors [10,11] considered a domain-wall moving in the 5-dimensional ‘Schwarzshild-AdS’ (Sch-AdS) spacetimes [26] with the metric

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_K,
\]

where \( d\Sigma^2_K \) is a metric of a unit three-dimensional sphere, plane or hyperboloid for \( K = +1, 0 \) or \(-1\), respectively, and

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\(^1\) In this field, a lot of works related have been done so far [6,15–22].
\[ f(r) = K + \frac{r^2}{l^2} - \frac{\mu}{r^2}. \]  

Here, \( l (>0) \) and \( \mu (\geq 0) \) for \( K = +1 \) or \( K = 0, \geq -l^2/4 \) for \( K = -1 \) are constants. The constant \( l \) gives the curvature scale of the bulk spacetime. Hereafter, we denote coordinates on the three-dimensional manifold (sphere, plane or hyperboloid) by \( x^i \). Therein the domain wall is the 3-brane and the bulk spacetime is the 5-dimensional Sch-AdS spacetime. By assuming the \( Z_2 \)-symmetry which is inspired by the reduction from M-theory to \( E_8 \times E_8 \) heterotic string theory \cite{3}, we obtain the Friedmann equation on the brane:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho - \frac{K}{a^2} + \frac{\Lambda_4}{3} + \frac{\kappa^2_4}{36}\rho^2 + \frac{\mu}{3l^2}, \tag{3}
\]

where the orbit of the domain wall is described by \( r = a(\tau) \) with the proper time \( \tau \). Here, \( G_N = \kappa_4^3\lambda/48\pi \), \( \Lambda_4 = \kappa_4^2\lambda^2/12 - 3l^{-2} \) and \( \lambda \) is the vacuum energy on the brane. The deviation from the conventional Friedmann equation is expressed as the forth and fifth terms in the right-hand side of Eq. (3). The fifth term comes from the ‘electric’ part of the 5-dimensional Weyl tensor \cite{4}.

The form of Eq. (1) seems simple enough to handle while a metric derived by several authors \cite{12–14} looks rather complicated. However, as already explained above, it is expected that both of metrics express the same spacetime. Yet, the Weyl tensor for the metric derived in Refs. \cite{12–14} becomes zero in the limit of the infinite value of the fifth coordinate (or the affine parameter in the Gaussian normal coordinate). This implies that the limit of the infinite affine parameter does not correspond to the Cauchy horizon since the Weyl tensor cannot be zero except for the conformal infinity as long as the bulk spacetime is not exactly the anti-de-Sitter spacetime. In order to show these explicitly, we transform coordinates in the metric (1) so that the transformed coordinate system becomes a Gaussian normal coordinate system based on a hypersurface given by \( r = R(T) \).

First, because of the existence of the Killing field \((\partial/\partial T)\mu\) in the bulk spacetime, the unit tangent vector \( u^\mu \) of a geodesic should satisfy

\[
g_{\mu\nu}u^\mu(\partial/\partial T)^\nu = -E, \quad g_{\mu\nu}u^\mu u^\nu = 1, \tag{4}\]

where \( E \) is an integration constant. Hence, we obtain \( u^\mu \) as

\[
u^\mu \partial_\mu = \frac{E}{f(r)} \partial_\tau \pm \sqrt{f(r) + E^2} \partial_r. \tag{5}\]

In the above, we assumed \( u^i = 0 \) because we are interested only in geodesics whose tangent have zero \( x^i \)-components. The trajectory of the geodesic is given by

\[
dx^\mu = u^\mu, \tag{6}\]

where \( w \) is the affine parameter. For the case of \( 4\mu + l^2(E^2 + K)^2 > 0 \), the \( r \)-component can be integrated as

\[
2r^2 + l^2(E^2 + K) = \sqrt{4l^2\mu + l^4(E^2 + K)^2} \cosh \left[ 2l^{-1}(w + w_0) \right], \tag{7}\]

where \( w_0 \) is a constant. For the cases of \( 4\mu + l^2(E^2 + K)^2 = 0 \) and \( 4\mu + l^2(E^2 + K)^2 < 0 \), the \( r \)-component of Eq. (6) are integrated to give different expressions of \( r \) in terms of \( w \). However, the final form of the metric we shall obtain below is common for all cases. Hence, in the following arguments we show explicit calculation for the first case only. Let us determine the constants \( E \) and \( w_0 \) so that the geodesic intersects with the hypersurface \( r = R(T) \) perpendicularly at \( T = T_0 \) and that the affine parameter \( w \) is zero on the hypersurface:

\[
u^\mu \propto g^{\mu\nu} \partial_\nu(r - R(T)) \quad \text{at} \quad T = T_0, r = R(T_0), \quad 2R^2(T_0) + l^2(E^2 + K) = \sqrt{4l^2\mu + l^4(E^2 + K)^2} \cosh \left( 2l^{-1}w_0 \right). \tag{8}\]

These can be solved to give

\[
E = E(T_0) \equiv \pm R'(T_0) \sqrt{\frac{f(R(T_0))}{R'^2(R(T_0)) - R'^2(T_0))}}, \tag{9}\]

\[
w_0 = w_0(T_0) \equiv \frac{l}{2} \cosh^{-1} \left[ \frac{2R^2(T_0) + l^2(E^2(T_0) + K)}{\sqrt{4l^2\mu + l^4(E^2 + K)^2}} \right], \tag{10}\]

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where we have taken a convention such that $\cosh^{-1} X > 0$ for $X > 1$. Note that the set $(T_0, w, x^i)$ can be considered as a coordinate system.

Next, we can introduce a new coordinate $t$ so that $t$ becomes a proper time on the hypersurface $r = R(T)$:

$$
\left( \frac{\partial t}{\partial T_0} \right)_{w,x^i} = \frac{f(a(t))}{\sqrt{f(a(t)) + \dot{a}^2(t)}},
$$

$$
\left( \frac{\partial t}{\partial w} \right)_{T_0,x^i} = \left( \frac{\partial}{\partial x^i} \right)_{T_0,w} = 0,
$$

where $a(t) = R(T_0(t))$. Note that the function $a(t)$ is well-defined in a whole coordinate patch as well as on the hypersurface $r = r(T)$, or $w = 0$, since both $t$ and $T_0$ are constant along the geodesic. In this coordinate, $E(T_0)$ has a simple expression

$$
E(T_0) = \pm \dot{a}(t),
$$

where a dot denotes $(\partial/\partial t)_{w,x^i}$. Hence the original coordinate $r$ is given in terms of new coordinates $t$ and $w$ as

$$
r^2 = \varphi(t, w)a^2(t),
$$

where

$$
\varphi(t, w) = \cosh(2l-1w) + \frac{l^2}{2}(H^2 + Ka^{-2})(\cosh(2l-1w) - 1) \pm \sqrt{1 + l^2(H^2 + Ka^{-2} - \mu a^{-4})} \sinh(2l-1w).
$$

Here $H(t)$ is defined by

$$
H(t) = \frac{\dot{a}(t)}{a(t)}.
$$

Now let us confirm that the new coordinate system $(t, w, x^i)$ is actually a Gaussian normal coordinate system. For this purpose, it is sufficient to show that

$$
dw = g_{\mu\nu}u^\mu dx^\nu,
$$

or

$$
\left( \frac{\partial w}{\partial T} \right)_{r,x^i} = \mp \dot{a}(t),
$$

$$
\left( \frac{\partial w}{\partial r} \right)_{T,x^i} = \pm \frac{\sqrt{f(r) + \dot{a}^2(t)}}{f(r)} t,.
$$

$$
\left( \frac{\partial w}{\partial x^i} \right)_{T,r} = 0
$$

is integrable. The integrability condition $d^2w = 0$ is equivalent to

$$
\left( \frac{\partial t}{\partial T} \right)_{r,x^i} / \left( \frac{\partial t}{\partial r} \right)_{T,x^i} = -\frac{f(r)\sqrt{f(r) + \dot{a}^2}}{\dot{a}}.
$$

This condition is easily confirmed by using the relation

$$
\left( \frac{\partial t}{\partial T} \right)_{r,x^i} / \left( \frac{\partial t}{\partial r} \right)_{T,x^i} = -\left( \frac{\partial t}{\partial w} \right)_{T,x^i} / \left( \frac{\partial T}{\partial w} \right)_{T,x^i},
$$

and

$$
\left( \frac{\partial T}{\partial w} \right)_{T,x^i} = \left( \frac{\partial T}{\partial w} \right)_{T_0,x^i} = u^T = \pm \frac{\dot{a}(t)}{f(r)},
$$

$$
\left( \frac{\partial r}{\partial w} \right)_{T,x^i} = \left( \frac{\partial r}{\partial w} \right)_{T_0,x^i} = u^r = \pm \sqrt{f(r) + \dot{a}^2}.
$$

(13)
Thus, it is easily shown that

\[ t, w, x \]

In the following arguments we shall show that the Gaussian normal coordinate system (\( \psi \)) by defining \( k \) and \( C \) by

\[ k = 2 l^{-1}, \]
\[ C = -l^2 \mu, \]

the functions \( \psi(t, w) \) and \( \varphi(t, w) \) are rewritten as

\[ \psi(t, w) = \cosh(kw + 2k^{-2}(H^2 + \dot{H})(\cosh(kw) - 1)) \pm \frac{1 + 2k^{-2}(2H^2 + \dot{H} + Ka^{-2})}{\sqrt{1 + 4k^{-2}(H^2 + Ka^{-2}) + Ca^{-4}}} \sinh(kw), \]
\[ \varphi(t, w) = \cosh(kw + 2k^{-2}(H^2 + Ka^{-2})(\cosh(kw) - 1)) \pm \sqrt{1 + 4k^{-2}(H^2 + Ka^{-2}) + Ca^{-4}} \sinh(kw). \]

Using the Friedmann equation of Eq. (3) and setting \( C = Ca^{-4})^2 \), we can see that the metric (23) with (26) is equivalent to the expression obtained in Ref. [12]. As shown in Refs. [12–14] the lower signs should be taken in all equations if we glue two copies of the region \( w \geq 0 \) to obtain the brane-world with positive tension. Thus, hereafter, we take the lower signs.

From now on we show where the Gaussian normal coordinate covers in the Sch-AdS spacetime. For simplicity, we will consider only \( \Lambda_4 = 0 \) cases. We concentrate on the \( \mu \neq 0 \) cases since the \( \mu = 0 \) case is easier and can be understood in a similar way. Note that the metric (1) has an event horizon at \( r = r_h \), where \( r_h \) is given by

\[ r_h^2 = \frac{l^2}{2}(\sqrt{K^2 + 4l^{-2}\mu} - K). \]

In the following arguments we shall show that the Gaussian normal coordinate system \( (t, w, x') \) covers the region beyond the event horizon and the wormhole.

First, let us consider the case when the condition

\[ (H^2a^2 + K)^2 + 4l^{-2}\mu > 0 \]

is satisfied. For \( K = +1 \), this condition is automatically satisfied since \( \mu \geq 0 \) for the bulk black hole spacetimes. This condition is automatically satisfied also when \( K = 0 \) and \( Ha \neq 0 \). In this case, it is easily shown that

\[ r^2 = \varphi(t, w)a(t)^2 \rightarrow \infty \quad (w \rightarrow \infty) \]

and that

\[ r^2 = \varphi(t, w)a(t)^2 \geq r_{min}^2, \quad (w \geq 0) \]

where

\[ r_{min}^2 = \frac{l^2}{2} \left[ \sqrt{(H^2a^2 + K)^2 + 4l^{-2}\mu} - (H^2a^2 + K) \right]. \]
Note that \( r_{\text{min}}(t) \) has been determined by the variation of \( w \) under a fixed \( t \). The equality in Eq.(30) holds at \( w = w_{\text{min}}(t) \), where \( w_{\text{min}}(t) > 0 \) is given by

\[
\cosh(2t^{-1}w_{\text{min}}(t)) = \frac{2t^{-2}a^2 + H^2 a^2 + K}{\sqrt{(H^2 a^2 + K)^2 + 4t^{-2}\mu}}. \tag{32}
\]

We can easily show that \( r_{\text{min}} \to 0 \) and \( w_{\text{min}} \to 0 \) as \( a \to 0 \) by using the Friedmann equation on the brane. On the other hand, there is another minimum \( r^*_{\text{min}}(w) \) which is determined by the variation of \( t \) under a fixed \( w \). The orbit of \( r^*_{\text{min}}(w) \) is the same as \( \psi(t, w) = 0 \) because \( \psi \) is proportional to \( \delta r^2 \). These minimum will be important when we draw the conformal diagram.

Now it is easily shown by using the lower-limit of \( \mu \) (\( \mu \geq 0 \) for \( K = +1 \) and \( K = 0 \), \( \mu \geq -l^2/4 \) for \( K = -1 \), and we are considering the \( \mu \neq 0 \) case) that \( r^2_{\text{min}} \leq r^2_h \). The equality holds if and only if \( H a = 0 \). Hence, the coordinate \((t, w, x')\) actually covers the region beyond the event horizon and the wormhole.

Next, let us consider the case when \( K = -1 \) and the condition

\[
(H^2 a^2 - 1)^2 + 4t^{-2} \mu = 0 \tag{33}
\]

is satisfied. In this case, \( r^2 \) approaches to a finite value in the limit \( w \to \infty \):

\[
r^2 = \varphi(t, w)a(t)^2 \to \frac{l^2}{2}(1 - H^2 a^2) < \frac{l^2}{2} \quad (w \to \infty). \tag{34}
\]

On the other hand,

\[
r^2_h > \frac{l^2}{2}. \tag{35}
\]

Therefore, the coordinate \((t, w, x')\) reaches the region beyond the event horizon.

Finally, let us consider the case of \( K = -1 \) and the time when condition

\[
(H^2 a^2 - 1)^2 + 4t^{-2} \mu < 0 \tag{36}
\]

is satisfied. In this case, there exists a value of \( w > 0 \) such that \( r^2 = 0 \). Note that \( r = 0 \) corresponds to a singularity inside the horizon since tetrad components of the Weyl tensor for the metric (1) diverge at \( r = 0 \). Thus, the coordinate \((t, w, x')\) covers the region inside the event horizon and reaches the singularity inside the horizon.

Now it is easy to see the global structure. Figure 1 is for \( \mu > 0 \), \( K = +1 \). Figure 2 is for \( \mu > 0 \), \( K = -1 \). Figures 3 and 4 are for \( \mu = 0 \), \( K = +1 \) and for \( \mu = 0 \), \( K = 0, -1 \), respectively. Figure 5 is for \( \mu < 0 \), \( K = -1 \). In these figures, the region covered by the Gaussian normal coordinate is expressed as the shaded region. Note that in Figures 1, 2 and 5, for a non-zero small value of \( w \), \( \partial_t \) should be past directed for small \( t \), turns to the opposite direction at a time \( t^* \) when the orbit of \( \partial_t \) reaches the \( r^*_{\text{min}}(w) \), and future directed for large \( t \).

To obtain these figures, we have used the following two facts. First, the hypersurface \( t = t_0 \) is always spacelike, and should come in contact with the hypersurface \( r = r_{\text{min}}(t_0) \) at \( w = w_{\text{min}}(t_0) \), where \( t_0 \) is a constant. Secondly, for a fixed \( w, r \to \infty \) in the limits \( a \to 0 \) and \( a \to \infty \), providing some reasonable assumptions (eg. \( \Delta_4 = 0 \) and \( pa^2 \to 0 \) in the limit \( a \to \infty \)). This means that the constant-\( t \) hypersurface should become null in these limits.

In this brief note, we have given the coordinate transformation between the metric of Eq. (1) and of Eq. (23). As a result, we could see the region where the Gaussian normal coordinate covers. For general cases of single brane with the positive tension and \( \mu \neq 0 \), the coordinate \( w \) labelling the extra dimension does not terminate at the ‘Cauchy horizon’ and goes beyond the event horizon. For some cases, the coordinate goes through the wormhole and reaches another domain of communication where we can define the total energy well \([27]\). The energy should be \( \mu \) for cases which we considered and we can prove the positivity on the slice without naked singularities at least \( K = 0 \) cases \([28,29]\). This gives us the good news to the general brane with the positive tension. According to Ref. \([4]\), the effective 4-dimensional Einstein equation is

\[
(4) G_{\mu\nu} = -\Lambda q_{\mu\nu} + 8\pi G_N T_{\mu\nu} + \kappa^2 \Pi_{\mu\nu} - E_{\mu\nu}, \tag{37}
\]

where \( T_{\mu\nu} \) is the energy-momentum tensor on the brane, \( \Pi_{\mu\nu} = -(1/4)T^{\mu\nu}T_{\mu\nu} + (1/12)TT_{\mu\nu} + \cdots \) and \( E_{\mu\nu} = (5)G_{\mu\nu}/\sqrt{n^\alpha n^\beta} \) is the ‘electric’ part of the 5-dimensional Weyl tensor. For general issues such as the positivity of the ADM energy, cosmic no-hair conjecture \([31]\), singularity theorem \([32]\) and so on, it is important to clear the situation whether the right-hand side of Eq. (37) satisfies the local energy condition \([32]\) or not. For the Sch-AdS spacetime, \( E_{00} \sim -\mu/a^4 \) and then \( -E_{00} \geq 0 \) is guaranteed. On the other hand, the perturbation analysis on the Randall & Sundrum brane world suggests \( E_{00} \geq 0 \) \([33]\) and this may be bad news. The signature of \( E_{00} \) seems to depend on situations. The full non-linear analysis may be needed.
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FIG. 1. The global structure for $\mu > 0$, $K = +1$. The world volume of the brane starts at $r = 0$ and ends at $r = 0$. The function $r_{\min}(t)$ increases from zero to $r_h$, and decreases to zero. Accordingly, the hypersurface $w = w_{\min}(t)$ starts at $r = 0$ with the brane, passes through the bifurcating point of the horizon, and ends at $r = 0$ with the brane. Thus, the region covered by the Gaussian normal coordinate should be the shaded region. Two dashed lines in the figure are a constant-$t$ hypersurface and a constant-$w$ hypersurface.

FIG. 2. The global structure for $\mu > 0$, $K = 0, -1$. The world volume of the brane starts at $r = 0$ and ends at $r = \infty$. The function $r_{\min}(t)$ starts with the value zero and ends with the value $r_{\min}(\infty)$, where $r_{\min}(\infty) = r_h$ for $K = 0$ and $r_{\min}(\infty) = (l^2 \mu)^{1/4} < r_h$ for $K = -1$. (We have assumed that $\Lambda_4 = 0$ and $\rho a^2 \to 0$ in the limit $a \to \infty$.) Thus, the region covered by the Gaussian normal coordinate should be the shaded region. Note that hypersurfaces $w = w_{\min}(t)$ and $\psi(t, w) = 0$ start at the point where the brane starts, and end at the point where $r = r_{\min}(\infty)$ comes in contact with the horizon. Two dashed lines in the figure are a constant-$t$ hypersurface and a constant-$w$ hypersurface.

FIG. 3. The global structure for $\mu = 0$, $K = +1$. The world volume of the brane starts at $r = 0$ and ends at $r = 0$. The minimum of $r$ for a fixed $t$ is always zero. Thus, the region covered by the Gaussian normal coordinate should be the shaded region.

FIG. 4. The global structure for $\mu = 0$, $K = 0, -1$. The world volume of the brane starts at $r = 0$ and ends at $r = \infty$. The minimum of $r$ for a fixed value of $t$ is always zero. Thus, the region covered by the Gaussian normal coordinate should be the shaded region.

FIG. 5. The global structure for $\mu < 0$, $K = 1$. The world volume of the brane starts at $r = 0$ and ends at $r = \infty$. The minimum of $r$ for a fixed $t$ becomes zero in the limits $a \to 0$ and $a \to \infty$. Thus, the region covered by the Gaussian normal coordinate should be the shaded region. Note that hypersurfaces $w = w_{\min}(t)$ and $\psi(t, w) = 0$ start at the point where the brane starts, and end at $r = 0$. Two dashed lines in the figure are a constant-$t$ hypersurface and a constant-$w$ hypersurface.