SUBMERSIONS AND EQUIVARIANT QUILLEN METRICS

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Abstract

In this paper, we calculate the behaviour of the equivariant Quillen metric by submersions. We thus extend a formula of Berthomieu-Bismut to the equivariant case.

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Résumé. Dans cet article, on calcule le comportement de la métrique de Quillen équivariante par submersions. On étend ainsi une formule de Berthomieu-Bismut au cas équivariant.

Introduction

Let $\xi$ be a Hermitian vector bundle on a compact Hermitian complex manifold $X$. By Hodge theory, we can identify $H(M, \xi)$, the cohomology of $\xi$, with the corresponding harmonic elements in the Dolbeault complex $\Omega^* (M, \xi)$. Let $h^H(M,\xi)$ be the corresponding $L^2$-metric on $H(M,\xi)$.

Let $\lambda(\xi)$ be the inverse of the determinant of the cohomology of $\xi$. Quillen defined first a metric on $\lambda(\xi)$ in the case that $X$ is a Riemann surface. So we call “Quillen metric”. This is the product of the $L^2$ metric on $\lambda(\xi)$ by the analytic torsion of Ray-Singer of $\xi$. The analytic torsion of Ray-Singer [RS] is the regularized determinant of the Kodaira Laplacian on $\xi$. In [BGS3], Bismut, Gillet, and Soulé have extended it to complex manifolds. They have established the anomaly formulas for Quillen metrics, which tell us the variation of Quillen metric on the metrics on $x$ and $TX$ by using some Bott-Chern classes.

Later, Bismut and Köhler [BK6] have extended the analytic torsion of Ray-Singer to the analytic torsion forms $T$ for a holomorphic submersion. In particular, the equation on $\frac{\Delta}{2\pi} T$ gives a refinement of the Grothendieck-Riemann-Roch Theorem. They have established also the corresponding anomaly formulas.

In [GS1], Gillet and Soulé had conjectured an arithmetic Riemann-Roch Theorem in Arakelov geometry. In [GS2], they have proved it for the first Chern class. The analytic torsion forms are contained in their definition of direct image.

Let $i : Y \to X$ be an immersion of compact complex manifolds. Let $\eta$ be a holomorphic vector bundle on $Y$, and let $(\xi, \sigma)$ be a complex of holomorphic vector bundles which provides a resolution of $i_* \eta$. Then by [KM], the line $\lambda^{-1}(\eta) \otimes \lambda(\xi)$ has a nonzero canonical section $\sigma$. In [BL], Bismut and Lebeau have given a formula for the Quillen norm of $\sigma$ in terms of Bott-Chern currents on $X$ and of a genus $R$ introduced by Gillet and Soulé [GS1]. Recently, in [B6], Bismut has extended this result to a relative situation. This result and the precedent works have completed the proof of the arithmetic Riemann-Roch Theorem.

In [BerB], Bismut and Berthomieu solved a similar problem. In fact, let $\pi : M \to B$ be a submersion of compact complex manifolds. Let $\xi$ be a holomorphic vector bundle on $M$. Let $R^*\pi, \xi$ be the direct image of $\xi$. Then, by [KM], the line $\lambda(\xi) \otimes \lambda^{-1}(R^*\pi, \xi)$ has a nonzero canonical section $\sigma$. In [BerB], they have given a formula for the Quillen norm of $\sigma$ in terms of Bott-Chern classes on $M$ and the analytic torsion forms of $\pi$. Recently, Ma [Ma] has extended this result to a relative situation.

Another side, let $G$ be a compact Lie group acting holomorphically on every object on $X$. Then Bismut [B5] defined $\lambda_G(\xi)$ the inverse of the equivariant determinant of the cohomology of $\xi$ on $X$. He also defined an equivariant Quillen metric on $\lambda_G(\xi)$ which is a central function on $G$ (refer also §1a)). In [B5], Bismut calculated the equivariant Quillen metric of the nonzero canonical section of $\lambda_G^{-1}(\eta) \otimes \lambda_G(\xi)$ for a $G$-equivariant immersion $i : Y \to X$. In this way, he has generalized the result of [BL] to the equivariant case. In [B4], he also conjectured an equivariant arithmetic Riemann-Roch Theorem in Arakelov geometry. Recently, using the result of [B5], Köhler and Roessler [KRo] have given a version of this conjecture.

In this paper, we shall extend the result of Bismut and Berthomieu to $G$-equivariant case. This completes the picture on $G$-equivariant case.

Let $\pi : M \to B$ be a submersion of compact complex manifolds with fibre $X$. Let $\xi$ be a holomorphic vector bundle on $M$. Let $G$ be a compact Lie group acting holomorphically on $M$ and $B$, and commuting with $\pi$, whose actions lift holomorphically on $\xi$.

Let $R^k\pi_* \xi$ be the direct image of $\xi$. We assume that the $R^k\pi_* \xi (0 \leq k \leq \dim X)$ are locally
free.

Let $\sigma$ be the canonical section of $\lambda_G(\xi) \otimes \lambda_G^{-1}(R, \pi, \xi)$.

Let $h^TM, h^TB$ be $G$-invariant Kähler metrics on $TM$ and $TB$. Let $h^TX$ be the metric induced by $h^TM$ on $TX$. Let $h^\xi$ be a $G$-invariant Hermitian metric on $\xi$. Let $\omega^M$ be the Kähler form of $h^TM$.

Let $\|\|_{\lambda_G(\xi) \otimes \lambda_G^{-1}(R, \pi, \xi)}$ be the $G$-equivariant Quillen metric on the line $\lambda_G(\xi) \otimes \lambda_G^{-1}(R, \pi, \xi)$ attached to the metrics $h^TM, h^\xi, h^TB, h^R_{(X, \xi, X)}$ on $TM, \xi, TB, R^\pi, \xi$. The purpose of this paper is to calculate the $G$-equivariant Quillen metric $\|\|_{\lambda_G(\xi) \otimes \lambda_G^{-1}(R, \pi, \xi)}$.

For $g \in G$, let $\text{Td}_g(TM, g^TM)$ be the Chern-Weil Todd form on $M^g = \{x \in M, gx = x\}$ associated to the holomorphic hermitian connection on $(TM, h^TM)$ [B5, §2(a)], which appears in the Lefschetz formulas of Atiyah-Bott [ABo]. Other Chern-Weil forms will be denoted in a similar way. In particular, the forms $\text{ch}_d(\xi, h^\xi)$ on $M^g$ are the Chern-Weil representative of the $g$-Chern character form of $(\xi, h^\xi)$.

In this paper, by an extension of [BK6], we first construct the equivariant analytic torsion forms $T_g(\omega^M, h^\xi)$ on $B^g$, such that

$$\frac{\partial}{\partial \theta} T_g(\omega^M, h^\xi) = \text{ch}_d(H(X, \xi, X), h^R_{(X, \xi, X)} - \int_{X^g} \text{Td}_g(TX, h^TX) \text{ch}_d(\xi, h^\xi).$$

We also establish the corresponding anomaly formulas. Remark that in [KR0], they have also defined the forms $T_g(\omega^M, h^\xi)$.

Let $\text{Td}_g(TM, TB, h^TM, h^TB) \in P^{M^0}/P^{M^0,0}$ be the Bott-Chern class, constructed in [BGS1], such that

$$\frac{\partial}{\partial \theta} \text{Td}_g(TM, TB, h^TM, h^TB) = \text{Td}_g(TM, h^TM) - \pi^*(\text{Td}_g(TB, h^TB)) \text{Td}_g(TX, h^TX).$$

The main result of this paper is the following extension of [BerB, Theorem 3.1]. Namely, we prove in Theorem 3.1 the formula

$$\log(\|\|_{\lambda_G(\xi) \otimes \lambda_G^{-1}(R, \pi, \xi)}(g)) = -\int_{B^g} \text{Td}_g(TB, h^TB) T_g(\omega^M, h^\xi) + \int_{M^g} \text{Td}_g(TM, TB, h^TM, h^TB) \text{ch}_d(\xi, h^\xi).$$

We apply the methods and techniques in [BerB] and [B5], with necessary equivariant extensions, to prove Theorem 3.1. The local index theory [B1] and finite propagation speed of the solution of the hyperbolic equation [CP], [T] will also play an important role as in [BerB] and [B5].

This paper is organized as follows.

In Section 1, we recall the construction of the equivariant Quillen metrics [B5]. In Section 2, we construct the equivariant analytic torsion forms, and we prove the corresponding anomaly formulas. In Section 3, we extend the result of [BerB] to the equivariant case. In Section 4, we state eight intermediary results which we need for the proof of Theorem 3.1, and we prove Theorem 3.1. In Sections 5-9, by combining the techniques of [BerB] and [B5], we prove the eight intermediary results.

Throughout, we use the superconnection formalism of Quillen [Q1]. The reader is referred for more details to [B5, BGS1, BerB].

1 Equivariant Quillen metrics

This Section is organized as follows. In a), we recall the construction of the equivariant Quillen metrics of [B5, §1]. In b), we indicate the characteristic classes which we will often use.
a) Equivariant Quillen metrics [B5].

Let $X$ be a compact complex manifold of complex dimension $l$. Let $\xi$ be a holomorphic vector bundle on $X$.

Let $G$ be a compact Lie group. We assume that $G$ acts on $X$ by holomorphic diffeomorphisms and that the action of $G$ lifts to a linear holomorphic action on $\xi$.

Let $E = \oplus_{i=0}^{\dim X} E^i$ be the vector space of $C^\infty$ sections of $\Lambda(T^{*\langle 0,1 \rangle} X) \otimes \xi$ over $X$. Let $\overline{\partial}^X$ be the Dolbeault operator acting on $E$. Then $G$ acts on $(E, \overline{\partial}^X)$ by chain homomorphisms, and we have an identification of $G$-spaces

\[ H(E, \overline{\partial}^X) \simeq H(X, \xi). \]

Let $h^{TX}, h^\xi$ be $G$-invariant Hermitian metrics on $TX, \xi$.

Let $dv_X$ be the volume element on $X$ associated to $h^{TX}$. Let $\langle . , . \rangle_{\Lambda(T^{*\langle 0,1 \rangle} X) \otimes \xi}$ be the Hermitian product induced by $h^{TX}, h^\xi$ on $\Lambda(T^{*\langle 0,1 \rangle} X) \otimes \xi$. If $s, s' \in E$, set

\[ \langle s, s' \rangle = \left( \frac{1}{2\pi} \right)^{\dim X} \int_X \langle s, s' \rangle_{\Lambda(T^{*\langle 0,1 \rangle} X) \otimes \xi} dv_X. \]

Let $\overline{\partial}^X_s$ be the formal adjoint of $\overline{\partial}^X$ with respect to the Hermitian product (1.2). Set

\[ D^X = \overline{\partial}^X + \overline{\partial}^X_s, \]

\[ K(X, \xi) = \text{Ker} D^X. \]

By Hodge theory,

\[ K(X, \xi) \simeq H(X, \xi). \]

Clearly, for $g \in G$, $g$ commute to $D^X$, so (1.4) is an identification of $G$-spaces.

Clearly $K(X, \xi)$ inherits a $G$-invariant metric from $\langle . , . \rangle$. Let $h^{H(X, \xi)}$ be the corresponding metric on $H(X, \xi)$.

Let $F$ be a finite dimension $G$-vector space. Let $h^F$ be a $G$-invariant metric on $F$. Then we have the isotypical decomposition

\[ F = \oplus_{W \in \hat{G}} \text{Hom}_G(W, F) \otimes W, \]

and this decomposition is orthogonal with respect to $h^F$. Let

\[ \det(F, G) = \oplus_{W \in \hat{G}} \left( \det(\text{Hom}_G(W, F) \otimes W) \right)^{-1} \]

For $W \in \hat{G}$, set

\[ \lambda_W(\xi) = (\det(\text{Hom}_G(W, H(X, \xi)) \otimes W))^{-1}. \]

Put

\[ \lambda_G(\xi) = \oplus_{W \in \hat{G}} \lambda_W(\xi). \]

In the sequel, $\lambda_G(\xi)$ will be called the inverse of the equivariant determinant of the cohomology of $\xi$. Then $\lambda_G(\xi)$ is a direct sum of complex lines.
Let $|\lambda_W(\xi)|$ be the metric on $\lambda_W(\xi)$ induced by $h^{H(X, \xi)}$. Set

$$
(1.8) \quad \log(\||^{2}_{\lambda_W(\xi)}(g)) = \sum_{W \in G} \log(\||^{2}_{\lambda_W(\xi)}(g)^{2}) r_{k}(W).
$$

The symbol $|\lambda_W(\xi)|$ will be called the (equivariant) $L_2$ metric on $\lambda_G(\xi)$.

Take $g \in G$. Set

$$
(1.9) \quad X^{g} = \{x \in X, gx = x\}.
$$

Then $X^{g}$ is a compact complex totally geodesic submanifold of $X$.

Let $P$ be the orthogonal projection operator from $E$ on $K(X, \xi)$ with respect to the Hermitian product $(1.2)$. Set $P^{\perp} = 1 - P$. Let $N$ be the number operator of $E$, i.e. $N$ acts by multiplication by $i$ on $E^{i}$. Then by standard heat equation methods, we know that as $t \to 0$, for any $g \in G$, $k \in \mathbb{N}$,

$$
(1.10) \quad \text{Tr}_{k}[gN \exp(-tD^{X,2})] = \sum_{j=0}^{k} a_{j} t^{j} + O(t^{k}).
$$

**Definition 1.1.** For $s \in \mathbb{C}$, $\text{Re}(s) > \dim X$, set

$$
(1.11) \quad \theta^{X}(s)(g) = -\text{Tr}_{k}[gN(D^{X,2})^{-s} P^{\perp}].
$$

By (1.10), $\theta^{X}(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$.

**Definition 1.2.** For $g \in G$, set

$$
(1.12) \quad \log(\||^{2}_{\lambda_{G}(\xi)}(g)) = \log(\||^{2}_{\lambda_{G}(\xi)}(g)) - \frac{\partial \theta^{X}(s)}{\partial s}(0).
$$

The symbol $\||^{2}_{\lambda_{G}(\xi)}$ will be called a Quillen metric on the equivariant determinant $\lambda_{G}(\xi)$.

b) Some characteristic classes.

Let $X$ be a complex manifold. Let $h^{TX}$ be a Hermitian metric on $TX$. Let $L$ be a holomorphic vector bundle over $X$. Let $h^{L}$ be a Hermitian metric on $L$.

Let $\nabla^{L}$ be the holomorphic Hermitian connection on $(L, h^{L})$. Let $R^{L}$ be its curvature.

Let $g$ be a holomorphic section of $\text{End}(L)$. We assume that $g$ is an isometry of $L$. Then $g$ is parallel with respect to $\nabla^{L}$.

Let $1, \theta_{1}, \ldots , \theta_{q}(0 < \theta_{j} < 2\pi)$ be the locally constant distinct eigenvalues of $g$ acting on $L$ on $X$. Let $L$, $L_{\theta_{1}}, \ldots , L_{\theta_{q}}(\theta_{0} = 0)$ be the corresponding eigenspaces. Then $L$ splits holomorphically as an orthogonal sum

$$
(1.13) \quad L = L_{\theta_{0}} \oplus \cdots \oplus L_{\theta_{q}}.
$$

Let $h^{L_{\theta_{0}}, \ldots , h^{L_{\theta_{q}}}$ be the Hermitian metrics induced by $h^{L}$. Then $\nabla^{L}$ induces the holomorphic Hermitian connections $\nabla^{L_{\theta_{0}}, \ldots , \nabla^{L_{\theta_{q}}}$ on $(L_{\theta_{0}}, h^{L_{\theta_{0}}), \ldots , (L_{\theta_{q}}, h^{L_{\theta_{q}}).}$ Let $R_{\theta_{0}}, \ldots , R_{\theta_{q}}$ be their curvatures.

If $A$ is $(q, q)$ matrix, set

$$
(1.14) \quad \text{Td}(A) = \det(\frac{A}{1 - e^{-A}}), \quad e(A) = \det(A), \quad \text{ch}(A) = \text{Tr}[\exp(A)].
$$
The genera associated to \( Td \) and \( e \) are called the Todd genus and the Euler genus.

**Definition 1.3.** Set

\[
\begin{align*}
Td_g(L, h^L) &= Td(-\frac{R^{L_0}}{2i\pi})\Pi_j\frac{Td(-\frac{R^{L_j}}{2i\pi})}{e^{i\theta_j}}, \\
Td'_g(L, h^L) &= \frac{\partial}{\partial b}[Td^{-1}\left(-\frac{R^{L_0}}{2i\pi} + b\right)] \\
(Td^{-1})'(L, h^L) &= \frac{\partial}{\partial b}[Td^{-1}\left(-\frac{R^{L_0}}{2i\pi} + b\right)] \\
ch_g(L, h^L) &= \text{Tr}[g\exp(-\frac{R^L}{2i\pi})].
\end{align*}
\]

Then the forms in (1.15) are closed forms on \( X \), which lie in \( P^X \), and their cohomology class does not depend on the \( g \) invariant metric \( h^L \). We denote these cohomology classes by \( Td_g(L) \), \( Td'_g(L), \ldots, ch_g(L) \).

2 **Equivariant analytic torsion forms and anomaly formulas**

This Section is organized as follows. In a), we describe the Kähler fibrations. In b), we construct the Levi-Civita superconnection in the sense of \([\text{B}1]\). In c), we indicate results above the equivariant superconnection forms. In d), we construct the equivariant analytic torsion forms. In e), we prove the anomaly formulas, along the lines of \([\text{B}5], [\text{BK}0]\).

a) Kähler fibrations.

Let \( \pi : M \to B \) be a holomorphic submersion with compact fibre \( X \). Let \( TM, TB \) be the holomorphic tangent bundles to \( M, B \). Let \( TX \) be the holomorphic relative tangent bundle \( TM/B \). Let \( J^{TX} \) be the complex structure on the real tangent bundle \( T_RX \). Let \( h^{TX} \) be a Hermitian metric on \( TX \).

Let \( T^HM \) be a vector subbundle of \( TM \), such that

\[
T^HM = T^HM \oplus TX.
\]

We now define the Kähler fibration as in \([\text{BGS2}, \text{Definition 1.4}]\).

**Definition 2.1.** The triple \((\pi, h^{TX}, T^HM)\) is said to define a Kähler fibration if there exists a smooth real 2-form \( \omega \) of complex type (1,1), which has the following properties :

a) \( \omega \) is closed.

b) \( T^H_RX \) and \( T_RX \) are orthogonal with respect to \( \omega \).

c) If \( X, Y \in T_RX \), then \( \omega(X, Y) = \langle X, J^TXY \rangle_{h^{TX}} \).

Now we recall a simple result of \([\text{BGS2}, \text{Theorems 1.5 and 1.7}]\).

**Theorem 2.2.** Let \( \omega \) be a real smooth 2-form on \( M \) of complex type (1,1), which has the following two properties :

a) \( \omega \) is closed.

b) The bilinear map \( X, Y \in T_RX \to \omega(J^TX, Y) \) defines a Hermitian product \( h^{TX} \) on \( TX \).
For $x \in M$, set

\begin{equation}
T^H_x(M) = \{ Y \in T_xM; \text{ for any } X \in T_xX, \omega(X, \overline{Y}) = 0 \}.
\end{equation}

Then $T^H M$ is a subbundle of $TM$ such that $TM = T^H M \oplus TX$. Also $(\pi, h^TX, T^H M)$ is a Kähler fibration, and is an associated $(1,1)$-form.

A smooth real $(1,1)$-form $\omega'$ on $M$ is associated to the Kähler fibration $(\pi, h^TX, T^H M)$ if and only if there is a real smooth closed $(1,1)$-form $\eta$ on $B$ such that

\begin{equation}
\omega' - \omega = \pi^* \eta.
\end{equation}

b) The Bismut superconnection of a Kähler fibration.

Let $\omega^M$ be a real $(1,1)$ form on $M$ taken as in Theorem 2.2.

Let $\xi$ be a complex bundle on $M$. Let $h^\xi$ be a Hermitian metric on $\xi$. Let $\nabla^TX, \nabla^\xi$ be the holomorphic Hermitian connections on $(TX, h^TX), (\xi, h^\xi)$. Let $R^TX, L^\xi$ be the curvatures of $\nabla^TX, \nabla^\xi$. Let $\nabla^{\Lambda(T^{*0,1}X)}$ be the connection induced by $\nabla^TX$ on $\Lambda(T^{*0,1}X)$. Let $\nabla^{\Lambda(T^{*0,1}X)} \otimes \xi$ be the connection on $\Lambda(T^{*0,1}X) \otimes \xi$,

\[ \nabla^{\Lambda(T^{*0,1}X)} \otimes \xi = \nabla^{\Lambda(T^{*0,1}X)} \otimes 1 + 1 \otimes \nabla^\xi. \]

**Definition 2.3.** For $0 \leq p \leq \dim X$, $b \in B$, let $E^p_b$ be the vector space of $C^\infty$ sections of $(\Lambda^p(T^{*0,1}X) \otimes \xi)_{|X_b}$ over $X_b$. Set

\begin{equation}
E_b = \bigoplus_{p=0}^{\dim X} E^p_b, \quad E_b^+ = \bigoplus_{\text{even}} E^p_b, \quad E_b^- = \bigoplus_{\text{odd}} E^p_b.
\end{equation}

As in [B1, §1f], [BGS2, §1d]), we can regard the $E_b$'s as the fibres of a smooth $\mathbb{Z}$-graded infinite dimensional vector bundle over the base $B$. Smooth sections of $E$ over $B$ will be identified with smooth sections of $\Lambda(T^{*0,1}X) \otimes \xi$ over $M$.

Let $\langle \rangle$ be the Hermitian product on $E$ associated to $h^TX, h^\xi$ defined in (1.2).

If $U \in T^H_R B$, let $U^H$ be the lift of $U$ in $T^H_R M$, so that $\pi, U^H = U$.

**Definition 2.4.** If $U \in T^H_R B$, if $s$ is a smooth section of $E$ over $B$, set

\begin{equation}
\nabla^E_{U^H s} = \nabla^{\Lambda(T^{*0,1}X)} \otimes \xi s,
\end{equation}

By [B1, §1f], $\nabla^E$ is a connection on the infinite dimension vector bundle $E$. Let $\nabla^{E'}$ and $\nabla^{E''}$ be the holomorphic and antiholomorphic parts of $\nabla^E$.

For $b \in B$, let $\overline{\partial}^{X_b}$ be the Dolbeault operator acting on $E_b$, and let $\overline{\partial}^{X_b'}$ be its formal adjoint with respect to the Hermitian product (1.2).

The bundle $\Lambda(T^{*0,1}X) \otimes \xi$ is a $c(T_R Z)$-Clifford module. In fact, if $U \in TX$, let $U' \in T^{*0,1}X$ correspond to $U$ by the metric $h^TX$. If $U, V \in TX$, set

\begin{equation}
c(U) = \sqrt{2} U' \wedge, \quad c(V) = -\sqrt{2} i_V.
\end{equation}

Let $P^{TX}$ be the projection $TM \simeq T^H M \oplus TX \rightarrow TX$.

If $U, V$ are smooth vector fields on $B$, set

\begin{equation}
T(U^H, V^H) = -P^{TX}[U^H, V^H].
\end{equation}

Then $T$ is a tensor. By [BGS2], we know that as a 2-form, $T$ is of complex type $(1,1)$.
Let $f_1, \cdots, f_m$ be a base of $T_R B$, and let $f^1, \cdots, f^m$ be the dual base of $T^*_R B$.

**Definition 2.5.** Set

$$c(T) = \frac{1}{2} \sum f^\alpha f^\beta c(T(f^H, f^H_\beta)).$$

Then $c(T)$ is a section of $(\Lambda(T^*_R B) \otimes \text{End}(\Lambda(T^{(0,1)}_R X) \otimes \xi))^{\text{odd}}$. We also define $c(T^{(1,0)}), c(T^{(0,1)})$ by formulas similar to (2.8), so that

$$c(T) = c(T^{(1,0)}) + c(T^{(0,1)}).$$

**Definition 2.6.** For $u > 0$, let $B_u$ be the Bismut superconnection constructed in [B1, §3], [BGS2, §2a],

$$B_u^H = \nabla^{E^u} + \sqrt{u} \overline{\partial^X} - \frac{c(T^{(1,0)})}{2\sqrt{2u}},$$

$$B_u^{'1} = \nabla^{E^u} + \sqrt{u} \partial^X s - \frac{c(T^{(0,1)})}{2\sqrt{2u}},$$

$$B_u = B'_u + B''_u.$$

Let $N_V$ be the number operator defining the $\mathbb{Z}$-grading on $\Lambda(T^{(0,1)}_R X) \otimes \xi$ and on $E$. $N_V$ acts by multiplication by $p$ on $\Lambda^p(T^{(0,1)}_R X) \otimes \xi$. If $U, V \in T_R B$, set

$$\omega^{H\Pi}(U, V) = \omega^M(U^H, V^H).$$

**Definition 2.7.** For $u > 0$, set

$$N_u = N_V + \frac{i \omega^{H\Pi}}{u}.$$

c) Equivariant superconnection forms and double transgression formulas.

At first, we assume that the direct image $R^*\pi_4 \xi$ of $\xi$ by $\pi$ is locally free. For $b \in B$, let $H(X_b, \xi|_{X_b})$ be the cohomology of the sheaf of holomorphic sections of $\xi|_{X_b}$. Then the $H(X_b, \xi|_{X_b})$'s are the fibres of a $\mathbb{Z}$-graded holomorphic vector bundle $H(X, \xi|_X)$ on $B$, and $R^*\pi_4 \xi = H(X, \xi|_X)$. So we will write indifferently $R^*\pi_4 \xi$ or $H(X, \xi|_X)$.

By (1.4), the $K(X_b, \xi|_{X_b})$ are the fibres of a smooth bundle $K(X, \xi|_X)$ over $B$. By [BGS3, Theorem 3.5], the isomorphism of the fibre (1.4) induces a smooth isomorphism of $\mathbb{Z}$-graded vector bundles on $B$

$$H(X, \xi|_X) \simeq K(X, \xi|_X).$$

Then $K(X, \xi|_X)$ inherits a Hermitian product from $(E, \langle \quad \rangle)$. Let $h^{H(X, \xi|_X)}$ be the corresponding smooth metric on $H(X, \xi|_X)$. Let $P^{H(X, \xi|_X)}$ be the orthogonal projection operator from $E$ on $H(X, \xi|_X) \simeq K(X, \xi|_X)$. Let $\nabla^{H(X, \xi|_X)}$ be the holomorphic Hermitian connection on $(H(X, \xi|_X), h^{H(X, \xi|_X)}$).

Let $G$ be a compact Lie group. We assume that $G$ acts holomorphically on $M, B, \xi$, and that $\xi, M$ are $G$-equivariant (vector) bundles over $M, B$. Let $\omega^M, h^\xi$ be $G$-invariant.

Then $R^*\pi_4 \xi$ is also a $G$-equivariant vector bundle over $B$, and $h^{H(X, \xi|_X)}$ is also $G$-invariant.

**Definition 2.8.** Let $P^B$ be the vector space of real smooth forms on $B$, which are sums of forms of type $(p, p)$. Let $P^{B, 0}$ be the vector space of the forms $\alpha \in P^B$ such that there exist smooth forms $\beta, \gamma$ on $B$ for which $\alpha = \partial \beta + \overline{\partial} \gamma$. 

8
Let $\Phi$ be the homomorphism of $\Lambda^\text{even}(T^*_R B)$ into itself: $\alpha \to (2i\pi)^{-\deg_0/2}\alpha$.

**Theorem 2.9.** For $u > 0$, the forms $\Phi \text{Tr}_u[g \exp(-B_u^2)]$ and $\Phi \text{Tr}_u[gN_u \exp(-B_u^2)]$ lie in $P^{B_u}$. The forms $\Phi \text{Tr}_u[g \exp(-B_u^2)]$ are closed and that their cohomology class is constant. Moreover

$$\frac{\partial}{\partial u} \Phi \text{Tr}_u[g \exp(-B_u^2)] = -\frac{1}{u} \frac{\partial}{\partial u} \Phi \text{Tr}_u[gN_u \exp(-B_u^2)].$$

**Proof:** Since $g$ commutes with $N_u, B_u$, etc, by proceeding as in [BGS2, Theorem 2.9], we have Theorem 2.9.

For $g \in G$, we have also a holomorphic submersion $\pi : M^g \to B^g$ with compact fibre $X^g$. Put

$$C_{-1,g} = \int_{X^g} \frac{m}{2\pi i} \text{Td}_g(TX, h^{TX}) \cdot \text{ch}_g(\xi, h^\xi),$$

$$C_{0,g} = \int_{X^g} (-\text{Td}_g(TX, h^{TX}) + \dim X \text{Td}_g(TX, h^{TX})) \cdot \text{ch}_g(\xi, h^\xi).$$

Set

$$\text{ch}_g(H(X, \xi|X), h^{H(X, \xi|X)}) = \sum_{k=0}^{\deg_0} (-1)^k \text{ch}_g(H^k(X, \xi|X), h^{H^k(X, \xi|X)}),$$

$$\text{ch}_g(H(X, \xi|X), h^{H(X, \xi|X)}) = \sum_{k=0}^{\deg_0} (-1)^k \text{ch}_g(H^k(X, \xi|X), h^{H(X, \xi|X)}).$$

**Theorem 2.10.** As $u \to 0$

$$\Phi \text{Tr}_u[g \exp(-B_u^2)] = \int_{X^g} \text{Td}_g(TX, h^{TX}) \cdot \text{ch}_g(\xi, h^\xi) + O(u).$$

There are forms $C_{j, g} \in P^{B^g}(j \geq -1)$ such that for $k \in \mathbb{N}$, as $u \to 0$

$$\Phi \text{Tr}_u[gN_u \exp(-B_u^2)] = \sum_{k=1}^j C_{j,g} u^j + O(u^{k+1}).$$

Also

$$C_{-1,g} = C_{-1,g} - 1,$$

$$C_{0,g} = C_{0,g} \text{ in } P^{B_g}/P^{B^g,0}.$$

As $u \to +\infty$

$$\Phi \text{Tr}_u[g \exp(-B_u^2)] = \text{ch}_g(H(X, \xi|X), h^{H(X, \xi|X)}) + O(\frac{1}{\sqrt{u}}),$$

$$\Phi \text{Tr}_u[gN_u \exp(-B_u^2)] = \text{ch}_g(H(X, \xi|X), h^{H(X, \xi|X)}) + O(\frac{1}{\sqrt{u}}).$$

**Proof:** By combining the technique of [BGS2, Theorem 2.2, 2.16] and [B7, Theorem 4.9-4.11], we have the equations (2.17)-(2.18),(2.19).

Equation (2.20) was stated in [BKô, Theorem 3.4] if $g = 1$. By proceeding as in [BeGeV, Theorem 9.23], we also have (2.20).

**d) Higher analytic torsion forms.**

For $s \in \mathbb{C}, \text{Res} > 1$, set

$$\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \left( \Phi \text{Tr}_u[gN_u \exp(-B_u^2)] - \text{ch}_g(H(X, \xi|X), h^{H(X, \xi|X)}) \right) du.$$
For $s \in \mathbb{C}, \text{Res} < \frac{1}{2}$, set

$$
\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^{+\infty} u^{s-1} \left( \Phi\text{Tr}_s[gN_u \exp(-B^2_u)] - \text{ch}'_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}) \right) du.
$$

Then $\zeta_2(s)$ is a holomorphic function of $s$.

**Definition 2.11.** Set

$$(2.21) \quad T_g(\omega^M, \xi^\ell) = \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0).$$

Then $T_g(\omega^M, \xi^\ell)$ is a smooth form on $B^g$. Using (2.18)-(2.20), we get

$$(2.22) \quad T_g(\omega^M, \xi^\ell) = -\int_0^1 \left( \Phi\text{Tr}_s[gN_u \exp(-B^2_u)] - \frac{C_{1,g}}{u} - C_{0,g} \right) du + C_{-1,g} + \Gamma'(1) \left( C_{0,g} - \text{ch}'_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}) \right).$$

**Theorem 2.12.** The form $T_g(\omega^M, \xi^\ell)$ lies in $P^{B^g}$. Moreover

$$(2.23) \quad \frac{\overline{\partial}}{2\pi i} T_g(\omega^M, \xi^\ell) = \text{ch}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}) - \int_{X^g} \text{Td}_g(TX, g^{TX}) \text{ch}_g(\xi, \xi^\ell).$$

**Proof:** As we saw before, the forms $\Phi\text{Tr}_s[gN_u \exp(-B^2_u)]$ lie in $P^{B^g}$. So the form $T_g(\omega^M, \xi^\ell) \in P^{B^g}$. Using Theorem 2.10 and equation (2.14), the proof of our Theorem 2.12 proceeds as the proof of [ BGS2, Theorem 2.20].

**e) Anomaly formulas for the analytic torsion forms.**

Now let $(\omega'^M, \xi'^\ell)$ be another couple of objects similar to $(\omega^M, \xi^\ell)$. We denote with a ' the objects associated to $(\omega'^M, \xi'^\ell)$.

By [BGS1, § 1(f)], there are uniquely defined Bott-Chern classes $\text{Td}_g(TX, g^{TX}, g^{TX'})$, $\text{ch}_g(\xi, \xi'^\ell), h^{H(X,\xi'^{\ell})}) \in P^{M^g}/P^{M^g,0}, \text{ch}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}, h^{H(X,\xi_{|X})}) \in P^{B^g}/P^{B^g,0}$ such that

$$
\frac{\overline{\partial}}{2\pi i} \text{Td}_g(TX, g^{TX}, g^{TX'}) = \text{Td}_g(TX, g^{TX}) - \text{Td}_g(TX, g^{TX}),
$$

$$
\frac{\overline{\partial}}{2\pi i} \text{ch}_g(\xi, \xi'^\ell) = \text{ch}_g(\xi, \xi'^\ell) - \text{ch}_g(\xi, \xi'^\ell),
$$

$$
\frac{\overline{\partial}}{2\pi i} \text{ch}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}) = \text{ch}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}) - \text{ch}_g(H(X,\xi_{|X}), h^{H(X,\xi_{|X})}).
$$

Let $C$ be a smooth section of $T^*_R X \odot \text{End}(\Lambda(T^{*_{(0,1)}X}) \odot \xi)$. Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $T^*_R X$. We use the notation

$$
(\nabla_{e_i}^{\Lambda(T^{*_{(0,1)}X}) \odot \xi} + C(e_i))^2 = \sum_{i=1}^{2n} (\nabla_{e_i}^{\Lambda(T^{*_{(0,1)}X}) \odot \xi} + C(e_i))^2 - \nabla_{e_i}^{\Lambda(T^{*_{(0,1)}X}) \odot \xi} - C(\sum_{i=1}^{2n} \nabla_{e_i}^{\Lambda(T^{*_{(0,1)}X}) \odot \xi} e_i).
$$
Theorem 2.13. The following identity holds
\[
T_{\theta}(\omega^{M}, h^{\xi}) - T_{\theta}(\omega^{M}, h^{\xi}) = \widehat{\text{ch}}_{g}(H(X, \xi), h^{H(X, \xi)}, h^{H(X, \xi)}) \\
\left( - \int_{X^{n}} [\widetilde{T}_{d}g(TX, h^{TX}, h^{TX}) \text{ch}_{g}(\xi, h^{\xi}) + \widetilde{T}_{d}g(TX, h^{TX}) \partial \text{ch}_{g}(\xi, h^{\xi}, h^{\xi})] \right) \text{ in } P^{B_{\theta}} / P^{B_{\theta}, 0}.
\]

In particular, the class of \( T_{\theta}(\omega, h^{\xi}) \) in \( P^{B_{\theta}} / P^{B_{\theta}, 0} \) only depends on \( (h^{TX}, h^{\xi}) \).

Proof: Assume first that \( h^{\xi} = h^{\eta} \). Let \( c \in [0, 1] \to \omega^{c}_{\xi} \) be a smooth family of \( G \)-invariant \((1,1)\)-forms on \( M \) verifying the assumptions of Theorem 2.2 such that \( \omega^{M}_{0} = \omega^{M}, \omega^{1}_{M} = \omega^{J}_{M} \).

Then all the objects considered in Section 2 a)-d) now depend on the parameter \( c \). Most of the time, we will omit the subscript \( c \). The upperdot \( \cdot \) is often used instead of \( \frac{\partial}{\partial c} \).

Set
\[
Q = -u^{-1} \ast, \\
Q^{H(X, \xi)} = P^{H(X, \xi)} Q \, P^{H(X, \xi)}.
\]

Let \( e_{1}, \cdots, e_{n} \) be an orthonormal base of \( T_{R}Z \) with respect to \( h^{TX} \). Let \( f_{1}, \cdots, f_{2m} \) be a base of \( T_{R}B \), and that \( f^{1}, \cdots, f^{2m} \) is the corresponding dual base of \( T_{R}B \). Set
\[
M_{u} = \frac{i}{4} \omega(e_{i}, e_{j})c(e_{i})c(e_{j}) - \frac{i}{\sqrt{2}u} \omega(f^{H}, e_{i})f^{\alpha}c(e_{i}) \\
- \frac{\omega u \pi}{2u}(f_{\alpha}, f_{\beta})f^{\alpha}f^{\beta} - \frac{1}{4} \omega(e_{i}, f^{TX} e_{i}).
\]

By the arguments of [BGS2, Theorem 2.11], we know there is \( p \in \mathbb{N}, \mu_{j} \in P^{B_{\theta}}, (j \geq -p) \) such that as \( u \to 0 \), we have the asymptotic expansion
\[
\Phi Tr_{\ast}[g^{2m}] \exp(-B_{u}^{2}) = \Sigma_{j=-p}^{\infty} \mu_{j} u^{j} + O(u^{k+1}).
\]

By proceeding as in [BKö, § 2,3], we easily find an analogue of [BKö, Theorem 3.16],
\[
\frac{d}{d \theta} \omega^{\xi} = \mu_{0} + \Phi Tr_{\ast}[g^{H(X, \xi)} \exp(-(\nabla H(X, \xi))^{2})] \\
+ \frac{\partial}{\sqrt{2}u} \theta^{\xi}(0) + \frac{\partial}{\sqrt{2}u} \theta^{\xi}(0) + \frac{\partial}{\sqrt{2}u} \theta^{\xi}(0).
\]

In (2.28), the \( \theta^{\xi}(0) \) are universal formulas of \( g, \omega^{M}, h^{\xi} \) as in [BKö].

Let \( da, d\bar{a} \) be two odd Grassmann variables which anticommute with the other odd elements in \( L(T_{R}B) \) or \( c(T_{R}X) \). Set
\[
L_{u} = -B_{u}^{2} - du \frac{\partial B_{u}}{\partial u} - d\bar{a}[B_{u}, -M_{u}] + d\bar{a}[(-u M_{u})].
\]

If \( \alpha \in C(da, d\bar{a}) \), let \( [\alpha]^{da} d\bar{a} \in C \) be the coefficient of \( da d\bar{a} \) in the expansion of \( \alpha \). By an analogue formula of [BKö, Theorem 3.17], we know that the class of \( -\mu_{0} \) in \( P^{B_{\theta}} / P^{B_{\theta}, 0} \) coincides with the class of the constant term in the asymptotic expansion of \( \Phi Tr_{\ast}[g \exp(L_{u})]^{da} d\bar{a} \).

Let \( \nabla_{u} \) be the connection on \( \Lambda(da \otimes d\bar{a}) \otimes \Lambda(T_{R}B) \otimes \Lambda(T^{*}(0,1)X) \otimes \xi \) on fibre \( X \).

\[
\nabla_{u} = \nabla \Lambda(T^{*}(0,1)X) \otimes \xi + \frac{1}{u} \langle S(.) e_{j}, f^{H} \rangle \sqrt{\frac{u}{2}} c(e_{j}) f^{\alpha} + \frac{1}{2 u} \langle S(.) f^{H}, f^{\beta} \rangle f^{\alpha} f^{\beta} \\
- \frac{da}{2 u} \sqrt{\frac{u}{2}} c(.) - \frac{i}{u} \omega(e_{k}, .) d\bar{a} \sqrt{\frac{u}{2}} c(e_{k}) - i \frac{\omega}{u} (f^{H}, .) d\bar{a} f^{\alpha}.
\]

(2.30)
Let $K^X$ be the scalar curvature of $(X, h^{TX})$. Set

$$L^\xi = L^\xi + \frac{1}{2} \text{Tr}[R^{TX}].$$

By [BK9, Theorem 3.18], we get

$$L_u = \frac{u}{2} (\nabla_{u,e}^t)^2 - \nabla_{e_1} (\omega (e_j, J^T X e_j)) \frac{d\tau}{4 \sqrt{2}} - \nabla_{f^g} (\omega (e_j, J^T X e_j)) \frac{d\tau}{4}$$

$$+ \frac{u K^X}{8} - \frac{u}{4} c(e_j) L^\xi (e_i, e_j)$$

$$- \sqrt{\frac{u}{2}} c(e_i) f^\alpha L^\xi (f^H, f^H) - \frac{f^\alpha f^\beta}{2} L^\xi (f^H, f^H).$$

Let $P_b(x, x', y)(b \in B, x, x' \in X_b)$ be the smooth kernel associated to $\exp(L_u)$ with respect to $d\nu_{X(x'), b}$. Then

$$\Phi \text{Tr}_{s} [g \exp(L_u)] = \int_X \Phi \text{Tr}_{s} [g P_u (g^{-1}x, x, b)] \frac{d\nu_{X(x)}}{(2 \pi)^{dim X}}.$$

By standard estimates on heat kernels, for $b \in B$, the problem of calculating the limit of (2.33) when $u \to 0$ can be localized to an open neighborhood $\mathcal{U}$ of $X^g$ on $X_b$. Using normal geodesic coordinates to $X^g$ in $X_b$, we will identify $\mathcal{U}$ to an $\varepsilon$-neighbourhood of $X^g$ in $N_{X^g} / X$. Let $k(x, z)(x \in X^g, z \in N_{X^g} / X, |z| < \varepsilon)$ be defined by

$$d\nu_X = k(x, z) d\nu_{X^g}(x) d\nu_{N_{X^g}}(z).$$

Then

$$k(x, 0) = 1.$$

Clearly

$$\int_{\mathcal{U}} \Phi \text{Tr}_{s} [g P_u (g^{-1}x, x)] \frac{d\nu_{X(x)}}{(2 \pi)^{dim X}}$$

$$= \int_{x \in X^g} \int_{|z| < \varepsilon} u^{2 dim N_{X^g}} \Phi \text{Tr}_{s} [g P_u (g^{-1}(x, \sqrt{u}z), (x, \sqrt{u}z))]$$

$$k(x, \sqrt{u}z) \frac{d\nu_{X^g}(x) d\nu_{N_{X^g}}(z)}{(2 \pi)^{dim X}}.$$

Of course, since we have used normal geodesic coordinates to $X^g$ in $X$, if $(x, z) \in N_{X^g} / X$,

$$g^{-1}(x, z) = (x, g^{-1}z).$$

Take $x_0 \in X^g$, by using the finite propagation speed as in [B6, § 11b] , we may instead assume that $X_b$ by $(TX)_{x_0}$ with $0 \in (TX)_{x_0}$ representing $x_0$ and that the extended fibration over $C^m$ coincides with the given fibration over $B(0, \varepsilon)$.

Take $y \in C^m$, set $Y = y + \gamma$. We trivialize $\Lambda(\text{d}a \oplus \text{d}\tau) \otimes \Lambda(T \tilde{R}^X B) \otimes \Lambda(T^{(0, 1)} X) \otimes \xi$ by parallel transport along the curve $t \to t Y$ with respect to $\nabla^u$.

Let $\rho(Y)$ be a $C^\infty$ function over $C^m$ which is equal to 1 if $|Y| \leq \frac{\varepsilon}{2}$, and equal to 0 if $|Y| \geq \frac{\varepsilon}{2}$.

Let $H_{\rho^1}$ be the vector space of smooth sections of $(\Lambda(\text{d}a \oplus \text{d}\tau) \otimes \Lambda(T \tilde{R}^X B) \otimes \Lambda(T^{(0, 1)} X) \otimes \xi)_{x_0}$ over $(T \tilde{R}^X)_{x_0}$. For $u > 0$, let $L^1_u$ be the operator

$$L^1_u = (1 - \rho^2(Y)) \left( \frac{u \Delta^{TX}}{2} - \rho^2(Y) L_u \right).$$
For $u > 0$, $s \in H_{x_0}$, set

\begin{align}
F_u(s) &= s(\frac{X}{\sqrt{u}}), \\
L_u^2 &= F_u^{-1}L_u^2 F_u.
\end{align}

Let $e_1, \cdots, e_{2^r}$ be an orthonormal base of $(T_{R^*X})_{x_0}$, and let $e_{2^r + 1}, \cdots, e_{2n}$ be an orthonormal base of $N_{X^g/X', R, x_0}$.

Let $L_u^3$ be the operator obtained from $L_u^2$ by replacing the Clifford variables $c(e_j)(1 \leq j \leq 2^r)$ by the operators $\frac{\partial c(e_j)}{c(e_j)}$.

Let $P_u^j(z, z') (z, z' \in (T_{R^*X})_{x_0})$ be the smooth kernel associated to $\exp(-L_u^j)$ with respect to $\frac{dv_{X^g}}{(2\pi)^{\dim X^g}}$. If $\alpha \in C((e_j, i_e)(1 \leq j \leq 2^r))$, let $[\alpha]_{\max} \in C$ be the coefficient of $e_1 \wedge \cdots \wedge e_{2^r}$ in the expansion of $\alpha$. Then as in [B5, Proposition 11.12], if $z \in N_{X^g/X, R}$

\begin{equation}
\Tr_e[gP_u^j(g^{-1}z, z)] = (-i)^{\dim X^g} \frac{-\dim N_{X^g/X}}{\dim X^g} \left[ \Tr_e[gP_u^j(g^{-1}z, z)] \right]^{\dim X^g} \frac{dv_{X^g}}{(2\pi)^{\dim X^g}}.
\end{equation}

Let $R_{X^g/X, L_{x_0}^j, \cdots, X^g}$ be the restrictions of $R_{X^g/X, L_{x_0}^j, \cdots, X^g}$ over $X^g$. Let $\nabla e_i$ be the ordinary differentiation operator on $(T_{R^*X})_{x_0}$ in the direction $e_i$. By [ABoP, Proposition 3.7], and (2.32), as $u \to 0$,

\begin{align}
L_u^3 - L_0^3 &= -\frac{1}{2} \left( \nabla e_j + \frac{1}{2} \left\langle R_{X^g/X}^j e_j, e_j \right\rangle - d\alpha_1 + d\alpha_2 \left( \frac{j}{2} \omega(Y, e_j) \right) \right)^2 \\
d\alpha_2 - d\alpha_1 &= \frac{d\alpha_1}{4} \left( \omega(e_j, J^TX e_j) + L_{X^g}^j \right)
\end{align}

and $\alpha_1, \alpha_2$ are 1-forms of $\Lambda(T_{R^*X} \oplus T_{R^*}B)$.

Let

\begin{align}
L_0^3 &= -\frac{1}{2} \left( \nabla e_j + \frac{1}{2} \left\langle R_{X^g/X}^j e_j, e_j \right\rangle + d\alpha_2 \left( \frac{j}{2} \omega(Y, e_j) \right) \right)^2 \\
d\alpha_2 - d\alpha_1 &= \frac{d\alpha_1}{4} \left( \omega(e_j, J^TX e_j) + L_{X^g}^j \right)
\end{align}

Let $P_0^j(z, z')$ be the heat kernel of $\exp(-L_0^j)$ over $(T_{R^*X})_{x_0}$ with respect to $\frac{dv_{X^g}}{(2\pi)^{\dim X^g}}$. By proceeding as in [B5, §11g]- [§11i]), we have:

There exist $\gamma > 0, c > 0, C > 0, r \in \mathbb{N}$ such that for $u \in [0, 1]$, $z, z' \in (T_{R^*X})_{x_0}$, we have

\begin{align}
P_0^j(z, z') &\leq ce(1 + |z| + |z'|)^r \exp(-C|z - z'|^2), \\
(P_0^j - P_0^3)(z, z') &\leq cu(1 + |z| + |z'|)^r \exp(-C|z - z'|^2).
\end{align}

From (2.34), (2.39)-(2.42), we get

\begin{align}
\lim_{u \to 0} \int_{z \in N_{X^g/X, R}} \Phi \Tr_e[gP_u^3(g^{-1}z, z)] k(x, z) \frac{dv_{X^g/X}}{(2\pi)^{\dim X^g/X}} \\
= \int_{N_{X^g/X, R}} \left( -i \right)^{\dim X^g} \left\{ \Phi \Tr_e[gP_0^3(g^{-1}z, z)] \right\}^{\max} \frac{dv_{X^g/X}}{(2\pi)^{\dim X^g/X}} \\
= \int_{N_{X^g/X, R}} \left( -i \right)^{\dim X^g} \left\{ \Phi \Tr_e[gP_0^3(g^{-1}z, z)] \right\}^{\max} \frac{dv_{X^g/X}}{(2\pi)^{\dim X^g/X}}
\end{align}

Chirally for $U, V \in TX$,

\begin{equation}
\omega(U, V) = \left\langle U, J^TX (h^TX)^{-1} \frac{\partial h^TX}{\partial c} V \right\rangle.
\end{equation}
So

\begin{equation}
L^8_0 = -\frac{1}{2} \left( \nabla e_1 + \frac{1}{2} \left( \left( R_{TX} - i d \text{ad}_{TX} (h_{TX})^{-1} \frac{\partial h_{TX}}{\partial c} \right) Y, e_1 \right) \right)^2 + L^8_{iX^0} - \frac{1}{2} \left( \text{Tr} R_{TX}^{iX^0} + \text{ad}_{TX} \left[ (h_{TX})^{-1} \frac{\partial h_{TX}}{\partial c} \right] \right)
\end{equation}

By proceeding as in [B4, (3.16)-(3.21)],

\begin{equation}
(-1)^{\dim X^0} \int_{N^{iX^0}/X^0 \times \mathbb{R}} \left\{ \phi \text{Tr}_s [g P_{0, s}^s (g^{-1} z, z)]^{\max} \right\} \frac{d\text{vol}_{N^{iX^0}/X^0} (z)}{(2\pi)^{\dim X^0}} d\text{vol}_{X^0}
\end{equation}

\begin{equation}
= \left\{ \frac{\partial}{\partial b} \left[ \text{Td} \left( \frac{-R_{TX}^{iX^0}}{2\pi} - b (h_{TX})^{-1} \frac{\partial h_{TX}}{\partial c} \right) \right]^{\max} \prod_{j=1}^q \left[ \frac{\partial}{\partial c} \left( \frac{-R_{TX}^{iX^0}}{2\pi} - b (h_{TX})^{-1} \frac{\partial h_{TX}}{\partial c} + i \theta_j \right) \right]_{b=0} \cdot \text{ch}_g (\xi, h^k) \right\}^{\max}.
\end{equation}

By using [BGS1, Remark 1.28 and Corollary 1.30] and proceeding as in [BK6, §3h], we finish the proof of Theorem 2.13 in the case where \( h^k = h^\xi \).

To prove (2.24) in the full generality, one only needs to consider the case where \( \omega^M = \omega^\xi \). Then by using Theorem 2.12 and by proceeding as in [BGS1, §1f], i.e. by replacing \( B \) by \( B \times \mathbb{P}^1 \), one easily obtains (2.24) in this special case.

3 The equivariant Quillen norm of the canonical section \( \sigma \)

This Section is organized as follows. In a), we describe the canonical section \( \sigma \). In b), we announce a formula for the equivariant Quillen norm of \( \sigma \).

In this Section, we make the same assumptions as in Section 2c), and we use the same notation as in Sections 1,2.

a) The canonical section \( \sigma \).

Let \( M, B \) be compact complex manifolds of complex dimension \( n \) and \( m \). Let \( \pi : M \to B \) be a holomorphic submersion with fibre \( X \). Let \( \xi \) be a holomorphic vector bundle on \( M \). Let \( G \) be a compact Lie group. We assume that \( \xi, M \) are \( G \)-equivariant holomorphic bundles over \( M, B \).

We assume that the sheaves \( R^k \pi^* \xi (0 \leq k \leq \dim X) \) are locally free.

If given \( W \in \tilde{G}, \lambda_W, \mu_W \) are complex lines, if \( \lambda = \oplus_{W \in \tilde{G}} \lambda_W, \mu = \oplus_{W \in \tilde{G}} \mu_W \), set

\begin{equation}
\lambda^{-1} = \oplus_{W \in \tilde{G}} \lambda_W^{-1}, \quad \lambda \otimes \mu = \oplus_{W \in \tilde{G}} \lambda_W \otimes \mu_W.
\end{equation}

Now we use the notation of Section 1. Set

\begin{equation}
\lambda_G (\xi) = \det (H(M, \xi), G),
\lambda_G (R^k \pi^* \xi) = \det (H(B, R^k \pi^* \xi), G),
\lambda_G (R^* \pi^* \xi) = \oplus_{k=0}^{\dim X} (\lambda_G (R^k \pi^* \xi))^{(-1)^k}.\n\end{equation}

By proceeding as in [BerB, §1b] and [B5, §3b], for \( W \in \tilde{G}, \) the line \( \lambda_W (\xi) \otimes \lambda_W^{-1} (R^* \pi^* \xi) \) has canonical nonzero section \( \sigma_W \). Set

\begin{equation}
\sigma = \oplus_{W \in \tilde{G}} \sigma_W \in \lambda_G (\xi) \otimes \lambda_G^{-1} (R^* \pi^* \xi).
\end{equation}

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b) A formula for the Quillen norm of canonical section $\sigma$.

Let $h^T_M, h^T_B$ be $G$-invariant Kähler metrics on $TM$ and $TB$. Let $h^T_X$ be the metric induced by $h^T_M$ on $TX$. Let $h^G$ be a $G$-invariant Hermitian metric on $\xi$.

On $M^0$, we have the exact sequence of holomorphic Hermitian vector bundles

$$0 \rightarrow TX \rightarrow TM \rightarrow \pi^*TB \rightarrow 0.$$  \hfill (3.4)

By a construction of [BGS1, §1f], there is a uniquely defined class of forms $\overline{Td}_g(TM, TB, h^T_M, h^T_B) \in P^{M^0}/\Gamma(P^{M^0}, 0)$, such that

$$\frac{R_g}{2\pi} \overline{Td}_g(TM, TB, h^T_M, h^T_B) = Td_g(TM, h^T_M) - \pi^*(Td_g(TB, h^T_B))Td_g(TX, h^T_X).$$  \hfill (3.5)

Let $\omega_M$ be the Kähler form of $h^T_M$. Let $\| \|$ be the $G$-equivariant Quillen metric on the line $\lambda_G(\xi) \otimes \lambda^{-1}_G(R^*\pi_\xi)$ attached to the metrics $h^T_M, h^G, h^T_B, h^{H(X, \xi_X)}$ on $TM, \xi, TB, R^*\pi_\xi$.  

Now we state the main result of this paper, which extends [BerB, Theorem 3.1].

**Theorem 3.1.** For $g \in G$, the following identity holds,

$$\log \left( \|[\sigma]\|^2_{\lambda_G(\xi)\otimes\lambda^{-1}_G(R^*\pi_\xi)}(g) \right) = -\int_{B^0} Td_g(TB, h^T_B) T_g(\omega_M, h^G)$$

$$+ \int_{M^0} \overline{Td}_g(TM, TB, h^T_M, h^T_B) \text{ch}_g(\xi, h^G).$$  \hfill (3.6)

**Proof:** The proof of Theorem 3.1 will be given in Sections 4-9. \hfill $\blacksquare$

**Remark 3.2.** By Theorem 2.13, to prove Theorem 3.1 for any Kähler metric $h^T_M, h^T_B$, we only need to establish (3.6) for one given metrics $h^T_M, h^T_B$. So by replacing $h^T_M$ by $h^T_M + \pi^*h^T_B$, we may and we will assume that $\tilde{h}^T_M$ is a Kähler metric on $TM$ and

$$h^T_M = \tilde{h}^T_M + \pi^*h^T_B.$$  \hfill (3.7)

4 A proof of Theorem 3.1

This Section is organized as follows. In a), we introduce a 1-form on $\mathbb{R}_+ \times \mathbb{R}_+$ as in [BerB, § 3a)]. In b), we state eight intermediary results which we need for the proof of the Theorem 3.1 whose proofs are delayed to Sections 5-9. In c), we prove Theorem 3.1.

In this Section, we make the same assumption as in Section 3. Also, we assume that $h^T_M$ is given by formula (3.7). In the sequel, $g \in G$ is fixed once and for all.

a) A fundamental closed 1-form.

Recall that $N_V$ denotes the number operator of $\Lambda(T^{s(0,1)}X)$. Let $N_H$ be the number operator of $\Lambda(T^{s(0,1)}B)$. By (2.2), we have the identification of smooth vector bundles over $M$

$$TM \simeq TX \oplus T^HM,$$

$$T^HM \simeq \pi^*TB.$$  \hfill (4.1)

This identification determines an identification of $\mathbb{Z}$-graded bundles of algebra

$$\Lambda(T^{s(0,1)}M) = \Lambda(T^{s(0,1)}B) \otimes \Lambda(T^{s(0,1)}X).$$  \hfill (4.2)
So the operator \( N_V \) and \( N_H \) acts naturally on \( \Lambda(T^{s(0,1)}M) \). Of course, \( N = N_V + N_H \) defines the total grading of \( \Lambda(T^{s(0,1)}M) \otimes \xi \) and \( \Omega(M, \xi) \).

**Definition 4.1.** For \( T > 0 \), let \( h^T_M \) be the Kähler metric on \( TM \)

\[
h^T_M = \frac{1}{T^2} h^{TM} + \pi^* h^T_B.
\]

Let \( \langle \ , \ \rangle_T \) be the Hermitian product \((1.2)\) on \( \Omega(M, \xi) \) attached to the metrics \( h^T_M, h^k \). Let \( D^T_M \) be the corresponding operator constructed in \((1.3)\) acting on \( \Omega(M, \xi) \). Let \( *_T \) be the Hodge operator associated to the metric \( h^T_M \). Then \( *_T \) acts on \( \Lambda(T^R_M) \otimes \xi \).

**Theorem 4.2.** Let \( \alpha_{u,T} \) be the 1-form on \( \mathbb{R}_+^4 \times \mathbb{R}_+^4 \)

\[
\alpha_{u,T} = \frac{2du}{u} \text{Tr} \left[ gN \exp(-u^2 D^M_* \Pi^2) \right] + dT \text{Tr} \left[ g *_T \frac{\partial *_T}{\partial T} \exp(-u^2 D^M_* \Pi^2) \right].
\]

Then \( \alpha_{u,T} \) is closed.

**Proof:** Clearly \( g \) is an even operator which commutes with the operators \( \overline{\partial}_M, \overline{\partial}_M^*, *_T, N_V, N_H \). By using \([\text{BerB}, (4.27), (4.28), (4.30)]\), the proof of Theorem 4.2 is identical to the proof of \([\text{BerB}, \text{Theorem 4.3}]\). \( \blacksquare \)

Take \( \epsilon, A, T, 0 < \epsilon \leq 1 \leq A < +\infty, 1 \leq T_0 < +\infty \). Let \( \Gamma = \Gamma_{\epsilon, A, T_0} \) be the oriented contour in \( \mathbb{R}_+^4 \times \mathbb{R}_+^4 \)

The contour \( \Gamma \) is made of four oriented pieces \( \Gamma_1, \ldots, \Gamma_4 \) indicated above. For \( 1 \leq k \leq 4 \), set

\[
P_k = \int_{\Gamma_k} \alpha.
\]

**Theorem 4.3.** The following identity holds,

\[
\sum_{k=1}^4 P_k = 0.
\]

**Proof:** This follows from Theorem 4.2. \( \blacksquare \)

**b) Eight intermediate results.**

Let \( \overline{\partial}^B_* \) be the formal adjoint of the operator \( \overline{\partial}^B \) acting on \( \Omega(B, R^* \pi, \xi) \), with respect to the metrics \( h^B_T, h^B_\pi(\lambda \xi \phi) \). Set

\[
D^B = \overline{\partial}^B + \overline{\partial}^B_*,
\]

\[
F = \text{Ker}D^B.
\]
By Hodge theory,

\[ H^*(B, R^* \pi_\xi) \simeq F. \]  

Let \( Q \) be the orthogonal projection from \( \Omega(B, R^* \pi_\xi) \) on \( F \) with respect to the Hermitian product (1.2) attached to the metrics \( hT^B, hH^{(X, \Delta_X)} \). Set \( Q^\perp = 1 - Q \).

Let \( a \in [0, 1] \) be such that the operator \( D^{B,2} \) has no eigenvalues in \([0, 2a]\).

**Definition 4.4.** For \( T > 0 \), set

\[ E_T = \text{Ker} D^M_T. \]

Let \( P_T \) be the orthogonal projection operator from \( \Omega(M, \xi) \) on \( E_T \) with respect to \( \langle \ , \ \rangle_T \).

Let \( E^{[0,a]}_T \) (resp. \( E^{[0,a]}_T \)) be the direct sum of the eigenspaces of \( D^{M,2}_T \) associated to eigenvalues \( \lambda \in [0, a] \) (resp. \( \lambda \in [0, a] \)). Let \( D^{M,2,[0,a]}_T \) (resp. \( D^{M,2,[0,a]}_T \)) be the restriction of \( D^{M,2}_T \) to \( E^{[0,a]}_T \) (resp. \( E^{[0,a]}_T \)). Let \( P^{[0,a]}_T \) (resp. \( P^{[0,a]}_T \)) be the orthogonal projection operator from \( \Omega(M, \xi) \) on \( E^{[0,a]}_T \) (resp. \( E^{[0,a]}_T \)) with respect to \( \langle \ , \ \rangle_T \). Set \( P^\infty = 1 - P^{[0,a]}_T \).

For \( 0 \leq k \leq n, g \in G \), set

\[ \chi_g(\xi) = \text{Tr}_s H^{(M,\xi)}[g], \]
\[ \chi_g(R^k \pi_\xi) = \text{Tr}_s H^{(B, R^k \pi_\xi, \xi)}[g]. \]

Then by the Lefschetz fixed point formula of Atiyah-Bott [ABo],

\[ \chi_g(\xi) = \int_{M^g} Td_g(TM)ch_g(\xi), \]
\[ \chi_g(R^k \pi_\xi) = \int_{B^g} Td_g(TY)ch_g(R^k \pi_\xi). \]

We now state eight intermediary results contained in Theorems 4.5 - 4.12 which play an essential role in the proof of Theorem 3.1. The proof of Theorems 4.5 - 4.12 are deferred to Sections 5-9.

**Theorem 4.5.** For any \( u > 0 \),

\[ \lim_{T \to +\infty} \text{Tr}_s [gN \exp(-u^2 D^{M,2}_T)] = \text{Tr}_s [gN \exp(-u^2 D^{B,2}_T)]. \]

For any \( u > 0 \), there exists \( C > 0 \) such that for \( T \geq 1 \),

\[ |\text{Tr}_s [gN \exp(-u^2 D^{M,2}_T)] - \Sigma_{j=0} \dim X(-1)^j \chi_g(R^j \pi_\xi)| \leq \frac{C}{T}. \]

For any \( \varepsilon > 0 \), there exists \( C > 0 \) such that for \( u \geq \varepsilon, T \geq 1 \),

\[ |\text{Tr}[g \exp(-u^2 D^{M,2}_T)]| \leq C. \]

**Theorem 4.6.** For any \( u > 0 \),

\[ \lim_{T \to +\infty} \text{Tr}_s [gN \exp(-u^2 D^{M,2}_T) P^\infty] = \text{Tr}_s [gN \exp(-u^2 D^{B,2}_T) Q^\perp]. \]

There exist \( c > 0, C > 0 \) such that for \( u \geq 1, T \geq 1 \),

\[ \text{Tr}[gN \exp(-u^2 D^{M,2}_T) P^\infty] \leq c \exp(-Cu). \]
Theorem 4.7. The following identity holds,
\begin{equation}
\lim_{T \to +\infty} \text{Tr} \left[ g D_T^{M,2,[0,a]} \right] = 0.
\end{equation}

For $T \geq 1$ large enough, for $0 \leq i \leq \dim M,$
\begin{equation}
\text{Tr} \left[ g [h_T^{M,0} \phi_{T,i}] \right] = \sum_{j=0}^{i} \text{Tr} \left[ g H_i(B, R^{i-j} \pi, \xi) \right].
\end{equation}

Let $(E_r, d_r)$ $(r \geq 2)$ be the Leray spectral sequence associated to $\pi, \xi.$ By [Ma, Theorem 14.1], the Dolbeault complex $(\Omega(M, \xi), \overline{\partial}^M)$ filtered as in [BerB, §1a] calculate the Leray spectral sequence. Then as in [BerB, Section 4], for $r \geq 2,$ $E_r$ is equipped with a metric $h^{E_r}$ associated to $h^{TM}, h^{TB}, h^G.$ For $r \geq 2,$ let $r \mid \lambda_G(\xi)$ be the corresponding metric on $\lambda_G(\xi) \simeq (\det(E_r, G))^{-1}.$

For $r \geq 1,$ let $N_{|E_r}, N_{H|E_r}, N_{V|E_r}$ be the restriction of $N, N_H, N_V$ to $E_r.$

Theorem 4.8. The following identity holds,
\begin{equation}
\lim_{T \to +\infty} \left\{ \text{Tr}_s[g \log(D_T^{M,2,[0,a]})] + 2\sum_{r \geq 2} (r-1) \left( \text{Tr}_s[g N_{E_r}] - \text{Tr}_s[g N_{E_{r+1}}] \log(T) \right) \right\} = \log \left( \frac{1}{2} \left| \frac{\lambda_G(\xi)}{\lambda_G(\xi)} \right|^2 \right) (g).
\end{equation}

For $T \geq 1,$ let $|\lambda_G(\xi)|$ be the $L_2$ metric on the line $\lambda_G(\xi)$ associated to the metrics $h_T^{TM}, h^G$ on $TM, \xi.$

Theorem 4.9. The following identity holds,
\begin{equation}
\lim_{T \to +\infty} \left\{ \log \left( \frac{1}{|\lambda_G(\xi)|} \right)^2 (g) + 2 \left( -\dim X \chi_2(\xi) + \text{Tr}_s[g N_{V|E_r}] \log(T) \right) \right\} = \log \left( \frac{1}{|\lambda_G(\xi)|} \right)^2 (g).
\end{equation}

For $a > 0,$ let $B_a$ be the Bismut superconnection on $\Omega(X, \xi)|X|$ constructed in Definition 2.6 which is attached to $h^{TM}, h^G$ on $TM, \xi.$ Let $\tilde{N}_a$ be the operator defined in (2.12) associated to the metric $h_T^{TM}.$

Theorem 4.10. For any $T \geq 1,$
\begin{equation}
\lim_{\varepsilon \to 0} \text{Tr}_s \left[ g [\ast_{T/\varepsilon} \frac{\partial}{\partial T} (\ast_{T/\varepsilon}) \exp(-T^2 D_{T/\varepsilon}^{M,2})] \right] = \frac{2}{T} \int_B \text{Td}_g(TB, h^{TB}) \text{Tr}_s \left[ g N_{T/\varepsilon} \exp(-B^2_{T/\varepsilon}) \right] - \frac{2}{T} \dim X \chi_2(\xi).
\end{equation}

Let $\omega^{M}, \omega^{M}, \omega^B$ be the Kähler forms associated to $h^{TM}, h^{TM}, h^B.$ Let $\nabla_T^{TM}$ be the holomorphic Hermitian connection on $(TM, h^{TM}),$ and let $R_T^{TM}$ be its curvature.

Theorem 4.11. There exist $C > 0$ such that for $\varepsilon \in [0, 1], \varepsilon \leq T \leq 1,$
\begin{equation}
\left| \text{Tr}_s \left[ g [\ast_{T/\varepsilon} \frac{\partial}{\partial T} (\ast_{T/\varepsilon}) \exp(-T^2 D_{T/\varepsilon}^{M,2})] \right] - \frac{2}{T} \int_{M^{\varepsilon}} \text{Td}_g(TM) \chi_g(\xi) \right| + \int_{M^{\varepsilon}} \frac{\partial}{\partial T} \text{Td}_g \left( \frac{h^{T/\varepsilon}_{M^{\varepsilon}}}{2\pi} - b(h^{T/\varepsilon}_{M^{\varepsilon}})^{-1} \frac{\partial}{\partial T} (h^{T/\varepsilon}_{M^{\varepsilon}}) \right)_{b=0} \chi_g(\xi, h^F) \leq C.
\end{equation}
Theorem 4.12. There exist \( \delta \in [0, 1], C > 0 \) such that for \( \varepsilon \in [0, 1], T \geq 1 \),

\[
\left| \text{Tr}_x \left[ g * T/\varepsilon \frac{\partial}{\partial T} (g * T/\varepsilon) \exp(-\varepsilon^2 D^M_T/\varepsilon) \right] \right| - \frac{C}{\varepsilon^{2T/\pi}}.
\]

Besides, at a formal level, Theorems 4.5-4.9 can be obtained formally from [BerB, Theorem 4.8-4.12] by introducing in the right place the operator \( g \). This will permit us to transfer formally the discussion in [BerB, Section 4] to our situation.

c) Proof of Theorem 3.1.

By Theorem 2.12,

\[
\chi_g(R^* \pi_\varepsilon \xi) = \int_{X^\varepsilon} Td_g(TX) \chi_g(\xi).
\]

We also have the obvious equality

\[
Tg'(TM) = \pi^t \left( Tg'(TB) \right) Td_g(TX) + \pi^t \left( Td_g(TB) \right) Td'(TX).
\]

By Theorem 4.3, Theorems 4.5-4.12, and proceeding as in [BerB, § 4c,d]], using (4.24), (4.25), we get (3.6).

5 A proof of Theorems 4.5, 4.6 and 4.7

The proof of Theorems 4.5, 4.6 and 4.7 is essentially the same as the proof of [BerB, Theorem 4.8, 4.9 and 4.10] given in [BerB, § 5], where the corresponding results were established when \( G \) is trivial.

Now we use the notation of [BerB, §5].

At first, for each \( U \in TB, (gU)^H = gU^H \), so the operator \( C_T \) in [BerB, (5.7)] commute with the action of \( G \).

Let \( \langle \cdot \rangle_\infty \) be the Hermitian product on \( E_0 \) associated to the metrics \( \pi^* h^TB \oplus h^TX, h^\xi \) on \( TM, \xi \) defined by (1.2).

Let \( E_{1,T}, E_{1,T}', E_{1,T}^0 \) \((\mu \geq 0)\) be the vector spaces defined in [BerB, Definition 5.12]. Then for any \( T > 0 \), the linear isometric embedding \( J_T \) of \( E_{1,\infty} \) in \( E_{1,T} \) defined in [BerB, Definition 5.16] is \( G \)-equivariant. Let \( E_{1,T}^{0,1} \) be the orthogonal space to \( E_{1,T}^0 \) in \( E_{1,T}^0 \) with respect to \( \langle \cdot \rangle_\infty \). It follows from the previous considerations that for any \( T > 0 \), the orthogonal splitting \( E_{1,T}^0 = E_{1,T}^0 \oplus E_{1,T}^{0,1} \) of \( E_0 \) considered in [BerB, (5.29)] is \( G \)-invariant, i.e. \( G \) acts on \( E_{1,T}^0 \) and \( E_{1,T}^{0,1} \).

Therefore the matrix of the unitary operator \( g \) with respect to the splitting \( E_0^T = E_{1,T}^0 \oplus E_{1,T}^{0,1} \) can be written in the form

\[
g = \begin{bmatrix} g_{0,T} & 0 \\ 0 & g_{1,T} \end{bmatrix},
\]

and moreover

\[
g_{0,T} J_T = J_T g.
\]

The proof of Theorems 4.5, 4.6 and 4.7 then proceeds as in [BerB, § 5 c)-g)].
6 A proof of Theorems 4.8-4.9

In this Section, we give a proof of Theorems 4.8 and 4.9. These generalize [BerB, §6], where the corresponding results were proved in the case where $G$ is trivial.

We use the notation of [BerB, § 6].

a) Proof of Theorem 4.8.

At first we can verify the formulas of [BerB, Theorem 6.1-6.5] are $G$-equivariant. By using [B6, Theorem 1.4], and by proceeding as in [BerB, §6(d)], we obtain (4.16). This completes the proof of Theorem 4.8. □

b) Proof of Theorem 4.9.

For $W \in \widehat{G}$, let

\begin{align}
H^q_W &= \text{Hom}_G(W, H(X, \xi)) \otimes W, \\
E_{\infty, W} &= \text{Hom}_G(W, E_{\infty}) \otimes W.
\end{align}

Then

\begin{align}
H(X, \xi) &= \bigoplus_{W \in \widehat{G}} H^q_W, \\
\lambda_W(\xi) &= \det H^q_W.
\end{align}

By proceeding as in [BerB, §6(e)] for $H^q_W, \lambda_W(\xi)$, we deduce that as $T \to +\infty$

\begin{align}
\log \left( \frac{1}{|\lambda_W(\xi)|} \right) 2\chi(W)(g) \frac{\text{rk}(W)}{\text{rk}(T)} + 2 \left( -\dim X \text{Tr}_s[gJ_H^q] + \text{Tr}_s[gN_{E_{\infty, W}}] \right) \log(T)
\end{align}

Then Theorem 4.9 follows from (6.3). □

7 A proof of Theorem 4.10

This Section is organized as follows. In a), we show that the proof of (4.18) can be localized near $\pi^{-1}(B^\theta)$. In b), given $b_0 \in B^\theta$, we replace $M$ by $(T_R B)b_0 \times X_{b_0}$, and rescaling on certain Clifford variables. In c), we prove (4.18).

Recall that in this Section, we will calculate the asymptotics as $\varepsilon \to 0$ of certain supertraces involving $\varepsilon D^M_{T/\varepsilon}$ for a fixed $T \geq 1$.

a) The proof is local on $\pi^{-1}(B^\theta)$.

Let $dv_M$ (resp. $dv_B$, resp. $dv_X$) be the volume form on $M$ (resp. $B$, resp. on the fibre $X$) associated to the metric $h^TB \oplus h^{TX}$ on $TM \simeq \pi^*TB \oplus TX$ (resp. $h^TB$ on $TB$, resp. $h^{TX}$ on $TX$).

Let $\alpha, \alpha^M$ be the injective radius of $B, M$. In the sequel, we assume that given $0 < \alpha < \alpha_0 < \frac{1}{4} \inf\{\alpha^B, \alpha^M\}$ is chosen small enough so that if $y \in B$, $d^B(g^{-1}y, y) \leq \alpha$, then $d^B(y, B^\theta) \leq \alpha_0/4$, and if $x \in M, d^M(g^{-1}x, x) \leq \alpha$, then $d^M(x, M^\theta) \leq \alpha_0/4$. If $x \in B$, let $B^B(x, \alpha)$ be the open ball of center $x$ and radius $\alpha$ in $B$.
Let $f$ be a smooth even function defined on $\mathbb{R}$ with values in $[0, 1]$, such that
\begin{equation}
 f(t) = \begin{cases} 
 1 & \text{for } |t| \leq \alpha/2 \\
 0 & \text{for } |t| \geq \alpha. 
\end{cases}
\end{equation}

Set
\begin{equation}
 g(t) = 1 - f(t). 
\end{equation}

**Definition 7.1.** For $u \in [0, 1], a \in \mathbb{C}$, set
\begin{equation}
 F_u(a) = \int_{-\infty}^{+\infty} \exp(i\alpha \sqrt{2}) \exp\left(\frac{-t^2}{2}\right) f(ut) \frac{dt}{\sqrt{2\pi}}, \\
 G_u(a) = \int_{-\infty}^{+\infty} \exp(i\alpha \sqrt{2}) \exp\left(\frac{-t^2}{2}\right) g(ut) \frac{dt}{\sqrt{2\pi}}. 
\end{equation}

Clearly
\begin{equation}
 F_u(a) + G_u(a) = \exp(-a^2). 
\end{equation}

The functions $F_u(a), G_u(a)$ are even holomorphic functions. So there exist holomorphic functions $\tilde{F}_u(a), \tilde{G}_u(a)$ such that
\begin{equation}
 F_u(a) = \tilde{F}_u(a^2), \quad G_u(a) = \tilde{G}_u(a^2). 
\end{equation}

The restrictions of $F_u, G_u, \tilde{F}_u, \tilde{G}_u$ to $\mathbb{R}$ lie in $S(\mathbb{R})$.

From (7.4), we deduce that
\begin{equation}
 \exp(-\varepsilon^2 D_{T/\varepsilon} M_{\alpha}^2) = F_\varepsilon(\varepsilon D_{T/\varepsilon} M_{\alpha}) + G_\varepsilon(\varepsilon D_{T/\varepsilon} M_{\alpha}). 
\end{equation}

**Proposition 7.2.** For $\delta > 0$, there exist $c > 0$, $C > 0$ such that for $0 < \varepsilon \leq \delta, T \geq 1$,
\begin{equation}
 \left| \text{Tr}_S \left[ g \ast \frac{1}{\partial_T} (\ast T) G_\varepsilon(\frac{\varepsilon}{T} D_{T/\varepsilon}^M) \right] \right| \leq C \exp\left(-\frac{CT^2}{\varepsilon^2}\right). 
\end{equation}

**Proof.** The proof of our Theorem is essentially the same as the proof of [BerB, Proposition 8.3].

For $T \geq 1$ fixed, we use (7.7) with $\varepsilon = T$ and $T$ replace by $T/\varepsilon$, we find
\begin{equation}
 \left| \text{Tr}_S \left[ g \ast \frac{1}{\partial_T} (\ast T) G_\varepsilon(\frac{\varepsilon}{T} D_{T/\varepsilon}^M) \right] \right| \leq C \exp\left(-\frac{CT^2}{\varepsilon^2}\right). 
\end{equation}

Let $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, x') (x, x' \in M)$ be the smooth kernel associated to $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)$ with respect to the volume $\frac{d_{DM}}{(2\pi)^{d_{DM}}}$. Using (7.3) and finite propagation speed [CP §7.8], [T, §4.4], it is clear that for $\varepsilon \in [0, 1]$, $T \geq 1$, $x, x' \in M$, if $d_B(x, x') \geq \alpha$, then
\begin{equation}
 F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, x') = 0 
\end{equation}
and moreover, given $x \in M$, $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, \cdot)$ only depends on the restriction of $D_{T/\varepsilon}^M$ to $\pi^{-1} B^B(\pi x, \alpha)$.

For $u \in T_{RB}V, V \in T_{RB}X$, let $c(U), c(V)$ denote the corresponding Clifford multiplication operators acting on $\pi^* \Lambda(T^{\ast(0, 1)}B), \Lambda(T^{\ast(0, 1)}X)$ associated to $h^B$, $h^{TX}$ defined as in (2.6). Set
\begin{equation}
 A_{\varepsilon} = \frac{T}{\varepsilon} N_{\varepsilon} D_{T/\varepsilon}^{M} \left( \frac{T}{\varepsilon} \right)^{-N_{\varepsilon}}. 
\end{equation}
Then by (7.9), we get
\begin{equation}
(7.10) \quad \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T}(sT/s)F_c(\varepsilon D^M_{T/s}) \right] = \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T}(sT/s)F_c(A^e_{v,T}) \right].
\end{equation}

Let \( F_c(A^e_{v,T})(x,x') (x,x' \in M) \) be the smooth kernel associated to the operator \( F_c(A^e_{v,T}) \) with respect to \( \frac{dv_M}{(2\pi)^{\dim M}}. \)

For \( \alpha > 0 \), let \( \mathcal{U}_{0\alpha}(B^g) \) be the set of \( b \in B \) such that \( d^B(b,B^g) < \alpha_0 \). We identify \( \mathcal{U}_{0\alpha}(B^g) \) to \( \{(b,Y); b \in B^g, Y \in N_{B^g/B,R}, |Y| \leq \alpha_0 \} \) by using geodesic coordinates normal to \( B^g \) in \( B \), then
\begin{equation}
(7.11) \quad \int_{B^g} \int_{Y \in N_{B^g/B,R}} \int_X \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T}(sT/s)F_c(A^e_{v,T})(g^{-1}(b,Y,x),(b,Y,x)) \right] \frac{dv_M}{(2\pi)^{\dim M}}.
\end{equation}

By (7.11), we see that the proof of Theorem 4.10 is local near \( \pi^{-1}(B^g) \).

b) Rescaling of the variable \( Y \) and of the Clifford variables.

Taking \( b \in B^g \), we identify \( B^B(b_0,\alpha_0) \) with \( B(0,\alpha_0) \subset (TB)_{b_0} = \mathbb{C}^m \) by using normal coordinates.

Take \( y \in \mathbb{C}^m, |y| \leq \alpha_0 \), set \( Y = y + \overline{y} \). We identify \( TB_{b_0} \) to \( TB_{[0]} \) by parallel transport along the curve \( t \to tY \) with respect to the connection \( \nabla^{TB} \). We lift horizontally the paths \( t \in \mathbb{R}_+ \to tY \) into paths \( t \in \mathbb{R}_+ \to x_t \in M \) with \( x_t \in X_{b_0} \), \( \frac{dx_t}{dt} \in T^{B_{b_0}}M \). If \( x_0 \in X_{b_0} \), we identify \( TX_{x_0}, \xi_{x_0} \) to \( TX_{x_0}, \xi_{x_0} \) by parallel transport along the curve \( t \to x_t \) with respect to the connections \( \nabla^{TX}, \nabla^\xi \). These trivializations induce corresponding trivializations of \( \Lambda(T^*^{(0,1)}B), \Lambda(T^*^{(0,1)}M) \) \( \otimes \xi \).

Let \( \Omega_{b_0} = \Omega(X_{b_0},\xi|_{X_{b_0}}) \) be the vector space of smooth sections of \( (\Lambda(T^*^{(0,1)}X) \otimes \xi)|_{X_{b_0}} \) on \( X_{b_0} \). Then \( \Omega_{b_0} \) is naturally equipped with a Hermitian product \( \langle \ , \ \rangle \) attached to \( h^{TX|_{X_{b_0}}}, h^\xi|_{X_{b_0}} \).

There is also a smooth \( \mathbb{Z} \)-graded vector bundle \( K \subset \Omega_{b_0} \) over \( (TB)_{b_0} \approx \mathbb{R}^{2m} \) which coincide with \( \text{Ker}D^X \) on \( B(0,2\alpha_0) \), with \( \text{Ker}D_{b_0}^X \) over \( TR\bar{B}|B(0,3\alpha_0) \) and such that if \( K^\perp \) is the orthogonal bundle to \( K \) in \( \Omega_{b_0} \),
\begin{equation}
(7.12) \quad K^\perp \cap \text{Ker}D^X_{b_0} = \{0\}.
\end{equation}

Let \( P_b \) be the orthogonal projection operator from \( \Omega_{b_0} \) on \( K_b \). Set \( P_b^\perp = 1 - P_b \).

Let \( \varphi : R \to [0,1] \) be a smooth function such that
\begin{equation}
(7.13) \quad \varphi(t) = 1 \quad \text{for} \quad |t| \leq \alpha_0 \quad \quad \quad 0 \quad \text{for} \quad |t| \geq 2\alpha_0.
\end{equation}

Let \( \Delta^{TB} \) be the standard Laplacian on \( (TR\bar{B})_{b_0} \) with respect to the metric \( h^{TB_{b_0}} \). Let \( \mathcal{H}_{b_0} \) be the vector space of smooth sections of \( \pi^*\Lambda(T^*^{(0,1)}B)_{b_0} \otimes (\Lambda(T^*^{(0,1)}X) \otimes \xi) |_{X_{b_0}} \) over \( (TR\bar{B})_{b_0} \times X_{b_0} \). Let \( L^e_{v,T} \) be the operator
\begin{equation}
(7.14) \quad L^e_{v,T} = \varphi^2(|Y|)A^e_{v,T} + (1 - \varphi^2(|Y|)) \left( \frac{-\varepsilon^2 \Delta^{TB}}{2} + T^2 P^\perp_{b_0} D^X_{b_0} P^\perp_{b_0} \right).
\end{equation}

For \( \varepsilon > 0, s \in H_{b_0} \), set
\begin{equation}
(7.15) \quad S_{\varepsilon,s}(Y,x) = s(Y/\varepsilon, x).
\end{equation}
Put

\[ L^2_{\varepsilon,T} = S^{-1}_{\varepsilon} L^{1}_{\varepsilon,T} S_{\varepsilon}. \]

Let \( O_p \) be the set of differential operators acting on smooth sections of \((\Lambda(T^{r(0,1)} \otimes \xi))_{X_{b0}}\) over \( \mathbb{R}^{2m} \times X_{b0} \). Then we find that

\[ L^2_{\varepsilon,T} \in c(T_{\mathbb{R}} B) \otimes O_p. \]

Let \( f_1, \ldots, f_{2m'} \) be an orthonormal basis of \((T_{\mathbb{R}} B)_{b0} \), let \( f_{2m'+1}, \ldots, f_{2m} \) be an orthonormal basis of \( N_{B'/B, \mathbb{R}}_{b0} \).

**Definition 7.3.** For \( \varepsilon > 0 \), set

\[ c_{\varepsilon}(f_j) = \frac{\sqrt{2}}{\varepsilon} f_j \wedge -\frac{\varepsilon}{\sqrt{2}} i f_j, \quad 1 \leq j \leq 2m'. \]

Let \( L^3_{\varepsilon,T}, M^3_{\varepsilon,T} \) be obtained from \( L^2_{\varepsilon,T} \), \( \ast_{T/\varepsilon} \frac{\partial}{\partial T} (\ast_{T/\varepsilon}) \) by replacing the Clifford variables \( c(f_j)(1 \leq j \leq 2m') \) by the operators \( c_{\varepsilon}(f_j) \).

Let \( P^3_{\varepsilon,T}(Y, x), (Y', x')) \), \( F_{\varepsilon}(L^1_{\varepsilon,T})(Y, x), (Y', x') \) \((Y, x), (Y', x') \in (T_{\mathbb{R}} B)_{b0} \times X_{b0} \) be the smooth kernels associated to \( \exp(-L^1_{\varepsilon,T}), \overline{F}_{\varepsilon}(L^1_{\varepsilon,T}) \) calculated with respect to \( \frac{dv}{(2\pi)^{2m}} \). Using finite propagation speed, we see that if \((Y, x) \in (T_{\mathbb{R}} B)_{b0} \times X_{b0}, |Y| < a_0/4 \), then

\[ F_{\varepsilon}(A^1_{\varepsilon,T})(g^{-1}(Y, x), (Y, x)) = \overline{F}_{\varepsilon}(L^1_{\varepsilon,T})(g^{-1}(Y, x), (Y, x)) \]

We observe that for any \( k \in \mathbb{N}, c > 0 \), there is \( C > 0, C' > 0 \) such that for \( \varepsilon > 0 \),

\[ \sup_{\|a\| \leq \varepsilon} |a|^k \left| \overline{F}_{\varepsilon}(a^2) - \exp(-a^2) \right| \leq c \exp\left(-\frac{C}{\varepsilon^2}\right). \]

Using (7.19), and proceeding as in Proposition 7.2, we find for \( |Y| < a_0/4 \)

\[ \left| (\overline{F}_{\varepsilon}(L^1_{\varepsilon,T}) - \exp(-L^1_{\varepsilon,T}))(Y, x), (Y', x') \right| \leq \exp\left(-\frac{C}{\varepsilon^2}\right). \]

By (7.18), (7.19), we can replace \( F_{\varepsilon}(L^1_{\varepsilon,T}) \) by \( \exp(-L^1_{\varepsilon,T}) \) in (7.11).

We know that \( P^3_{\varepsilon,T}(Y, x), (Y', x') \) lies in \( \left( \text{End}(\Lambda(T_{\mathbb{R}} B^*) \otimes c(N_{B'/B, \mathbb{R}})_{b0} \otimes c(T_{\mathbb{R}} X_{b0}) \otimes \text{End}(\xi) \right) \).

Then \( M^3_{\varepsilon,T} P^3_{\varepsilon,T}(g^{-1}(Y, x), (Y, x)) \) can be expanded in the form

\[ M^3_{\varepsilon,T} P^3_{\varepsilon,T}(g^{-1}(Y, x), (Y, x)) = \sum_{1 \leq i_1 < \cdots < i_{2m'} \leq 2m'} f^{i_1} \wedge \cdots \wedge f^{i_{2m'}} \wedge i f_{j_1} \cdots i f_{j_{2m'}} \otimes R^{i_1 \cdots i_{2m'}} \]

\[ R^{i_1 \cdots i_{2m'}}(g^{-1}(Y, x), (Y, x)) \in c(N_{B'/B, \mathbb{R}})_{b0} \otimes c(T_{\mathbb{R}} X_{b0}) \otimes \text{End}(\xi). \]

Set

\[ \left[ M^3_{\varepsilon,T} P^3_{\varepsilon,T}(g^{-1}(Y, x), (Y, x)) \right]_{\text{max}} = R^{1, \cdots, 2m'}(g^{-1}(Y, x), (Y, x)). \]

**Proposition 7.4.** If \( Y \in N_{B'/B, \mathbb{R}}_{b0} \), the following identity holds

\[ \text{Tr}_x \left[ g \ast_{T/\varepsilon} \frac{\partial}{\partial T} (\ast_{T/\varepsilon}) P^3_{\varepsilon,T}(g^{-1}(Y, x), (Y, x)) \right] = (-i)^{\dim B'} e^{-2 \dim N_{B'/B, \mathbb{R}}} \text{Tr}_x \left[ g [M^3_{\varepsilon,T} P^3_{\varepsilon,T}(g^{-1}(e^{-1} Y, x), (e^{-1} Y, x))]_{\text{max}} \right]. \]
PROOF. Since \( g \) acts like the identity on \( \Lambda((T^+(0,1)B)^g) \), \( g \in c(N_{B^+/B,R})_{0,0} \otimes c(T_R X_{0,0}) \otimes \text{End}(\xi) \). Therefore the rescaling of the Clifford variable in (7.17) has no effect on \( g \). Identity (7.23) is now a trivial consequence of [Ge].

**c) Proof of Theorem 4.10.**

Recall that for \( u > 0 \), the Bismut superconnection \( B_u \) associated to \( h^T \) and \( h^\xi \) was constructed in Section 2b). Also we observe that \( B_u \) is unchanged if \( h^T \) is changed into \( \widehat{h}^T \).

Let \( R^T_B \) be the curvature of \( \nabla^T_B \). Also \( \nabla f_a \) denote the ordinary differentiation operator on \( (T_R B)_{0,0} \) in the direction \( f_a \).

Then as in [BerB, (7.30), (7.35)], we have as \( \varepsilon \to 0 \)

\[
(7.24) \quad L^3_{\varepsilon,T} \to L^3_{0,T}.
\]

and

\[
(7.25) \quad e^{-\frac{\varepsilon \pi}{2\pi}} L^3_{0,T} e^{-\frac{\varepsilon \pi}{2\pi}} = -\frac{1}{2} \left( \nabla f_a + \frac{1}{2} \left( R^T_B Y^T f_a \right)_{h^T_B} \right)^2 + \frac{1}{2} \text{Tr}(R^T_B) + B^2_{T/2}.
\]

By [BerB, (7.38)], we get, as \( \varepsilon \to 0 \)

\[
(7.26) \quad M^3_{\varepsilon,T} \to M^3_{0,T} = \frac{2}{T} \left( N_Y - \text{dim} X \right) + \frac{2i\zeta h^T_B}{T}.
\]

By [B4, (3.16)-(3.21)], [BerB, § 7d)], we have

\[
\int_{N_{B^+/B,R})_{0,0}} \int_{X_{0,0}} \text{Tr} \left[ g[M^3_{0,T} P^3_{0,T} (g^{-1}(Y,x),(Y,x))] \right]^{max} \frac{dv_{N_{B^+/B,R})_{0,0}}(x)}{(2\pi)^{dim M}}
\]

\[
\times \left( -i \right)^{dim M} \text{Tr} \left[ g(\hat{N} - \text{dim} X) \exp(-B^2_{T/2}) \right]^{max}
\]

\[
(7.27) \quad = \left( -i \right)^{dim M} \left\{ \text{Tr} \left[ g(\hat{N} - \text{dim} X) \exp(-B^2_{T/2}) \right] \right\}^{max}.
\]

**Theorem 7.5.** For \( T \geq 1 \) fixed, there exist \( C > 0, C > 0, r \in \mathbb{N} \) such that for \( \varepsilon \in [0,1] \), \( (Y,x),(Y',x') \in (T_R B)_{0,0} \times X_{0,0} \),

\[
(7.28) \quad \left| (P^3_{\varepsilon,T} - P^3_{0,T})(Y,x),(Y',x') \right| \leq c(1 + |Y| + |Y'|)^r \exp(-C|Y - Y'|^2).
\]

To prove Theorem 7.5, we establish at first a uniform estimate on the kernel \( P^3_{\varepsilon,T} \).

**Theorem 7.6.** There is \( C > 0 \) such that for \( m \in \mathbb{N} \), there exist \( c > 0, r \in \mathbb{N} \) such that for any \( \varepsilon \in [0,1] \), \( (Y,x),(Y',x') \in (T_R B)_{0,0} \times X_{0,0} \),

\[
(7.29) \quad \sup_{|Y| + |Y'| \leq m} \left| \frac{\partial^{|Y| + |Y'|}}{\partial Y \partial Y'} P^3_{\varepsilon,T} ((Y,x),(Y',x')) \right| \leq c(1 + |Y| + |Y'|)^r \exp(-C|Y - Y'|^2).
\]

**PROOF of Theorem 7.6.** Set

\[
(7.30) \quad g_\varepsilon(Y) = 1 + (1 + |Y|^2)^{\frac{\varepsilon}{2}} \varphi\left( \frac{|Y|^2}{2} \right).
\]

Let \( E_0 \) be the vector space of square integrable sections of \( \Lambda((T^+(0,1)B)^g) \otimes c(N_{B^+/B,R})_{0,0} \otimes (\Lambda(T^+(0,1)X) \otimes \xi)_{\|X_{0,0}} \) over \((T_R B)_{0,0} \times X_{0,0} \). For \( 0 \leq q \leq 2m = 2\text{dim }B^o \), let \( E_0^q \) be the vector
space of square integrable sections of \((\Lambda^q(T^*_{\mathbf{R}^p}B^q) \otimes c(N_{B^p/B,R}))_{k_0} \otimes (\Lambda(T^{(0,1)}_{\mathbf{R}}) \otimes \xi)|_{X_{k_0}}\). Then \(E^0 = \oplus_{q=0}^{2m} E_q^0\). Similarly, if \(p \in \mathbf{R}, E^p_0\) and \(E^q_0\) denote the corresponding \(p\)th Sobolev spaces.

If \(s \in E^0_0\), set

\[
(7.31) \quad |s|_{\varepsilon,0}^2 = \int_{(\mathbf{R}^p B)_{k_0} \times X_{k_0}} |s(Y, x)|^{2(m-q)} g_c(Y) \frac{dv_B(Y)dv_X(x)}{(2\pi)^{\text{dim} M}}.
\]

Let \(\langle \cdot, \cdot \rangle_{\varepsilon,0}\) be the Hermitian product attached to \(|\cdot|_{\varepsilon,0}\). If \(s \in E^1\), put

\[
(7.32) \quad |s|_{\varepsilon,1}^2 = |s|_{\varepsilon,0}^2 + \sum_{t=1}^{2m} |\nabla f_{s,0} s|_{\varepsilon,0}^2 + \sum_j |\nabla c_j s|_{\varepsilon,0}^2.
\]

Using the technique in [BerB, §91)], special [BerB, (9.51)] (in our situation, \(T\) is fixed), the bounds in (7.29) with \(C = 0\) are easily obtained. To get the required \(C > 0\), we proceed as in the proof of [B5, Theorem 11.14].

**Proof of Theorem 7.5.** Using Theorem 7.6, and proceeding as in [B5, §11 i), [BL, §11 q)], we have Theorem 7.5.

For \(b \in B^q, Y \in N_{B^p/B,R,b}, |Y| \leq \alpha_0\), let \(k(b,Y)\) be defined by

\[
(7.33) \quad dv_B(b, Y) = k(b, Y)dv_B(b)dv_{N_{B^p/B}}(Y).
\]

Using Theorem 7.5, (7.18), (7.20), (7.23), (7.27), we get over \(B^q\)

\[
\lim_{\varepsilon \to 0} \int_{\int_{\varepsilon}^{\varepsilon}} \int_{X} \text{Tr}_{s} \left[ g^{-1}(st \varepsilon) \frac{\partial}{\partial T}(st \varepsilon) F_{s}(A_{s}', T) \right] dv_X(x) = \frac{1}{\varepsilon} \left\{ Td_g(TB, h^{-1}B) Tr (g(N_{T} - \dim X) \exp(-B_{T}^2)) \right\}^\text{max.}
\]

By (7.7), (7.34), the proof of Theorem 4.10 is complete.

8 A proof of Theorem 4.11

This Section is organized as follows. In a), we reformulate Theorem 4.11. In b), we indicate that the proof is localized near \(\pi^{-1}(B^q)\) by Proposition 7.2. In c), we prove the estimate (8.1).

We make the same assumption and use the same notation as in Sections 4 and 7.

a) A reformulation of Theorem 4.11.

**Theorem 8.1.** There exists \(C > 0\) such that for \(0 < u \leq 1, T \geq 1\),

\[
\left| \text{Tr}_{s} \left[ g^{-1}(st \varepsilon) \frac{\partial}{\partial T} \exp(-\frac{u^2}{T^2}D_{T}^{M,2}) \right] - \frac{2}{u^2} \int_{M^p} \frac{\omega_{T}^{M}}{2\pi T} Td_g(TM)ch_g(\xi) \right| \leq Ch^2.
\]

By (8.1), the proof of Theorem 8.2 is complete. In fact, for \(0 < \varepsilon \leq 1, \varepsilon \leq T \leq 1\) we use (8.1), with \(u = T\) and \(T\) replaced by \(T^\varepsilon\), then we find that the right-hand side of (8.1) is dominated by

\[
CT^2 \frac{\varepsilon}{T} = C\varepsilon T \leq C\varepsilon.
\]
So we have proved (4.19).

**b) Localization of the problem near \( \pi^{-1}(B^g) \).**

By Proposition 7.2 and the argument in Section 7b), the proof of (8.1) can be localized near \( B^g \).

Thus, we are entitled to choose \( b_0 \in B^g \) as in Section 7b), to replace \( M \) by \( C^m \times X_{b_0} \) and to trivialize the vector bundles as indicated in Section 7b). Then we will prove (8.1) in this situation.

**c) Proof of Theorem 8.1.**

By (7.9)

\[
A'_{1/T,1} = T^{N_{Y^1}} \frac{1}{T} D_M T^{-N_{Y^1}}.
\]

Therefore

\[
\text{Tr}_x \left[ g * \pi^{-1} \frac{\partial}{\partial T} (s_T) \exp \left( -\frac{u^2}{T^2} D_T^{M^2} \right) \right] = \text{Tr}_x \left[ g * \pi^{-1} \frac{\partial}{\partial T} (s_T) \exp \left( -u^2 A'_{1/T,1} \right) \right].
\]

We will use the notation of Section 7 with \( \varepsilon \) replaced by \( \frac{1}{T} \), and \( T \) by 1. By (7.24), we see that as \( T \to +\infty \)

\[
L^3_{1/\varepsilon,1} \to L^3_{0,1}.
\]

Let \( P_{\varepsilon,T,u}^g((Y,x),(Y',x')) \) be the smooth kernel associated to the operator \( \exp(-u^2 L_{1/T}^g) \) calculated with respect to \( \frac{dv_{T,B_{b_0}}}{(2\pi)^{\dim M}} \). For \( Y \in N_{B^g/B,\text{R},b_0}, x \in X_{b_0} \), set

\[
Q_{\varepsilon,u}(Y,x) = \text{Tr}_x \left[ g \left( M_{\varepsilon,1}^3 P_{\varepsilon,1,u}^g(g^{-1}(Y,x),(Y',x)) \right) \right].
\]

By (7.23), for \( Y \in N_{B^g/B,\text{R},b_0}, x \in X_{b_0} \), we have

\[
\text{Tr}_x \left[ g * \pi^{-1} \frac{\partial}{\partial T} s_T P_{1/T,1,u}^g(g^{-1}(Y,x),(Y',x)) \right] = (-i)^{\dim B^g} \pi^{2\dim N_{B^g/B}} \frac{1}{T} Q_{1/T,u}(TY,x).
\]

By (8.6) and the argument of Section 7b), to calculate the asymptotics of (8.3) as \( u \to 0 \) uniformly in \( T \geq 1 \), we have to find the asymptotics as \( u \to 0 \) of

\[
\int_{Y \in N_{B^g/B,\text{R}}} dX Q_{1/T,u}(Y,x) \frac{dv_{X_{b_0}}}{(2\pi)^{\dim M}} \frac{dW_{N_{B^g/B}}}{(2\pi)^{\dim M}}.
\]

Let \( d^X(x,x') \) be the distance function on \( (X,h^{T_{X_{b_0}}}) \). Then \( d((Y,x),(Y',x')) = (|Y - Y'|^2 + d^X(x,x')^2)^{1/2} \) be a distance function on \( (T_{\text{R}}B)_{b_0} \times X_{b_0} \).

**Proposition 8.3.** There exist \( c,C > 0,p,r \in \mathbb{N} \) such that for any \( (Y,x),(Y',x') \in (T_{\text{R}}B)_{b_0} \times X_{b_0}, \ v \in [0,1], \ u \in [0,1], \)

\[
\left| u^p P_{\varepsilon,1,u}^g((Y,x),(Y',x')) \right| \leq c(1 + |Y| + |Y'|)^r \exp \left( -C\frac{|Y - Y'|^2 + d^X(x,x')^2}{u^2} \right).
\]

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PROOF. By proceeding as in the proof of Theorem 7.6, the bounds in (8.8) with $C = 0$ are easily obtained. To get the required $C > 0$, we proceed as in the proof of [B5, Theorem 11.14].

Let $u \in \mathbb{R} \to k(u)$ be a smooth even function such that

$$
(8.9) \quad k(u) = \begin{cases} 
0 & \text{for } |u| \leq 1/2, \\
1 & \text{for } |u| \geq 1.
\end{cases}
$$

For $q \in \mathbb{R}^*_+$, $a \in \mathbb{C}$, set

$$
(8.10) \quad K_q(a) = 2 \int_0^{+\infty} \cos(t \sqrt{2a}) \exp(-t^2/2) k(t/q) \frac{dt}{\sqrt{2\pi}}.
$$

Clearly, $K_q(a)$ is an even holomorphic function of $a$, therefore, there is a holomorphic function $a \in \mathbb{C} \to \tilde{K}_q(a)$ such that

$$
(8.11) \quad K_q(a) = \tilde{K}_q(a^2).
$$

Using finite propagation speed for the solution of hyperbolic equations for $\cos(s \sqrt{E_{L_1}})$ [CP, §7.8], [T;§4.4], we find there is a fixed constant $c' > 0$ such that

$$
(8.12) \quad P_{e,1,u}^q((Y, x), (Y', x')) = \tilde{K}_{q/u}(u^2L_{e,1}^q)((Y, x), (Y', x'))
$$

if $d((Y, x), (Y', x')) \geq c' q$.

By using the proof of Theorem 7.6, and [B5, Theorem 11.14], there is a $C > 0$ such that there exist $c > 0, p, r \in \mathbb{N}$ for which given $q \in \mathbb{N}$, $(Y, x), (Y', x') \in (T \mathbb{R}B)_{b_0} \times X_{b_0}, \varepsilon \in [0,1], u \in [0,1]$, then

$$
(8.13) \quad \left| u^p \tilde{K}_{q/u}(u^2L_{e,1}^q)((Y, x), (Y', x')) \right| \leq c(1 + |Y| + |Y'|)^r \exp(-Cq'^2/u^2).
$$

From (8.13), we have (8.8). \hfill \Box

By (8.8), to calculate the asymptotics of (8.7) as $u \to 0$, we can localize near $\{0\} \times X_{b_0}$. We identify $\mathcal{U}_{x_0}(\{0\} \times X_{b_0})$ to $\{(Y, x, X), Y \in (TB)_{b_0}, x \in X^g, X \in N_{X^g/X}; |Y|, |X| \leq \alpha_0\}$ by geodesic coordinates normal to $\{0\} \times X_{b_0}$ in $TB \times X$.

For $Y \in T \mathbb{R}B, x \in X^g, X \in N_{X^g/X} \mid X \leq \alpha_0/4$, let $k'(Y, x, X)$ be defined by

$$
(8.14) \quad dv_X(Y, x, X) = k'(Y, x, X)dv_{N_{X^g/X}}(X)dv_{X^g}(x).
$$

Using (8.4), we find there exist smooth functions $a'_{T,-n}(x), \ldots, a'_{T,0}(x) (x \in M^g)$ such that as $u \to 0$, for $x \in X_{b_0}$

$$
(8.15) \quad \int_{X \in N_{X^g/X}} Q_1/T_u((0, Y), (x, X))k'(Y, x, X) \frac{dv_{N_{X^g/X}}(X)dv_{N_{Y^g/Y}}(Y)}{(2\pi)^{\dim M}} = \sum_{j=-n}^{0} a'_{T,j}(x)u^{2j} + O(u^2).
$$

By (7.11),(7.26),(8.4)-(8.8),(8.15), we know that there exist $a_{T,j}$ depending continuously on $T \in [1, +\infty]$ such that for any $u \in [0,1], T \in [1, +\infty]$

$$
(8.16) \quad \left| \text{Tr}_x \left[ \varphi_T \frac{\partial}{\partial T} (\ast_T) \exp(-u^2D_T^{M,2}) \right] - \sum_{j=-\dim M}^{0} a_{T,j}u^{2j} \right| \leq \frac{cu^2}{T}.
$$

Set

$$
(8.17) \quad b_{-1,q} = \int_{M^g} \frac{h_T}{2\pi} Td_g(TM) \varphi_T(\xi), \\
b_{0,q} = \int_{M^g} \frac{\partial}{\partial \xi} \left[ Td_g\left( \frac{h_T}{2\pi} - b(h_T^{TM})^{-1} \frac{\partial h_T^{TM}}{\partial T} \right) \right]_{b=0} \varphi_T(\xi, h^2).
$$
By \[ B^5, (2.44), (2.63) \], for \( T \geq 1 \), as \( u \to 0 \)

\[
(8.18) \quad \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \exp(-u^2 D^M_T) \right] = \frac{2}{u^2} \frac{b-1}{T^3} - k_{0,a} + O(u^2).
\]

By (8.16), (8.18), we get (8.1).

\section{A proof of Theorem 4.12}

This Section is organized as follows. In a), as in \[ \text{BerB}, \S 9 \], we reduce the problem to a local problem near \( B^g \). In b), we summarize very briefly the content of \[ \text{BerB}, \S 9 c \]. In c), we establish key estimates on the kernel of \( F_\varepsilon(L^3_{\varepsilon,T}) \). In d), we prove Theorem 4.12.

\begin{itemize}
\item[a)] Finite propagation speed and localization.
\end{itemize}

\textbf{Proposition 9.1.} \textit{There exists \( C > 0 \), such that for \( 0 < \varepsilon \leq 1 \), \( T \geq 1 \)

\[
(9.1) \quad \left| \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \right] \right| \leq \frac{C}{T^3}.
\]

\textbf{Proof.} By an analogue of the McKean-Singer formula \[ \text{MKS} \], we find that

\[
(9.2) \quad \text{Tr}_s [g N_Y H_\varepsilon(D^B)] = \sum_{j=0}^{\dim X} (-1)^j j \chi_\varepsilon(R^j \pi_\varepsilon) H_\varepsilon(0).
\]

Using (9.2) and proceeding as in \[ \text{BerB}, \text{Proposition 9.1} \], we have (9.1). \( \blacksquare \)

By (7.6) and (9.1), to establish Theorem 4.12, we only need to establish the following result.

\textbf{Theorem 9.2.} \textit{If \( \alpha > 0 \) is small enough, there exist \( \delta > 0, C > 0 \), such that for \( 0 < \varepsilon \leq 1 \), \( T \geq 1 \)

\[
(9.3) \quad \left| \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \right] \right| \leq \frac{C}{T^{1+\delta}}.
\]

\textbf{Proof.} The remainder of the Section is devoted to the proof of Theorem 9.2. \( \blacksquare \)

Using (7.1), we deduce that

\[
(9.4) \quad \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \right] = \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \right] F_\varepsilon(A^2_{\varepsilon,T}).
\]

Let \( \tilde{F}_\varepsilon(A^2_{\varepsilon,T})(x,x') \) be the smooth kernel associated to \( F_\varepsilon(A^2_{\varepsilon,T}) \) with respect to \( dv_M/(2\pi)^{\dim M} \). Using finite propagation speed, it is clear that if \( x \in M \), \( \tilde{F}_\varepsilon(A^2_{\varepsilon,T})(x,) \) only depends on the restriction of \( A^2_{\varepsilon,T} \) to \( \pi^{-1}(B^g) \).

As in Section 7, the proof of (9.3) is local near \( \pi^{-1}(B^g) \).

\begin{itemize}
\item[b)] The matrix structure of the operator \( L^3_{\varepsilon,T} \) as \( T \to +\infty \).
\end{itemize}

We use the same trivialization and notation as in Section 7.

Also by using (7.18), (7.23), for \( Y \in (N_{B^g \cap B})_{0_0} \), we get

\[
(9.5) \quad \text{Tr}_s \left[ g^{-1} \frac{\partial}{\partial T} (\ast_T) \right] F_\varepsilon(L^1_{\varepsilon,T})(g^{-1}(Y,x), (Y,x)) = (-1)^{\dim B \varepsilon - 2 \dim N_{B^g \cap B}} \text{Tr}_s \left[ M^{3}_{\varepsilon,T} F_\varepsilon(L^3_{\varepsilon,T})(g^{-1}(\varepsilon^{-1} Y, x), (\varepsilon^{-1} Y, x)) \right]_{\text{max}}.
\]
Let $F^0_\varepsilon$ be the vector space of square integrable sections of $\Lambda(T_R^0 B^g) \otimes c(N_{B^g/B, R}) \otimes S^{-1}_\varepsilon K$ over $(T_R B)_{b_0}$. Then $F^0_\varepsilon$ is a Hilbert subspace of $E^0$. Let $F^0_{\varepsilon, \perp}$ be its orthogonal in $E^0$. Let $p_\varepsilon$ be the orthogonal projection operator from $E^0$ on $F^0_\varepsilon$.

For a fixed $\varepsilon > 0$, the analysis of the matrix structure of $L^3_{\varepsilon, T}$ as $T \to +\infty$ is the same as in [BerB, §9 c]. Of course, the rescaling on the Clifford variables which depends on $\varepsilon > 0$, is different, but this does not introduce any extra difficulty.

Then [BerB, Theorem 9.3] still holds for essentially the same reasons as in [BerB].

c) Uniform bounds on the kernel of $\tilde{F}_\varepsilon(L^3_{\varepsilon, T})$.

We now establish an extension of [BerB, Theorem 9.6].

Theorem 9.3. There exist $C > 0, r \in \mathbb{N}$, for which if $m' \in \mathbb{N}$, there exists $C' > 0$ such that if $|\alpha|, |\alpha'| \leq m'$, $\varepsilon \in [0, 1]$, $T \geq 1$, $(Y, x), (Y', x') \in (T_R B)_{b_0} \times X_{b_0}$,

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^\alpha \partial Y'^{\alpha'}} \tilde{F}_\varepsilon(L^3_{\varepsilon, T})((Y, x), (Y', x')) \right| \leq c(1 + |Y| + |Y'|)^r \exp(-C|Y - Y'|^2).$$

Proof. Recall $\langle \cdot, \cdot \rangle_{T, 0}$ the Hermitian product on $E^0$ defined by (7.31). If $s \in E^1$, put

$$|s^2|_{T, 1} = T^2|P_{\overline{Y}} s|^2_{\varepsilon, 0} + |P_{\overline{Y}} s|^2_{\varepsilon, 0} + \Sigma^m_i |\nabla_{f_i} s|^2_{\varepsilon, 0} + T^2 \sum |\nabla_{f_i} P_{\overline{Y}} s|^2_{\varepsilon, 0}.$$

The bounds in (9.6) with $C = 0$ are easily obtained by proceeding as in [BerB, Theorem 9.6]. To get the required $C > 0$, we proceed as in the proof of [B5, Theorem 11.14 and 13.14].

d) Proof of Theorem 9.2.

Let $\Xi_\varepsilon$ be the analogue of the elliptic second order differential operator considered in [BerB, Definition 9.7]. The minor difference with [BerB] is that here only the Clifford variables $c(f_l)$ ($1 \leq l \leq 2\dim B^g$) are rescaled, while in [BerB], the Clifford variables $c(f_l)$ ($1 \leq l \leq 2\dim B$) were rescaled. Because our Clifford rescaling introduces fewer diverging terms than in [BerB, §9], the analogue of [BerB, Theorem 9.8] still holds.

Proceeding as in [BerB, §9f), e)] and [B5, §13 j)], we obtain Theorem 9.2.

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