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EFFECTIVE DEGREES OF FREEDOM IN STRONG INTERACTION PROCESSES

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ABSTRACT

We propose a scheme of approximation where strong interaction processes with a purely longitudinal configuration of external momenta are described by fields which depend on longitudinal space-time variables only, but retain their full four-dimensional spin structure. The evolution of these fields is governed by an effective Lagrangian, which can be obtained from the four-dimensional Lagrangian using a variational method. We discuss briefly the main features of the effective Lagrangians for one-dimensional and two-dimensional configurations of the external momenta.

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1. INTRODUCTION

The theoretical study of strongly interacting particles cumulates the complexities of the many-body problem, together with the very stringent requirements of relativity and locality (analyticity for bootstrap people !).

The fact that most approaches lead to a limited predictive power depends on the difficulty of producing an approximation scheme with a good degree of consistency. The general requirements that a good solution should satisfy are so tight, that sooner or later we are bound to get into trouble with some of them. Since standard perturbation theory is out of question for strong interactions, one can think of two possible kinds of theoretical schemes :

- i) one can assume that a limited number of particles play a fundamental role or, in a more sophisticated way, that only a selected class of graphs is important;
- ii) another possibility, which has been until now much less popular, is to introduce a scheme, where one assumes that for a definite category of processes, only a limited number of degrees of freedom can be excited, whereas the remaining ones are frozen. In terms of graphs, this would correspond to the following prescription : keep all the diagrams, but make a unique "extremely" drastic approximation on all of them.

Needless to say, this second approach has been used in all fields of physics. Its application to particle physics has been pioneered by Tomonaga ¹⁾ and Pauli ²⁾, and its most beautiful and convenient application has been made by Christian, Freedman and Lee ³⁾, in the framework of the static model. In this article, we shall propose a novel application of the method to the study of strong interactions.

When we make the simplifying assumption that only few degrees of freedom are excited in a process, the actual choice of which modes should be considered as effective can be guided by different considerations, based both on the kinematical situation and on our ideas about the underlying dynamics. Here, we shall mainly emphasize the kinematical aspects of such a choice.

To be more specific, we shall concentrate our attention on processes where the momenta of the external particles obey some collinearity or coplanarity condition, so that they span a subspace L of the physical four-dimensional space. For convenience, we shall always refer to L as the longitudinal space, and to the orthogonal subspace T as the transverse space.

We shall show that for such processes it is possible to introduce a computational scheme based on an effective Lagrangian, where the fields depend on longitudinal space-time variables only and in which therefore the transverse degrees of freedom are frozen.

An amusing feature will be present when our approximation scheme is applied to spinor or vector fields. In this case the fields, while depending on longitudinal variables only, retain their full four-dimensional spin dependence. The situation is strongly reminiscent of what happens in the dual resonance model, where the whole physical spectrum is built with vector "gluons" having collinear momenta, but four-dimensional spin structure ⁴⁾⁻⁶⁾.

The effective Lagrangian of reduced dimensionality will lead in general to a finite (super-renormalizable) theory. If this is not soluble, one can go further in the scheme of approximation, and introduce some more specific assumptions on the state of the system. One example of this is to attribute [following Ref. 3)] a common state of excitation to all the particles of the same kind, whose form shall be determined variationally. Of course, this further approximation goes beyond the purely kinematical motivations, and must be based on dynamical considerations.

The plan of the article is as follows. In Section 2, we shall discuss the general aspects of the reduction of the number of effective dimensions of the fields. In Sections 3 and 4 we shall then illustrate this reduction in the particularly interesting cases of a one-dimensional or two-dimensional configuration of the external momenta.

In Section 3 we shall consider one-dimensional processes where all the external four-momenta are parallel. In this case the scheme of approximation leads to a set of fields depending on a time variable only, and the model has the complexity of a quantum system consisting of a small number of non-linearly coupled oscillators. To give an explicit example, we shall apply the method to a system of quarks and gluons, and shall see that it produces a reasonable description of the static properties of baryonic states.

In Section 4, we shall study the application of our scheme of approximation to two-dimensional processes, where the external momenta are collinear. We shall proceed by considering a definite four-dimensional Lagrangian, namely a Lagrangian describing the gauge invariant interaction of a massless vector gluon with a fermionic system. We shall reduce this Lagrangian to an effective two-dimensional Lagrangian, and we shall discuss the most important features of the field theory which emerges from the reduction process.

2. DERIVATION OF THE EFFECTIVE LAGRANGIAN

We consider strong interaction processes where the external momenta have non-vanishing components only in a longitudinal space of dimension less than four. If we consider any tree graph of the theory, because of momentum conservation at the vertices, the intermediate states will also have purely longitudinal momenta. This corresponds to the fact that the tree graphs constitute a classical approximation to the transition amplitude, and that, as long as we consider a classical system, transverse degrees of freedom will never be excited starting from an initial longitudinal configuration. Quantum mechanical effects come in with the loop graphs. Through these effects it becomes possible to find intermediate states with non-vanishing transverse momenta.

At this stage it is hard to make a fully theoretical assessment about the average size of transverse momenta. However, in the case of collinear processes (where the longitudinal space is bidimensional), relevant information can be obtained from experiment.

It is an established experimental fact that in high energy collisions there is very low production of particles with large transverse momenta. Since the loop graphs can be thought of as unitarity corrections to the tree graphs, it follows that, when the external momenta have only longitudinal components, the main contribution to the quantum mechanical transition amplitudes will also come from intermediate states with low transverse momenta. We shall assume that this suppression of strong transversal effects is a general feature, not limited to two-dimensional configurations.

The previous considerations open the possibility of giving an approximate treatment of strong interaction processes where, corresponding to external configurations of lower dimensionality, only longitudinal excitations are kept in the theory. We shall therefore approximate the complete four-dimensional Lagrangian with an effective Lagrangian where the fields depend only on longitudinal co-ordinates.

The most convenient procedure is to make use of a variational formulation. We start from a field theoretical Lagrangian with external sources. We extract the relevant amplitudes by means of appropriate functional derivatives of the vacuum to vacuum transition amplitude.

The vacuum state satisfies the equation

$$\oint \int dt \langle \psi(t) | \left[i \frac{\partial}{\partial t} - H - H_{\text{ext}} \right] | \psi(t) \rangle = 0 \quad (2.1)$$

H_{ext} contains the external time dependent sources and has the form

$$H_{\text{ext}} = \int \sum_i \phi_i(x) j_{\text{ext}}^\nu(x) d\vec{x} \quad (2.2)$$

We wish to postpone the discussion of spin effects, so we assume for the moment that the fields $\phi_i(x)$ are scalar fields. The index i labels then possible intrinsic degrees of freedom.

Since the external momenta, by hypothesis, belong to the longitudinal space, the sources in Eq. (2.2) will be functions of longitudinal co-ordinates only

$$j_{\text{ext}}(x) = j_{\text{ext}}(x_L) \quad (2.3)$$

It becomes quite natural then to separate the dependence of the fields $\phi_i(x)$ on the longitudinal and transverse variables. This can be done by introducing a complete set of orthonormal functions of the transverse co-ordinates and expanding

$$\phi(x) = \phi(x_L, x_T) = \sum_{n=0}^{\infty} F_n(x_T) \phi_n(x_L) \quad (2.4)$$

where the index of ϕ_i has been dropped for simplicity.

Until now no approximation has been made. We introduce our main assumption by requiring that all components of the vacuum state have a unique, common dependence on the transverse variables, expressed by one of the functions $F_n(x_T)$, say $F_0(x_T)$. Symbolically, we write

$$\phi(x_L, x_T) \longrightarrow F_0(x_T) \phi_0(x_L) \quad (2.5)$$

The squared Fourier transform of $F_0(x)$: $F_0^2(k_T)$ represents a distribution peaked around $k_T \approx 0$. We notice of course that after approximation (2.5) has been performed, our states do not have definite transverse momenta.

Equation (2.5) can be expressed more precisely by saying that the variation in Eq. (2.1) is taken only within states satisfying

$$\phi_n^{(-)}(x_L) | \psi \rangle = 0 \quad n = 1, 2, 3 \dots \quad (2.6)$$

where $\phi_n^{(-)}$ represents the negative frequency part of the component $\phi_n(x_L)$. It is obvious then that in all matrix elements of the form

$$\langle \psi | : \phi(x_1) \dots \phi(x_n) : | \psi \rangle \quad (2.7)$$

(: : stands for normal ordering), the field $\phi(x_1)$ can be substituted by its zero mode component :

$$\begin{aligned} \langle \psi | : \phi(x_1) \dots \phi(x_n) : | \psi \rangle &= \\ = F_0(x_{1T}) \dots F_0(x_{nT}) \langle \psi | : \phi_0(x_{1L}) \dots \phi_0(x_{nL}) : | \psi \rangle \end{aligned} \quad (2.8)$$

Before proceeding further, let us insist on the point that the reduction of the original four-dimension operator into a longitudinal one is obtained from a "weak" operator identity, valid only when applied on the zero state. Equation (2.5), stated as an operator identity, would change in a drastic way the fundamental operator algebra of the theory. The weak character of Eq. (2.5) allows us to write Eq. (2.8) for the ordered product of the original operators. In terms of Feynman graphs, this would correspond to make our approximation on connected graphs only.

We now apply Eq. (2.8) to the evaluation of the "vacuum" expectation value of the Lagrangian. We illustrate the reduction in the specific case of the $\lambda\phi^4$ theory, where

$$\begin{aligned} \mathcal{L} = & : \int d^3x \{ \partial_\mu \phi \partial_\mu \phi + \mu^2 \phi^2 + \lambda \phi^4 + \\ & + J_{ext} \phi \} : \end{aligned} \quad (2.9)$$

Using our approximation, we get

$$\begin{aligned} \langle \psi | \mathcal{L} | \psi \rangle = & \langle \psi | : \int d^3x_L \{ A \partial_L \phi_0(x_L) \partial_L \phi_0(x_L) + \\ & + (B + A\mu^2) \phi_0^2(x_L) + \lambda C \phi_0^4(x_L) + \\ & D J_{ext}(x_L) \phi_0(x_L) \} : | \psi \rangle \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} A &= \int dx_T F_o^2(x_T) = 1 \quad ; \quad B = \int dx_T [\partial_T F_o(x_T)]^2 \\ C &= \int dx_T F_o^4(x_T) \quad ; \quad D = \int dx_T F_o(x_T) \end{aligned} \quad (2.11)$$

We see that, in so far as the evaluation of $|\psi\rangle$ is concerned, the Lagrangian $\mathcal{L}$ is equivalent to an effective Lagrangian in the longitudinal space :

$$\begin{aligned} \mathcal{L}_{eff} = & : \int dx_L \left\{ \partial_L \phi_o(x_L) \partial_L \phi_o(x_L) + \right. \\ & + \mu_{eff}^2 \phi_o^2(x_L) + g_{eff} \phi_o^4(x_L) + \\ & \left. + J_{ext}(x_L) \phi_o(x_L) \right\} : \end{aligned} \quad (2.12)$$

where the effective mass

$$\mu_{eff}^2 = \mu_o^2 + \int dx_T (\partial_T F_o)^2 \quad (2.13)$$

receives a contribution from the average spread in transverse momentum, the effective coupling constant

$$g_{eff} = \lambda \int F_o^4(x_T) dx_T \quad (2.14)$$

has the correct dimensions $[\text{mass}]^{\text{no. of transverse dimensions}}$, and the external current has to be rescaled according to

$$J_{ext} = j_{ext} \int dx_T F_o(x_T) \quad (2.15)$$

Our approach will be to use the effective Lagrangian to compute all processes in which the external momenta have only longitudinal components. The quantities μ_{eff}^2 , g_{eff} , and $D = J_{ext}/j_{ext}$ will be considered as phenomenological parameters. They are related to the function $F_o(x_T)$ through the integrals of Eq. (2.11). In principle, one could also envisage a variational determination of $F_o(x_T)$.

At this stage, however, it is wise to consider it as an empirical quantity. Indeed, its form depends on the way through which the original four-dimensional theory manages to cut off large transverse momenta. We are at present quite ignorant about that mechanism. It is likely that in the case of renormalizable theories (with dimensionless coupling constants) transverse momenta are not suppressed strongly enough, so that one has to introduce an extra cut-off function. In this case, the function F_0 is a convenient expression for the effect of such a cut-off.

The procedure to derive the effective Lagrangian can be extended to the case of particles with spin. If the original Lagrangian contains a field $\phi_\lambda(x)$ with a Lorentz index, the natural generalization of our ansatz is

$$\phi_\lambda(x) = \phi_\lambda(x_L, x_T) \rightarrow \varphi_\lambda(x_L) F_0(x_T) \quad (2.16)$$

Our reduced fields recall in their spin indices their four-dimensional origin. In the particular case of a four-vector field $A_\mu(x)$, we are led to a reduced field

$$a_\mu(x_L) \quad \mu = 0, 1, 2, 3$$

which, for $x_L \rightarrow (t, z)$ is strongly reminiscent of fundamental "field" appearing in dual models. Specific examples of theories with spinning particles will be considered in the next chapter.

Before closing this section, it is instructive to see how our approximation looks in terms of Feynman diagrams. Let us consider, e.g., a process with n external spinless particles carrying momenta $k_1, k_2, \dots, k_n$. The contribution from any connected graph will be of the form :

$$A(k) = \int F(k, q) \pi d^4 q \quad (2.17)$$

where we denote by $q_1, \dots, q_\ell$ an independent set of internal momenta.

Let us suppose that the external momenta have non-vanishing components only in the longitudinal space, so that $k_T = 0$.

It follows that

$$F = F(k_L, q_L, q_T) \quad (2.18)$$

Because of its invariance under rotations and reflections in the transverse space, F can be a function only of the products $\vec{q}_{Ti} \cdot \vec{q}_{Tj}$. This shows that the point $q_{T,i}=0$ is a stationary point in the Feynman integrand. In many cases, this point is actually a maximum of the integrand because the denominators take their smallest value at that point.

This happens for example for all graphs of a multiperipheral kind ⁴⁾. On the other hand, this is also true if we work in the framework of a Euclidean field theory. When the external momenta belong to a restricted longitudinal space, one can approximate the connected Feynman graphs by taking $\vec{q}_{T,i}$ in the integrand, by writing

$$\int F(k_L, q_L, q_T) d^n \vec{q}_T = \Delta^n F(k_L, q_L, 0) \quad (2.19)$$

where n is the number of dimensions of the transverse space and Δ is a phenomenological parameter corresponding to the spread in transverse momenta. Our prescription leads to replace the exact connected graphs with the effective graphs evaluated in a space of lower multiplicity.

Starting from those effective graphs, one can construct an effective Hamiltonian in terms of fields defined in the longitudinal space. One can compare the results of Feynman graphs with those obtained with the variational method. One can check that, apart from minor details, everything is constant if one remembers that the variational states have a spread in transverse momentum represented by $F_0(k_T)$.

3. ONE-DIMENSIONAL PROCESSES

This section and the following one contain an illustration of our scheme of approximation in the cases where the space of the external momenta is one-dimensional or two-dimensional. One-dimensional processes will be considered in this section and two-dimensional processes in Section 4.

Situations where the four-dimensional external momenta are all parallel, so that they span a one-dimensional space, are not commonly found in laboratory experiments, but there are many interesting experimental quantities that can be related to amplitudes with parallel external momenta. Mass differences, vector or axial-vector coupling constants, scattering lengths are examples of such quantities. In general, one-dimensional processes give information on all physical quantities that are of static nature.

In most of the cases of interest, the external momenta are time-like, so that we can assume that they have only the energy component :

$$k_i^\mu = (E_i, \vec{0}) \quad (3.1)$$

The longitudinal space coincides then with the time axis, and the transverse space is the ordinary space.

When acting on a physical state, the fields are approximated by

$$\varphi(t, \vec{x}) \rightarrow \phi(t) F(\vec{x}) \quad (3.2)$$

and the effective Lagrangian will be of the form

$$L_{eff.} = L_{eff} [\phi(t), \dot{\phi}(t)] \quad (3.3)$$

where the phenomenological constants that appear in L_{eff} are related to integrals of the functions $F(\vec{x})$ [cf., Eqs. (2.12)-(2.15)].

Equivalently, we can say that the creation operators $a^\dagger(\vec{k})$ appearing in the expansion of $\varphi(t, \vec{x})$ are replaced, when acting on physical states, by $\tilde{F}(\vec{k}) A^\dagger$:

$$a^\dagger(\vec{k}) \rightarrow A^\dagger \tilde{F}(\vec{k}) \quad (3.4)$$

so that the effective Hamiltonian takes the form

$$H_{eff} = H_{eff}(A^\dagger, A) \quad (3.5)$$

We see that the approximate field theory, used to describe one-dimensional processes, has the complexity of a quantum system, consisting of a finite number of non-linearly coupled harmonic oscillators.

If some of the original fields had Lorentz indices, these will survive as spin indices of the creation and annihilation operators $A^\dagger$ and A . The invariance of the theory under rotations in space is preserved by the passage to the effective Hamiltonian, and the indices of spin will be saturated so as to give origin to a rotationally invariant H_{eff} .

Notice that the presence of fields with spin in the effective theory allows us to construct states with non-vanishing values of angular momentum, even if the orbital part of J is suppressed by our approximation. In

particular, the excitations of vector and axial vector fields represent the analogue of orbital degrees of freedom of the system.

We exemplify all that by considering a system of quarks and vector and axial vector gluons. According to Eq. (3.4) these will be described by operators :

$$\begin{array}{lll} B_{\alpha,i,I}, & B_{\alpha,i,I}^{\dagger} & \text{the quarks} \\ D_{\alpha,i,I}, & D_{\alpha,i,I}^{\dagger} & \text{the antiquarks} \\ V_{\ell,c}, & V_{\ell,c}^{\dagger} & \text{the vector gluons} \\ A_{\ell,c}, & A_{\ell,c}^{\dagger} & \text{the axial vector gluons,} \end{array}$$

where $\alpha = 1, 2$ and $\ell = 1, 2, 3$ are spin indices, $i = 1, 2, 3$ is an $SU(3)$ index and I, C are colour indices. To have a realistic model, we assume that quarks and gluons transform according to an $SU(3)$ group of colour, with the requirements that the physical states are colour singlets. This leads to the correct symmetry properties of the quark wave function ⁵⁾. The gluons are assumed to be octets of colour. There are motivations for this choice, independently of our work ⁶⁾; in the simple model we consider the choice is favoured because, as we shall see, it leads to a correct $SU(3)$ classification of the baryon states with higher spin.

On grounds of invariance under rotations and reflections, the effective Hamiltonian must be of the form

$$\begin{aligned} H_{\text{eff}} = & m (B^{\dagger} B + D^{\dagger} D) + \mu_V V^{\dagger} V + \mu_A A^{\dagger} A + \\ & + \frac{g}{2} (B^{\dagger} \sigma_e \Lambda_c B A_{ec} + D^{\dagger} \sigma_e \Lambda_c D A_{ec} + h.c.) + \quad (3.6) \\ & + g' (B^{\dagger} \sigma_e \Lambda_c D^{\dagger} V_{ec} + h.c.) + \text{interaction terms of higher order} \\ & \text{or trilinear in the gluonic operators} \quad , \end{aligned}$$

where by Λ_c we denote the λ matrices of colour and m, μ_V and μ_A are the effective masses of the different particles. In this example, we will assume that the only interaction terms are those explicitly written in Eq. (3.6), which we would obtain starting from a four-dimensional Lagrangian with trilinear coupling, among quarks and gluons ^{*)}.

*) We see in Eq. (3.6) the reason why we included axial vector gluons in the model. With vector gluons only, an interaction term generating a quark-quark transition must be at least quadrilinear in the reduced fields.

Let us discuss the main features of the baryonic states that result from the Hamiltonian of Eq. (3.6). As long as we do not consider the interaction terms, these are given by :

$$\begin{aligned} & B^+ B^+ B^+ |0\rangle, \text{ (ground state baryons)} \\ & B^+ B^+ B^+ V^+ |0\rangle, \\ & B^+ B^+ B^+ A^+ |0\rangle, \\ & B^+ B^+ B^+ V^+ V^+ |0\rangle, \text{ etc.} \end{aligned}$$

As we stressed before, the presence of the gluons, which retain their spin index in the one-dimensional theory, allows to incorporate states of higher spin in the spectrum. For example, adding one vector gluon to the three quarks of the ground state, we reproduce all the baryonic states obtained by coupling $L=1$ with $S=\frac{1}{2}, \frac{3}{2}$. The requirement that the over-all wave function be a colour singlet implies that the quark part of the colour wave function is an octet, so that we obtain all the states belonging to the representation 70, $L=1$ of $SU(6) \otimes O(3)$. This nice correlation between the gluonic excitations and the symmetry properties of the quark wave function is preserved in the states of higher spin.

Turning our attention to the ground state baryons, we see that these consist of the states of the fundamental octet and decuplet, for which we find the results of static $SU(6)$ as long as the interaction is not considered (together with the instructions of using it for static processes only. In our scheme, we would not apply the one-dimensional approximation, for instance, to the computation of a P wave decay, which reflects a two-dimensional configuration of the external momenta.)

In presence of the interaction, the results of static $SU(6)$ are modified. Acting on the ground state wave function, H_{eff} is equivalent to the simpler

$$\hat{H}_{\text{eff}} = 3m + \mu_A A^+ A + \frac{g}{2} \sum_{ec} (\sigma_e \Lambda_c A_{ec} + \text{h.c.}) \quad (3.7)$$

where $\sigma_e \Lambda_c$ is a matrix that acts on the quark part of the wave function.

With this Hamiltonian we find, for instance, mass splittings

$$\Delta M_{\text{octet}} = -\frac{16 g^2}{\mu_A} + O(g^4) \quad (3.8a)$$

and

$$\Delta M_{\text{decuplet}} = -\frac{8 g^2}{\mu_A} + O(g^4) \quad (3.8b)$$

and the following modifications of the g_A/g_V value for the nucleons

$$\frac{g_A}{g_V} = \frac{5}{3} \left[1 - \frac{16}{3} \left(\frac{g}{\mu_A} \right)^2 + O(g^4) \right] \quad (3.9)$$

(screening effect due to the axial vector gluons). The octet decuplet axial transition coupling constant is also modified according to :

$$g^* = \frac{4}{3} \left[1 - \frac{16}{3} \left(\frac{g}{\mu_A} \right)^2 + O(g^4) \right] \quad (3.10)$$

The ratio of magnetic moments $\mu_P/\mu_N = -\frac{3}{2}$ is not changed up to order g^4 .

It is nice to see that our scheme of approximation generates corrections that go in the right direction, even if we have predictions for a too limited number of quantities to make a real test of the model.

This scheme could be elaborated further and applied to the mesons, but we presented it for illustration purposes and shall not insist on it. Let us only stress once more that, in our method of approximation, the excitations of vector and axial vector fields can play a role analogous to the orbital excitations which are suppressed due to the limitation of the momenta within the wave function. This idea of a possible substitution of vector and axial vector excitations for orbital angular momentum might find applications beyond our model.

Let us finally remark that the values reproduced in Eqs. (3.8)-(3.10) do not represent more than perturbation theory to second order. But they can be obtained by a method which allows much easier computations to higher orders and which makes very manifest the dressing of the bare wave function.

We wish to illustrate this method here. We shall use the notation :

$$A = \sum_{ec} \frac{\sigma_e}{2} \Lambda_c A_{ec} \quad (3.11)$$

$$\hat{H}_{eff} = 3m + \mu_A A^+ A + g(A + A^*) \quad (3.12)$$

can be solved by introducing an operator U such that $H' = U^\dagger \hat{H}_{eff} U$ commutes with $A^+ A$. H' conserves then the number of axial vector gluons and is therefore a direct product of operators acting on finite-dimensional spaces, that can be diagonalized separately.

U can be found by a perturbation expansion in g . Up to fourth order terms we have ^{*)}

$$U = U_1 U_3 U_4 \quad (3.13)$$

with

$$U_1 = \exp \frac{g}{\mu_A} (A - A^*) \quad (3.14a)$$

$$U_3 = \exp -\frac{2}{3} \left(\frac{g}{\mu_A} \right)^3 \left[[A, A^+], (A + A^*) \right] \quad (3.14b)$$

$$U_4 = \exp -\frac{1}{8} \left(\frac{g}{\mu_A} \right)^4 \left\{ [[[A A^*] A] A] - h.c. \right\} \quad (3.14c)$$

and

$$H' = 3m + \mu_A A^+ A - \frac{g^2}{\mu_A} [A A^*] - \frac{g^4}{12 \mu_A^2} \left\{ [[[A A^*] A] A^*] + h.c. \right\} \quad (3.15)$$

*) The expansion of Eq. (3.13) must not be confused with the standard perturbative expansion of the operator that diagonalizes H . If the matrices M_r in the definition of $A = \sum_r M_r A_r$ commute, for instance, the whole operator U is given by the first factor U_1 in Eq. (3.13).

For any operator we have, of course,

$$\langle \psi | O | \psi \rangle = \langle 3q | O' | 3q \rangle \quad (3.16)$$

with

$$O' = U^\dagger O U$$

where $|\psi\rangle$ and $|3q\rangle$ represent the true and the bare wave functions of the ground state baryons. Equations (3.16) and

$$M = \frac{\langle 3q | H' | 3q \rangle}{\langle 3q | 3q \rangle}$$

can be used to compute the quantities of Eqs. (3.8)-(3.10).

4. TWO-DIMENSIONAL PROCESSES

In this section, we consider processes where the longitudinal space is two-dimensional. The external momenta will, in general, have an energy component, and space parts aligned along a common direction, so that the process is collinear.

There are many measurable quantities that can be expressed in terms of the amplitudes for collinear processes. Some examples are the forward cross-sections, the structure functions for inelastic lepton-hadron scattering, and also quantities like the rapidity distribution of particles produced in a high energy collision, if we are willing to neglect the small transverse momentum of the secondaries.

According to the general scheme introduced in Section 2, collinear processes will be described in terms of fields depending on a time variable t and a space variable x , and the evolution of these fields will be governed by an effective two-dimensional Lagrangian. The resulting theory is still much more complex than the one-dimensional theory, which we illustrated in Section 3, but has a much simpler structure than the original four-dimensional one. In particular, the theory will be in general super-renormalizable, and it will be possible to perform numerical computations based on further applications of variational principles.

We shall illustrate here the method of reduction of the effective number of degrees of freedom by applying it to a theory with a Lagrangian density

$$\mathcal{L} = -\bar{\psi}[(i\partial_\mu - gV_\mu)\gamma_\mu - m]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4.1)$$

where a spinor field $\psi(x)$ is coupled in a gauge invariant way to a massless vector field $V_\mu(x)$ ($F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$).

We consider a Lagrangian density of this form because we want to discuss some features of the method, connected with the gauge invariance of the theory (by which some of the components of the fields are redundant and can be eliminated), and also because we think it may exemplify a possible strong interaction among quarks and vector gluons^{*)}.

It is well known that the longitudinal components of the vector field can be eliminated from the theory, and that only their transverse part represents physical excitations. Since the variational ansatz of Section 2 is as assumption on the type of excitations of the fundamental fields present in the physical states, we argue that the replacement of Eq. (2.5) :

$$\varphi(x_\mu) \rightarrow F(x_T) \Phi(x_L) \quad (4.2)$$

can be made only for the independent components.

We shall therefore use the four-dimensional equations of motion that follow from the Lagrangian density $\mathcal{L}$ to express some of the components of the fields in terms of a set of independent ones, which will be then approximated according to Eqs. (2.5), (2.6) and (2.8).

It is very convenient to eliminate the dependent variables in the light cone frame, where the evolution variable is $x^+ = t+x/\sqrt{2}$ ^{**)} . A light cone formulation of the equations of motion obtained with the Lagrangian density of Eq. (4.1) has been given by Kogut and Soper⁷⁾, and we reproduce in the following equations some of their results.

*) Of course, even if $\mathcal{L}$ has the same form as an electromagnetic Lagrangian density, we do not conceive an application of our method to electromagnetic processes. In realistic computations, the fields should have $SU(3)$ and colour indices.

**) As usually, we will employ the notation x^- for $t-x/\sqrt{2}$, x_T or x_i , for the remaining space variables.

Independent dynamical variables are the components of the fields $\psi_+ = \frac{1}{2} \gamma^- \gamma^+ \psi$ and V_T . The "+" component of V^μ is eliminated by choosing a gauge where $V^+ = 0$. The dependent dynamical variables are then given by

$$\psi_- = \frac{1}{2} \gamma^+ \gamma^- \psi = \psi_{-,0} + \psi_{-,I} \quad (4.3)$$

and

$$V_- = V_0^- + V_I^- \quad (4.4)$$

with

$$\psi_{-,0}(x^-, x_i) = -\frac{i}{4} \int dy^- \epsilon(x^- - y^-) \{ i \partial_j \gamma^j + m \} \gamma^+ \psi_+(y^-, x_i) \quad (4.5a)$$

$$\psi_{-,I}(x^-, x_i) = \frac{ig}{4} \int dy^- \epsilon(x^- - y^-) \gamma^j V_j(y^-, x_i) \gamma^+ \psi_+(y^-, x_i) \quad (4.5b)$$

$$V_0^-(x^-, x_i) = -\frac{1}{2} \int dy^- |x^- - y^-| \partial_- \partial_j V^j(y^-, x_i) \quad (4.6a)$$

and

$$V_I^-(x^-, x_i) = -\frac{g}{\sqrt{2}} \int dy^- |x^- - y^-| \psi_+^+ \psi_+(y^-, x_i) \quad (4.6b)$$

In these equations a purely kinematical part of the dependent variables has been separated from the terms indexed by the interaction.

The evolution in the light cone frame is generated by the "-" component of the total momentum of the system, which we denote by H^- . The "Hamiltonian" H^- is given by

$$\begin{aligned} H^- = : \int dx_i dx^- \{ & \sqrt{2} \psi_0^+ i \frac{\partial}{\partial x^-} \psi_0 + \frac{1}{2} F_{12} F'^2 + \frac{1}{2} V_0^- \partial_- \partial_j V^j_+ \\ & + \frac{g}{2} V^i_+ (\psi_+^+ \gamma^0 \gamma^i \psi_{-,0} + h.c) + \sqrt{2} \psi_{-,I}^+ i \frac{\partial}{\partial x^-} \psi_{-,I} + \\ & \frac{g}{\sqrt{2}} V^- \psi_+^+ \psi_+ \} \end{aligned} \quad (4.7)$$

Inserting into Eq. (4.7) the equations (4.3)-(4.6), we can express H^- as a function of the independent fields alone. We can now make the replacements

$$\Psi_+(x^-, x_T) \rightarrow F(x_T) \psi_+(x^-) \quad (4.8)$$

$$V^i(x^-, x_T) \rightarrow f(x_T) v^i(x^-) \quad (4.9)$$

We obtain an effective Hamiltonian

$$\begin{aligned} H_{eff}^- = & -\frac{i\sqrt{2}}{g} \int dx^- dy^- \varepsilon(x^- - y^-) \left\{ \bar{m}^2 \psi_+^\dagger(x^-) \psi_+(y^-) + \right. \\ & + \bar{g}_1 \psi_+^\dagger(x^-) [v^i(x^-) - v^i(y^-)] \gamma^i \psi_+(y^-) + \\ & + \bar{g}_2^2 \psi_+^\dagger(x^-) v^i(x^-) \gamma_i v^j(y^-) \gamma_j \psi_+(y^-) \left. \right\} \\ & + \frac{\mu^2}{2} \int dz v^i(z) v_i(z) - \\ & - \frac{\bar{g}_3^2}{2} \int dx^- dy^- |x^- - y^-| \psi_+^\dagger \psi_+(x^-) \psi_+^\dagger \psi_+(y^-) \end{aligned} \quad (4.10)$$

with

$$\begin{aligned} \bar{m}^2 &= m^2 + \int dx_T (\nabla_T F)^2 \\ \bar{g}_1 &= g \int dx_T f F^2 \\ \bar{g}_2^2 &= g^2 \int dx_T f^2 F^2 \\ \bar{g}_3^2 &= g^2 \int dx_T F^4 \end{aligned} \quad \bar{\mu}^2 = \int dx_T (\nabla_T f)^2 \quad (4.11)$$

Notice that this Hamiltonian can be derived from a two-dimensional effective Lagrangian

$$\begin{aligned}
 L_{eff} = \int dx_- \left\{ - \bar{\psi} \left[\gamma_0 \partial_- - \bar{g}_3 \gamma_3 \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \\
 + \int dx_- \left\{ (\partial_- v^i \partial^+ v_i - \mu^i v_i v^i + g_1 \bar{\psi} \gamma^i v_i \psi) \right. \\
 \left. - \frac{i\sqrt{2}}{16} (g_2^2 - g_1^2) \int dx_- dy_- \left[\bar{\psi}_+^+(x_-) v_i(x_-) \gamma^i v_j(y_-) \right. \right. \\
 \left. \left. \cdot \gamma^j \psi_+(y_-) \right] \right\}
 \end{aligned}
 \tag{4.12}$$

where a longitudinal field U^L has been introduced and $F_{LL} = \partial_L U_L = \partial_L U_L$. This achieves the reduction of the original four-dimensional Lagrangian to an effective Lagrangian in two space-time dimensions. The first term in the right-hand side of Eq. (4.12) is identical to the Lagrangian for two-dimensional quantum electrodynamics (apart from the presence of a four-dimensional spin index in the fields ψ). The longitudinal vector field is introduced in that Lagrangian to have a manifestly covariant expression, but does not represent an independent dynamical variable and can be eliminated from the equations of motion.

The two-dimensional effective field theories that we propose for the description of collinear processes are still too complex to allow for an elementary solution. But they are much more manageable than the original four-dimensional theory. In particular, they will be in most cases finite theories, where it is possible to perform numerical computations based on some definite approximations.

A very simple type of approximation that can be used is to assume that also the longitudinal part of the excitations has a common wave function, so that in the expression of the physical states we shall replace the excitation operators $a^\dagger(k^+)$ with $F(k^+) a^\dagger^*$). Then, by a variational principle, we can determine both the function $F(k^+)$ and the approximate expression of the state in the Hilbert space spanned by the $a_i^\dagger$ operators.

*) We still use the light cone frame, so that the fields $\phi(x^-)$ are expanded into creation and annihilation operators of definite "+" momentum.

Since the assumption

$$a^+(k^+) \rightarrow F(k^+) a^+ \quad (4.13)$$

is not compatible with the conservation of the "+" component of momentum, we shall require that the states have only a definite expectation value of P^+ momentum, and formulate the variational principle in the form

$$\delta \langle \psi | H^- - P^- + \gamma (H^+ - P^+) | \psi \rangle = 0 \quad (4.14)$$

subject to

$$\langle \psi | \psi \rangle = 1 \quad (4.15)$$

and

$$\langle \psi | H^+ | \psi \rangle = P^+ \quad (4.16)$$

We have introduced in these equations the total "+" momentum operator H^+ , which, contrary to H^- , is a kinematical operator and does not depend on the interaction.

γ is a Lagrange multiplier which must be determined in such a way that Eq. (4.16) is satisfied. To give an example of a trivial application of Eqs. (4.14)-(4.16), we notice that if H^- does not contain an interaction term, they predict that the states consist of definite numbers N of excitations, each carrying a fraction $1/N$ of the total P^+ momentum, so that $|F(k^+)|^2$ becomes a δ function centred on P^+/N .

Equations (4.14)-(4.16) could be used also together with more elaborate variational assumptions on the structure of the wave function, other than the rather simplistic ansatz of a common state of excitation.

Whatever assumption is made, it must be based on dynamical considerations and goes beyond the mainly kinematical motivation we followed in this article. We shall not insist on the discussion of these possible further methods of approximation. We mentioned them to point out that, once the reduction to an effective two-dimensional theory has been made, the fundamental feature that this is a finite theory and the simplifications introduced by the reduction in number of degrees of freedom open the possibility of performing definite computations to be compared with experimental results.

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