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SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS
OF THE FIRST AND SECOND KINDS

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APPLICATION OF SPLINE FUNCTIONS TO
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ABSTRACT

Continuous approximations to the solution of systems of Volterra integral equations of the first and second kinds are sought by methods using spline functions of degree m , deficiency $(k-1)$ and a fixed quadrature rule of degree $\rho \geq m-1$. The resulting method is called an (m,k) -method. The stability behaviour of the $(m,1)$ - and the (m,m) -method is studied for arbitrarily finite m . Also studied is the stability of the $(3,2)$ -method for second-kind systems. Convergence results and asymptotic formulae for the discretization error are obtained.

INTRODUCTION

In this paper an attempt is made to develop further the methods and analysis of our works [1] and [2] where procedures using quadratic and cubic spline functions, respectively, were discussed. The main features of this further development are the following.

Firstly, the methods analysed use spline functions of arbitrarily finite degree m . Secondly, the analysis is carried throughout the paper for systems of equations rather than for the one Volterra equation case considered in the above-cited works. Thirdly, a discussion of the stability behaviour of the methods presented is given. Finally, whereas in [1] and [2] only equations of the second kind were considered, in the present paper, systems of the first as well as the second kind are treated.

For ease of presentation, the paper is divided into two parts. The first part is devoted to systems of the second kind and the second part to those of the first kind. In Sections 2, 3, and 4 of both parts, we describe the method, analyse the stability behaviour, and give convergence results, respectively. Although the procedures presented for both systems use a fixed quadrature formula and give rise to a spline function of degree m , deficiency- $(k-1)$, they possess rather contrasting stability behaviour. It is shown, for instance, that methods using spline functions with full continuity (deficiency-0) are divergent for all $m > 2$ for second-kind systems and for all $m > 1$ for systems of the first kind. More interestingly however, we find that whereas methods using spline functions with deficiency- $(m-1)$ are numerically stable for all positive m for second-kind systems, they exhibit differing stability behaviour when applied to first-kind systems depending on whether m is even or odd. This latter fact, however, is proved for $m \leq 6$ only. It is further shown that for these methods the order of convergence is, in general, $m+1$ and m for second- and first-kind equations, respectively. In particular, if exact integration is used then the order of convergence is $m+1$. Moreover, for second-kind systems, the $(2,1)$ -method and the (m,m) -method for $m = 1, 2$ are shown to be A-stable; and the stability of the $(3,2)$ -method is also proved.

It should be noted that the application of spline functions to Volterra integral equations has also been considered by Hung [3], Brunner [4], and more recently by Netravali [5]. A common feature of all three works is the use of exact integration of products of polynomials and the kernel of the considered integral equation. Furthermore, the problem of stability is not treated in [4] and [5]. However, Hung gives an A-stable cubic spline deficiency-1 method, for second-kind equations, which uses the differentiated form of the equations and

further shows that the use of cubic splines with full continuity gives rise to divergent methods. He also considers the solution of first-kind equations by linear spline functions. His results for first-kind equations are special cases of our results in Part II of this paper.

PART I

SYSTEMS OF THE SECOND KIND

1. PRELIMINARIES

In this part, we consider a system of Volterra integral equations of the second kind given by

$$\underline{\phi}(x) = \underline{f}(x) + \int_0^x K(x,y) \underline{\phi}(y) dy, \quad 0 \leq x \leq X, \quad (1.1)$$

where $\underline{\phi}(x)$ and $\underline{f}(x)$ are column vectors of order n and where $K(x,y)$ is a square matrix of order n . Conditions under which (1.1) has a unique, bounded solution in $[0, X]$ are exactly the same as in the theory of one Volterra equation (see, for example, [6]). For our purposes, however, we shall require the kernel $K(x,y)$ and the given function $\underline{f}(x)$ to satisfy more than is sufficient to guarantee the existence of a unique continuous solution. Indeed, we assume

- a) $\underline{f}(x)$ is continuous in $[0, X]$;
- b) $\partial K(x,y)/\partial x$ is continuous in the region $0 \leq y \leq x \leq X$;
- c) $\underline{\phi}^{(i)}(0)$ exist for $i = 0(1)m-k$; k being as in the next section.

For the approximation of the integral in (1.1), our methods use a fixed rule of approximate integration:

$$\left\{ \begin{array}{l} R(f) = \sum_{i=1}^M w_i f(t_i), \quad 0 \leq t_1 < t_2 < \dots < t_M \leq 1; \\ \sum_{i=1}^M w_i = 1, \quad w_i > 0 \quad \text{for all } i; \end{array} \right. \quad (1.2)$$

and we assume that it is of degree

$$\rho \geq m-1, \quad (1.3)$$

where m is a fixed positive integer to be specified later. We denote by β the quantity

$$\beta = \sum_{i=1}^M w_i t_i^m . \quad (1.4)$$

Given the mesh

$$0 < u_1 < u_2 < \dots < u_n$$

we shall denote by $R_n(f)$ the composite rule arising out of (1.2), and we write

$$E_n = R_n(f) - \int_0^{u_n} f(x) dx . \quad (1.5)$$

Assuming the integrand to be sufficiently differentiable, we can find an integer $\rho^* \geq \rho$ such that

$$E_n = O(h^{\rho^*}) ; \quad (1.6)$$

$$h = \max_{1 \leq i \leq n} (u_i - u_{i-1}) ,$$

and we note that

$$E_{n+1} - E_n = O(h^{\rho^*+1}) . \quad (1.7)$$

Also observe that if exact integration is used, then $E_n = 0$ for any n .

2. DESCRIPTION OF THE METHOD

Given two positive integers m and $k \leq m$, we construct a function $\underline{S}(x)$ in $C^{m-k}[0, X]$ in the following way. Construct a mesh

$$\Delta_N : 0 = x_0 < x_1 < \dots < x_{Nk} = X \quad (2.1)$$

and let

$$h_i = x_{i+1} - x_i , \quad i = 0(1)Nk-1 , \quad (2.2)$$

$$h = \max_i h_i .$$

For $x \in [x_{kr}, x_{k(r+1)}]$, $r = 0(1)N-1$, define

$$\underline{S}(x) = \sum_{i=0}^{m-k} \frac{(x - x_{kr})^i}{i!} \underline{S}^{(i)}(x_{kr}) + \sum_{i=1}^k (x - x_{kr})^{m-k+i} \underline{a}_i^{(r)}, \quad (2.3)$$

where $\underline{a}_i^{(r)}$ is determined so that

$$\underline{S}_{kr+p} = \underline{f}_{kr+p} + R_{kr+p} \left[K(x_{kr+p}, y) \underline{S}(y) \right], \quad p = 1(1)k, \quad (2.4)$$

$$\underline{S}^{(i)}(0) = \underline{\phi}^{(i)}(0), \quad i = 0(1)m-k;$$

and where $\underline{S}_t = \underline{S}(x_t)$; $\underline{f}_t = \underline{f}(x_t)$, and

$$\underline{a}_i^{(r)} = \left(a_{1i}^{(r)} a_{2i}^{(r)} \dots a_{ni}^{(r)} \right)^T. \quad (2.5)$$

In order to show that the above construction is well defined, we need to introduce some further notation. For each r , we define the vector

$$\underline{a} \equiv \underline{a}^{(r)} = \left(\underline{a}_1^{(r)} \underline{a}_2^{(r)} \dots \underline{a}_k^{(r)} \right)^T, \quad (2.6)$$

and for $p = 1(1)k$, we define

$$H_p \equiv H_p^{(r)} = x_{kr+p} - x_{kr}, \quad (2.7)$$

$$F_{p,i} = \sum_{j=0}^{p-1} h_{kr+j} R \left[K(x_{kr+p}, x_{kr+j} + h_{kr+j} y) \right. \\ \left. \left(H_p^{-1} (H_j + h_{kr+j} y) \right)^{m-k+i} \right]. \quad (2.8)$$

We denote by $\tilde{H}$ the block matrix of order k whose $(p,q)^{th}$ element is

$$\tilde{H}_{p,q} = H_p^{q-1} \left(1 - F_{p,q} \right) I, \quad p, q = 1(1)k, \quad (2.9)$$

I being the identity matrix of order n .

Theorem 1. Let $K(x,y)$ be uniformly bounded in $0 \leq y \leq x \leq X$. Then, for h sufficiently small, the function $\underline{S}(x)$ exists and is uniquely defined by (2.3) and (2.4) as a spline function of degree m , deficiency- $(k-1)$, i.e. $\underline{S}(x) \in C^{m-k}$.

Proof. The proof is by induction. Thus it is sufficient to observe that the coefficient matrix of $\underline{a}$ resulting from (2.3) and relations (2.4) is $\tilde{H}$ which, in view of (2.9), has an inverse for all $h < h_0$.

Q.E.D.

Henceforward, we shall refer to the method (2.3)-(2.4) as an (m,k) -method.

We now give some more notation which we need in subsequent sections of both parts of the paper.

For each integer p and for each pair of integers (i,j) we denote by $\omega_{ij}^{(p)}$ the diagonal matrix of order n :

$$\omega_{ij}^{(p)} = \begin{pmatrix} j \\ i \end{pmatrix} p^{j-i} I. \quad (2.10)$$

Denote by $\Omega^{(p)}$ the block upper triangular matrix of order $(m-k+1)$

$$\Omega_{ij}^{(p)} = \begin{cases} \omega_{ij}^{(p)} & i \leq j ; \quad i, j = 0(1)m-k ; \\ 0 & i > j ; \end{cases} \quad (2.11)$$

and by $\Gamma^{(p)}$ the block matrix with $(i,j)^{th}$ element

$$\Gamma_{ij}^{(p)} = \omega_{i, m-k+j}^{(p)}, \quad i = 0(1)m-k ; \quad j = 1(1)k. \quad (2.12)$$

We also define the block matrices

$$\begin{aligned} \Omega &= \left(\Omega^{(1)} \quad \Omega^{(2)} \quad \dots \quad \Omega^{(k)} \right)^T, \\ \Gamma &= \left(\Gamma^{(1)} \quad \Gamma^{(2)} \quad \dots \quad \Gamma^{(k)} \right)^T. \end{aligned} \quad (2.13)$$

Further, let $\underline{\zeta}_r^{(p)}$ denote the column vector whose i^{th} element is

$$\frac{h^i}{i!} \underline{S}^{(i)}(x_{kr+p}), \quad i = 0(1)m-k,$$

and let

$$\underline{\zeta}_{r+1} = \left(\underline{\zeta}_r^{(1)} \quad \underline{\zeta}_r^{(2)} \quad \dots \quad \underline{\zeta}_r^{(k)} \right)^T. \quad (2.14)$$

If we now assume that the mesh (2.1) is uniform, i.e.

$$x_i = ih, \quad i = 0(1)Nk, \quad (2.15)$$

and if we denote by $\tilde{\underline{a}}^{(r)}$ the vector

$$\tilde{\underline{a}}^{(r)} = h^{m-k} \left(h \underline{a}_1^{(r)} \quad h^2 \underline{a}_2^{(r)} \quad \dots \quad h^k \underline{a}_k^{(r)} \right)^T, \quad (2.16)$$

we easily find on differentiating (2.3) that

$$\underline{\zeta}_{r+1} = \Omega \underline{\zeta}_r^{(0)} + \Gamma \tilde{\underline{a}}^{(r)}. \quad (2.17)$$

Furthermore, if we denote by Y the block matrix whose $(p,q)^{th}$ element is

$$Y_{pq} = \begin{cases} 0, & q = 0; \quad p = 1(1)k, \\ \omega_{0,q}^{(p)}, & q = 1(1)m-k \end{cases} \quad (2.18)$$

and by Z the block matrix of order k with

$$Z_{pq} = \omega_{0,m-k+q}^{(p)}, \quad p, q = 1(1)k, \quad (2.19)$$

then (2.3) gives

$$\left[\underline{s}(x_{kr+1}) \quad \dots \quad \underline{s}(x_{kr+k}) \right]^T = \left[\underline{s}(x_{kr}) \quad \dots \quad \underline{s}(x_{kr}) \right]^T + Y \underline{\zeta}_r^{(0)} + Z \tilde{\underline{a}}^{(r)}. \quad (2.20)$$

Finally, for each pair of integers (i,j) , we define the matrix u_{ij} of order n by

$$u_{ij} = \frac{i^{j+1}}{j+1} I. \quad (2.21)$$

Let U be the block matrix with $(i,j)^{th}$ element

$$U_{ij} = u_{ij}, \quad i = 1(1)k; \quad j = 0(1)m-k; \quad (2.22)$$

and let V be the block matrix of order k defined by

$$\begin{cases} V_{ij} = u_{i,m-k+j}; & j = 1(1)k-1; \\ V_{ik} = \beta_i I; & i = 1(1)k; \\ \beta_i = \frac{i^{m+1}}{m+1} + i \left(\beta - \frac{1}{m+1} \right); \end{cases} \quad (2.23)$$

β being given by (1.4).

3. AN INVESTIGATION OF NUMERICAL STABILITY

In this section we study the application of (2.4), for particular values of the integer k , to the system

$$\begin{cases} \phi(x) = \underline{1} + A \int_0^x \phi(y) dy, \\ \operatorname{Re}(\lambda_i) < 0, \quad i = 1(1)n, \end{cases} \quad (3.1)$$

where A is an $n \times n$ matrix with eigenvalues λ_i .

Definition 1. An (m,k) -method defined by (2.3) and (2.4) is said to be numerically stable if all solutions $\{\underline{s}(x_{kr})\}$ tend to zero as $h \rightarrow 0$, $r \rightarrow \infty$ (x_{kr} fixed) when the method is applied to any system of the form (3.1).

Two special cases will now be discussed. Firstly, the case $k = 1$ corresponding to the use of spline functions with full continuity. Secondly, the case $k = m$ giving rise to a spline function in the continuity class C only. The contrasting stability behaviour of these two cases is the content of the following two theorems.

Theorem 2. The $(m,1)$ -method is divergent for all values of the degree $m > 2$.

Theorem 3. The (m,m) -method is numerically stable for all values of the degree m .

Before giving the proofs, let us observe that there is no loss of generality in confining our attention to the case where the mesh (2.1) has a constant mesh

interval described by (2.15). In this case, the application of (2.4) to (3.1) yields

$$(Z - h\tilde{A}V)\tilde{a}^{(r)} = (h\tilde{A}U - Y)\zeta_r^{(0)}, \quad (3.2)$$

where $\tilde{a}$, Y , Z , U , and V are, respectively, given by (2.16), (2.18), (2.19), (2.22) and (2.23); and where $\tilde{A}$ is the block diagonal matrix of order k

$$\tilde{A} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}.$$

Now, (2.17) and (3.2) immediately give

$$\zeta_{r+1} = G\zeta_r^{(0)}, \quad (3.3)$$

where

$$G = G(h) = \Omega + \Gamma(Z - h\tilde{A}V)^{-1} (h\tilde{A}U - Y). \quad (3.4)$$

Proof of theorem 2. It is sufficient to show that the matrix

$$G(0) = \Omega - \Gamma Z^{-1} Y \quad (3.5)$$

has an eigenvalue μ^* such that

$$|\mu^*| > 1.$$

To this end, let us note that when $k = 1$, (2.19) and (2.10) show that

$$Z = I;$$

(2.12), (2.18) and (2.10) give

$$(\Gamma Y)_{ii} = \begin{cases} 0, & i = 0, \\ \omega_{i,m}^{(1)}, & i = 1(1)m-1, \end{cases} \quad (3.6)$$

and hence

$$\text{tr}(\Gamma Y) = n(2^m - 2) .$$

However, (2.11) and (2.10) give

$$\text{tr}(\Omega) = nm .$$

We thus have

$$\text{tr}(G(0)) = n(m - 2^m + 2) \geq n(3 - m(m-1)) , \quad \text{for all } m > 2 .$$

Furthermore, in view of (3.6), it is readily seen that $\mu^* = 1$ is an eigenvalue of $G(0)$ of multiplicity n , for all m .

Q.E.D.

Proof of theorem 3. When $k = m$, (2.18) gives

$$Y = 0 ;$$

(2.12) and (2.19) give

$$\Gamma = Z ;$$

thus (3.4) gives

$$G = \Omega + h\tilde{A}U + O(h^2) . \quad (3.7)$$

In particular, (2.11), (2.21), (2.22), and (3.3) give

$$\zeta_{r+1}^{(m)} = (I + mhA) \zeta_r^{(0)} + O(h^2) .$$

Q.E.D.

We now note that the manipulation leading to (3.7) shows that for $m = 1$

$$\underline{S}_{r+1} = [I + (1-\beta)hA] (I - \beta hA)^{-1} \underline{S}_r , \quad (3.8)$$

and for $m = 2$

$$\underline{S}_{2r+2} = \left[I - hA + \frac{1}{2} (1-\beta)h^2A^2 \right]^{-1} \left[I + hA + \frac{1}{2} (1-\beta)h^2A^2 \right] \underline{S}_{2r} . \quad (3.9)$$

We have thus shown that for $m = 1, 2$ the (m, m) -method is A-stable in the sense of the following definition.

Definition 2. An (m, k) -method defined by (2.3) and (2.4) is said to be A-stable if all solutions $\{\underline{S}(x_{kr})\}$ tend to zero as $r \rightarrow \infty$ (x_{kr} fixed) when the method is applied with fixed positive h to any system of the form (3.1).

Whether the (m, m) -method is A-stable for all m is an open question.

We now study two further cases:

The quadratic case. We now consider the $(2, 1)$ -method -- a case not dealt with by the above theorem. For one Volterra equation, the use of quadratic splines is discussed in reference [1].

When $m = 2$, $k = 1$, relations (2.10)-(2.13), (2.18)-(2.23) yield

$$\begin{aligned} \Omega &= \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} ; \quad \Gamma = \begin{pmatrix} I \\ 2I \end{pmatrix} ; \\ Y &= (0 \quad I) ; \quad Z = I ; \\ U &= (I \quad \frac{1}{2}I) ; \quad V = \beta I ; \end{aligned}$$

and we therefore have in view of (3.3), (3.4) and (2.14)

$$\underline{S}_{r+1} = (I - \beta hA)^{-1} \left[I + (1-\beta)hA \quad \left(\frac{1}{2} - \beta \right) hA \right] \underline{\zeta}_r^{(0)} . \quad (3.10)$$

Theorem 4. The $(2, 1)$ -method is numerically stable for any choice of the quadrature rule (1.2) with degree $\rho > 0$. In particular, the method is A-stable if the rule (1.2) is the trapezoidal rule.

Proof. For the trapezoidal rule $\beta = \frac{1}{2}$.

Q.E.D.

In the remaining part of this section, we consider

The cubic case. Theorem 2 tells us, on the one hand, that the $(3, 1)$ -method is divergent, while on the other hand theorem 3 says that the $(3, 3)$ -method is stable. We now propose to investigate the stability of the intermediate case: the $(3, 2)$ -method. Indeed, this method is recommended in conjunction with Simpson's rule in reference [2].

For $m = 3$, $k = 2$, we find

$$\begin{aligned} Z - h\tilde{A}V &= \begin{pmatrix} I - \frac{1}{3}hA & I - \beta hA \\ 4I - \frac{8}{3}hA & 8I - \beta_2 hA \end{pmatrix}; \\ h\tilde{A}U - Y &= \begin{pmatrix} hA & \frac{1}{2}hA - I \\ 2hA & 2hA - I \end{pmatrix}; \\ \Omega^{(2)} &= \begin{pmatrix} I & 2I \\ 0 & I \end{pmatrix}; \quad \Gamma^{(2)} = 4 \begin{pmatrix} I & 2I \\ I & 3I \end{pmatrix}; \\ \beta_2 &= \frac{7}{2} + 2\beta. \end{aligned}$$

If we now look at the first element of the second block in (3.3), we find by direct computation and for sufficiently small h

$$\underline{S}_{2r+2} = \begin{bmatrix} I + 2hA & \left(\beta + \frac{5}{4}\right)hA \end{bmatrix} \underline{\zeta}_r^{(0)} + O(h^2). \quad (3.11)$$

We have thus proved

Theorem 5. The (3,2)-method is numerically stable for all quadrature rules (1.2) having degree $\rho > 1$.

4. CONVERGENCE AND ERROR ESTIMATES

In this section, the techniques we use in order to obtain an order-of-convergence result are straightforward extensions of those used in reference [2] to obtain a corresponding convergence result and we shall therefore give no details here. We assume

- $\phi(x) \in C^{m+1}[0, X]$;
- For each $x \in [0, X]$, $K(x, y)$ is continuously differentiable through order $(m+1)$ with respect to y , for all $0 \leq y \leq x$.

Let

$$\nu = \begin{cases} 0, & \rho^* = m-1; \\ 1, & \text{otherwise,} \end{cases} \quad (4.1)$$

where ρ^* is as in (1.6), and let us set in (2.3)

$$\underline{a}_i^{(r)} = \frac{1}{(m-k+i)!} \phi^{(m-k+i)}(x_{kr}) + h^{k-i+v} \underline{b}_i^{(r)}, \quad i = 1(1)k, \quad (4.2)$$

where $\underline{b}_i = \underline{b}_i(h)$. We also write

$$\underline{e}(x) = \underline{s}(x) - \phi(x). \quad (4.3)$$

Lemma 1. The coefficients $\underline{b}_i$ in (4.2) are uniformly bounded, and for $r = 0(1)N-1$ we have

$$\|\underline{e}^{(i)}(x_{kr})\| = O(h^{m+v-i}), \quad i = 0(1)m-k. \quad (4.4)$$

Proof. The proof is by induction on r . The boundedness part follows in a manner similar to that of theorem 1, and we only need to difference the error equation and use (1.7) when $v = 0$. As for (4.4), observe that it is trivially satisfied for $r = 0$, and Taylor's expansion, (2.3), and (4.2) give

$$\frac{1}{j!} \underline{e}^{(j)}(x_{kr+k}) = \sum_{i=j}^{m-k} \binom{i}{j} H_k^{i-j} \underline{e}^{(i)}(x_{kr}) + h^{m+v-j} \sum_{i=1}^k C_k^i \binom{m-k+i}{j} \underline{b}_i + O(h^{m+v-j}),$$

where $H_k = C_k h$.

Q.E.D.

In view of lemma 1 and the fact that $\underline{s}(x)$ is a spline function, we have

Theorem 6. Let $\phi(x)$ and $K(x,y)$ satisfy assumptions (4.a) and (4.b). Then for all $x \in [0, X]$ we have

$$\underline{s}^{(i)}(x) - \phi^{(i)}(x) = O(h^{m+v-i}), \quad i = 0(1)m+v \quad (4.5)$$

where

$$\underline{s}^{(i)}(x_{kr}) = \frac{1}{2} \left[\underline{s}^{(i)}\left(x_{kr} + \frac{1}{2} H_k\right) + \underline{s}^{(i)}\left(x_{kr} - \frac{1}{2} H_k\right) \right], \quad (4.6)$$

$$r = 1(1)N-1; \quad i > m-k.$$

We end this section by deriving an asymptotic formula for the discretization error. The application of (1.2) to (1.1); (2.4), (4.3), and (1.5) yield for $p = 1(1)k$

$$\underline{e}_{kr+p} = R_{kr+p} \left[K(x_{kr+p}, y) \underline{e}(y) \right] + \underline{E}_{kr+p} . \quad (4.7)$$

As in Linz [7], we assume there exists a function $\underline{E}(x)$ such that

$$\underline{E}_i = h^{m+\nu} \underline{E}(x_i) + O(h^{m+\nu+1}) , \quad i = 1(1)Nk ; \quad (4.8)$$

[typically, $\underline{E}(x)$ will involve the integral of the $(m+\nu)^{th}$ order partial derivative with respect to y of $K(x, y) \phi(y)$]. We now have

Theorem 7. Let $\underline{\eta}(x)$ be the solution of the integral equation

$$\underline{\eta}(x) = \underline{E}(x) + \int_0^x K(x, y) \underline{\eta}(y) dy , \quad 0 \leq x \leq X . \quad (4.9)$$

Then the error function (4.3) satisfies

$$\underline{e}(x) = h^{m+\nu} \underline{\eta}(x) + O(h^{m+\nu+1}) , \quad 0 \leq x \leq X . \quad (4.10)$$

Proof. Let

$$\underline{\delta}(x) = \underline{e}(x) - h^{m+\nu} \underline{\eta}(x) .$$

Then, it is not difficult to see that

$$\underline{\delta}_{kr+p} = \sum_{i=0}^{kr+p-1} h_i \sum_{j=1}^M w_j K(x_{kr+p}, x_i + h_i t_j) \underline{\delta}(x_i + h_i t_j) + O(h^{m+\nu+1}) .$$

If we now write

$$K = \max_{0 \leq y \leq x \leq X} \|K(x, y)\| ,$$

$$\|\underline{\delta}_j^*\| = \max_{x_{kj} \leq x \leq x_{kj+k}} \|\underline{\delta}(x)\| ; \quad j = 0(1)N-1 ,$$

then we can find a constant L such that

$$\|\delta_r^*\| \leq \frac{hkK}{1 - hkK} \sum_{j=0}^{r-1} \|\delta_j^*\| + Lh^{m+\nu+1}.$$

Since $\|\delta_0^*\| = O(h^{m+\nu+1})$, the result of the theorem follows by lemma 1.1 in Linz [7; p. 7].

Q.E.D.

5. CONCLUDING REMARK

In order to decrease appreciably the computational effort involved in the application of an (m,k) -method, we may apply over $[0, x_{kr}]$ a quadrature formula that uses the mesh points only, but having the same order as (1.2). Such a change will naturally have no effect on the stability and the order of convergence of the method.

In particular, for the $(3,2)$ -method Simpson's rule is an obvious choice over $[0, x_{2r}]$.

PART II

SYSTEMS OF THE FIRST KIND

1. PRELIMINARIES

In this part we discuss some direct methods for the numerical solution of systems of Volterra integral equations of the first kind of the form

$$\underline{g}(x) = \int_0^x K(x,y) \underline{\phi}(y) dy, \quad 0 \leq x \leq X, \quad (1.1)$$

where $\underline{g}(x)$ is a given vector-valued function, $K(x,y)$ a square matrix of order n , and $\underline{\phi}(x)$ is the unknown function. We assume

- a) $\underline{g}(0) = \underline{0}$;
- b) $\underline{g}(x)$ is continuously differentiable in $[0, X]$;
- c) $\det [K(x, x)] \neq 0$, for all $0 \leq x \leq X$;
- d) $K(x, y)$ is continuously differentiable, with respect to each variable, in the region $0 \leq y \leq x \leq X$;
- e) $\underline{\phi}^{(i)}(0)$ exists for $i = 0(1)m-k$, where m and k are as specified in the next section.

It is known that assumptions (a)-(d) guarantee that (1.1) has a unique, continuous solution in $[0, X]$. See reference [6].

In this part, the results and equations of Part I are prefixed by I.

2. DESCRIPTION OF THE METHOD

Given the mesh Δ_N described by (I.2.1), we define $\underline{S}(x)$ by (I.2.3) and require that the coefficients $\underline{a}^{(r)}$ be chosen according to the relations

$$\underline{g}_{kr+p} = R_{kr+p} \left[K(x_{kr+p}, y) \underline{S}(y) \right]; \quad p = 1(1)k, \quad (2.1)$$

$$\underline{S}^{(i)}(0) = \underline{\phi}^{(i)}(0); \quad i = 0(1)m-k,$$

where $R_n(f)$ is given by (I.1.2).

Theorem 1. The construction described by (I.2.3) and (2.1) is well defined and indeed gives rise to a unique spline function of degree m , deficiency $-(k-1)$.

Proof. We assume, without loss of generality, that the mesh Δ_N is uniform, and we write for $i = 1(1)k$ and for each r

$$\underline{a}_i^* = \frac{h^i}{m-k+i} \underline{a}_i^{(r)}.$$

We denote by V the block matrix of order k with $V_{ij} = i^j I$; $i, j = 1(1)k$.

Now, (I.2.3.), (2.1), and Taylor's expansion of $K(x, y)$ about (x_{kr}, x_{kr}) show, in view of assumptions (b)-(d), that the coefficient matrix of $(\underline{a}_1^*, \underline{a}_2^*, \dots, \underline{a}_k^*)^T$ may be written in the form

$$V + hW,$$

where W is uniformly bounded. Since V is non-singular, the theorem follows by induction on r .

Q.E.D.

3. NUMERICAL STABILITY

We discuss in this section the stability behaviour of the (m, k) -method, described in the previous section, when applied to the system

$$\underline{g}(x) = A \int_0^x \underline{\phi}(y) dy, \quad (3.1)$$

where $\det(A) \neq 0$, and where $\underline{g}(x)$ satisfies assumptions (a) and (b).

Definition. The (m, k) -method defined by (I.2.3) and (2.1) is said to be numerically stable if all solutions $\{\underline{S}(x_{kr})\}$ remain bounded as $h \rightarrow 0$, $r \rightarrow \infty$ such that x_{kr} remains fixed, when the method is applied to any system of the form (3.1).

In what follows we shall continue to assume that the mesh Δ_N is uniform.

For each r , denote by $\underline{d}_r$ the vector

$$\underline{d}_r = \begin{bmatrix} \underline{g}_{kr+1} - \underline{g}_{kr} \\ \underline{g}_{kr+2} - \underline{g}_{kr} \\ \vdots \\ \underline{g}_{kr+k} - \underline{g}_{kr} \end{bmatrix} h^{-1} \quad (3.2)$$

Then, the application of (2.1) to (3.1) gives

$$\tilde{A}^{-1} \underline{d}_r = U \underline{\zeta}_r^{(0)} + V \tilde{\underline{a}}^{(r)}, \quad (3.3)$$

where U and V are given by (I.2.22) and (I.2.23), respectively, $\tilde{A}$ being the block diagonal matrix of order k with $\tilde{A}_{ii} = A$, $i = 1(1)k$. If we now write

$$\begin{aligned} G &= \Omega - \Gamma V^{-1} U, \\ B_r &= \Gamma V^{-1} \tilde{A}^{-1} \underline{d}_r, \end{aligned} \quad (3.4)$$

where Ω and Γ are given by (I.2.13), we immediately obtain from (3.3) and (I.2.17)

$$\underline{\zeta}_{r+1} = G \underline{\zeta}_r^{(0)} + B_r. \quad (3.5)$$

Observe that in view of assumption (b) and (3.2), $\|B_r\|$ is uniformly bounded.

Corresponding to theorem I.2, we have

Theorem 2. The $(m,1)$ -method is divergent for all $m > 1$ for all quadrature formulae (I.1.2) for which

$$\begin{aligned} \beta(m+1) &= 1 + \alpha, \\ -\frac{1}{2} &< \alpha < \frac{1}{2}, \end{aligned} \quad (3.6)$$

where β is given by (I.1.4). (Note: if the degree $\rho \geq m$ then $\alpha = 0$.)

Proof. When $k = 1$, we easily find

$$\text{tr } (\Omega) = nm;$$

$$(\Gamma V^{-1} U)_{ii} = \beta^{-1} \binom{m}{i} \frac{1}{i+1} I; \quad i = 0(1)m-1,$$

and hence

$$\text{tr } (G) = n \left[m - \frac{2}{1+\alpha} (2^m - 1) \right].$$

Q.E.D.

We now go on to consider the case $k = m$. If we assume that the quadrature formula is of degree $\rho \geq m$, then by (I.2.23)

$$\beta_i = \frac{i^{m+1}}{m+1}, \quad (3.7)$$

and consequently

$$\Gamma V^{-1}U = \Omega + \Gamma D \Gamma^{-1} \Omega, \quad (3.8)$$

where D is a block diagonal matrix with

$$D_{ii} = iI, \quad i = 1(1)m; \quad (3.9)$$

and since $\Omega = [I \ I \ \dots \ I]^T$ when $k = m$. Thus

$$G = -\Gamma D \Gamma^{-1} \Omega. \quad (3.10)$$

If we now denote by G_m the m^{th} block of G and use the fact that Γ is a Vandermonde matrix, we find by direct computation that

$$G_m = \begin{cases} -I, & m = 1, 3, 5; \\ I, & m = 2, 4, 6. \end{cases} \quad (3.11)$$

Moreover, if $\underline{g}(x)$ has a uniformly bounded second derivative in $[0, X]$, then it follows from (3.2) and (3.4) that

$$B_r = \Gamma V^{-1} U A^{-1} \underline{g}_{mr} + O(h),$$

and we therefore obtain, in view of (3.8) and (3.11),

$$\left(B_r \right)_m = 0, \quad m = 2, 4, 6. \quad (3.12)$$

Now, in view of (3.5), (3.10), (3.11), and (3.12) we have established

Theorem 4. The (m, m) -method is numerically stable for $m = 1, 3$, and 5 . If $\underline{g}(x)$ has a uniformly bounded second derivative in $[0, X]$, then the method is also stable for $m = 2, 4$, and 6 .

The practical implication of (3.11) is that methods for even m are to be preferred to those for m odd since the latter are likely to produce small oscillations.

4. CONVERGENCE AND ERROR BOUNDS

For each r , and for $p = 1(1)k$, we write

$$\underline{L}_{r,p} = \sum_{q=0}^{kr-1} h_q \sum_{j=1}^M w_j \frac{\partial K}{\partial x} \left(\xi_{p,r}, x_q + h_q t_j \right) \underline{e} \left(x_q + h_q t_j \right), \quad x_{kr} < \xi_{p,r} < x_{kr+p} \quad (4.1)$$

$$\underline{J}_{r,p} = \sum_{q=0}^{p-1} h_q \sum_{j=1}^M w_j K \left(x_{kr+p}, x_{kr+q} + h_{kr+q} t_j \right) \sum_{i=0}^{m-k} \frac{1}{i!} \left(H_q + h_{kr+q} t_j \right)^i \underline{e}^{(i)}(x_{kr}), \quad (4.2)$$

where $\underline{e}(x)$ is defined by (I.4.3). Let the vectors $\underline{b}_i$ be defined by (I.4.2) where

$$v = \begin{cases} -1, & \rho^* = m-1; \\ 0, & \rho^* = m; \\ 1, & \text{otherwise.} \end{cases} \quad (4.3)$$

Then it is not difficult to see that

$$\begin{aligned} & \sum_{q=0}^{p-1} h_q \sum_{j=1}^M w_j K \left(x_{kr+p}, x_{kr+q} + h_{kr+q} t_j \right) \sum_{i=1}^k \left(H_q + h_{kr+q} t_j \right)^{m-k+i} h^{k-i+v} \underline{b}_i = \\ & = \underline{E}_{kr+p} - \underline{E}_{kr} - p \underline{H} \underline{L}_{r,p} - \underline{J}_{r,p} + O(h^{m+2}), \end{aligned} \quad (4.4)$$

where $\underline{E}_n = \underline{E}(K(x_n, y) \underline{\phi}(y))$ and where the $O(h^{m+2})$ term depends on $K(x, y)$, $\underline{\phi}(x)$, and the rule $R(f)$ only.

In view of (4.4) we may now proceed along lines identical to those that led to theorem I.6 and obtain

Theorem 5. Let $\underline{\phi}(x) \in C^{m+1}[0, X]$; let $\underline{g}(x)$ and $K(x, y)$ satisfy assumptions (a)-(d) and let the integer v be given by (4.3). Then for all $x \in [0, X]$ and for all $m \geq v$ we have

$$\underline{S}^{(i)}(x) = \underline{\phi}^{(i)}(x) + O(h^{m+v-i}), \quad i = 0(1)m+v, \quad (4.5)$$

where $\underline{S}^{(i)}(x_{kr})$, $i > m-k$, is defined by (I.4.6).

We now give a result on the growth of the discretization error. We assume the existence of a function $\underline{E}(x)$ satisfying

$$\underline{E}_{i+1} - \underline{E}_i = h^{m+\nu} \underline{E}(x_i) + O(h^{m+\nu+1}) . \quad (4.6)$$

Instances of this function are

$$\underline{E}(x) = -\frac{1}{12} \left[\underline{\psi}^{1,1}(x,x) - \underline{\psi}^{1,1}(x,0) \right] \quad (4.7)$$

for the trapezoidal rule and $m = 1$; and

$$\underline{E}(x) = -\frac{1}{16 \times 180} \left[\underline{\psi}^{1,3}(x,x) - \underline{\psi}^{1,3}(x,0) \right] \quad (4.8)$$

for Simpson's rule and $m = 3$, where

$$\underline{\psi}(x,y) = K(x,y)\underline{\phi}(y) , \quad (4.9)$$

$$\underline{\psi}^{i,j}(x,y) = \partial^{i+j} \underline{\psi}(x,y) / \partial x^i \partial y^j .$$

Now, for $i = 0(1)Nk-1$, we write

$$h_i = C_i h \quad (4.10)$$

and define the vectors

$$\underline{\delta}_i = \sum_{j=1}^M w_j \underline{e}(x_i + h_i t_j) . \quad (4.11)$$

In terms of these vectors, we easily find that the error equation for the (m,k) -method is given by

$$\begin{aligned} & h \sum_{i=0}^{kr-1} C_i \left[K(x_{kr+k}, x_i) - K(x_{kr}, x_i) \right] \underline{\delta}_i + hK(x_{kr}, x_{kr}) \sum_{i=0}^{k-1} C_{kr+i} \underline{\delta}_{kr+i} = \\ & = \underline{E}_{kr+k} - \underline{E}_{kr} + O(h^{m+\nu+2}) ; \end{aligned} \quad (4.12)$$

and if we write

$$\delta_{ki}^* = \sum_{j=0}^{k-1} c_{ki+j} \delta_{ki+j} ; \quad i = 0(1)r-1 , \quad (4.13)$$

we readily obtain

$$hH_k \sum_{i=0}^{r-1} K^{1,0}(x_{kr}, x_{ki}) \delta_{ki}^* + hK(x_{kr}, x_{kr}) \delta_{kr}^* = E_{kr+k} - E_{kr} + O(h^{m+\nu+2}) . \quad (4.14)$$

We have thus proved

Theorem 6. Let $\eta(x)$ denote the solution of the second-kind system

$$\eta(x) = [K(x, x)]^{-1} E(x) - \int_0^x [K(x, x)]^{-1} K^{1,0}(x, y) \eta(y) dy .$$

Then we have

$$\delta_{kr}^* = h^{m+\nu} \eta(x_{kr}) + O(h^{m+\nu+1}) ; \quad r = 0(1)N .$$

5. CONCLUDING REMARK

In addition to our remark in Section I.5, we note that for the (3,3)-method we may apply over $[0, x_{3r}]$ the three-eighth rule, and use Simpson's rule in the manner described over $[x_{3r}, x_{3(r+1)}]$. The resulting method is stable and is of order 4.

REFERENCES

1. M.E.A. El Tom, Application of spline functions to Volterra integral equations, J. Inst. Maths. Applics. 8 (1971).
2. M.E.A. El Tom, Numerical solution of Volterra integral equations by spline functions, BIT 13, No. 1 (1973); pp. 1-7.
3. Hing-Sum Hung, The numerical solution of differential and integral equations by spline functions, Tech. Summary Rep. No. 1053, Maths. Research Centre, U.S. Army, University of Wisconsin, 1970.
4. H. Brunner, The solution of non-linear Volterra integral equations by piecewise polynomials, Proceedings of the Manitoba Conf. on Numer. Maths., Winnipeg (1971), pp. 65-78.
5. A.N. Netravali, Spline approximation to the solution of the Volterra integral equations of the second kind, Maths. Comp. 27 (1973), pp. 99-106.
6. W. Pogorzelski, Integral Equations and Their Applications, Vol. I, Pergamon Press, Oxford, 1966.
7. P. Linz, The numerical solution of Volterra integral equations by finite difference methods, Tech. Summary Rep. No. 825, Maths. Research Centre, U.S. Army, University of Wisconsin, 1968.