MULTI-HADRON PRODUCTION IN A CASCADE MODEL FOR e^+e^- ANNIHILATION

I. - TOTAL CROSS-SECTION AND MULTIPLICITY DISTRIBUTION

N.S. Craigie

II. Institut für Theoretische Physik
der Universität Hamburg

and

K.D. Rothe ^{*)}

CERN -- Geneva

A B S T R A C T

We consider a simple dynamical model for e^+e^- annihilation in which the production of pions is the result of a kind of bremsstrahlung process, each step of the cascade being characterized by the decay of a (massive) spin-one system into a pion plus a similar spin-one system. We study the high-energy behaviour of the n particle and total cross-sections as well as the multiplicity distribution for a large class of transition form factors. The model is found to include features familiar from the uncorrelated jet, the multiperipheral and the thermodynamical models. In particular, it is able to accommodate a logarithmically increasing multiplicity as well as a constant total cross-section.

^{*)} On leave from the Institut für Theoretische Physik der Universität Heidelberg.

1. INTRODUCTION

The CEA data¹⁾ indicate that, at least in the intermediate energy region, the physics of e^+e^- annihilation may look quite different from what had been expected. In the range 5-25 GeV² of c.m. energy squared s , the experimental data for the ratio $R(s)$ of hadron to μ -pair production are easily fit by a curve rising linearly with s , contrary to the constant behaviour expected on the basis of the parton model and naive dimensional analysis. The uncorrelated jet picture²⁾ is thus excluded in this energy range.

The observed angular distribution in the once inclusive process indicates, on the other hand, that at present energies the annihilation takes place predominantly in the p-wave, in support of a one-photon exchange picture. However, if the rise of $R(s)$ observed at present should persist up to arbitrarily high energies, then higher-order effects in quantum electrodynamics are expected to become important at sufficiently high energies, and the one-photon exchange picture would have to be abandoned. Alternatively, if one considers the rise of $R(s)$ as an intermediate energy effect, one would have to understand the underlying dynamical structure which is responsible for the observed behaviour, i.e. the rather slow approach to scaling.

As for the transverse momentum distribution, it could well be quite different from that found in diffractive hadronic processes where the leading particle tends to single out a particular direction of the outgoing beam. Evidence for a possible production of secondaries with a large relative transverse momentum is already provided by the pure annihilation channel in low-energy $\bar{p}p$ collisions³⁾, which share some of the features characteristic of e^+e^- annihilation. This has been a motivation for also regarding the production of secondaries in e^+e^- annihilation as the statistical decay of a very heavy fireball⁴⁾ in the framework of the statistical bootstrap⁵⁾. Although the statistical view of e^+e^- annihilation does not lead to a prediction for $\sigma(s)$ itself, it does make predictions for the distribution (isotropic) and mean multiplicity (linear rise with the c.m. energy) of the secondaries.

On the other hand, one would probably favour a picture intermediate between the production of two uncorrelated jets and a statistical distribution; a picture in which some of the transverse momenta of secondaries are allowed to grow large, and mean particle multiplicities increase slowly with the total available c.m. energy. In particular, one might favour a logarithmic increase of the particle multiplicity as the counterpart of the q^2 plateau observed in deep inelastic electroproduction, a feature which seems difficult to realize within the framework of conventional field theories⁶⁾.

Our broad aim in this and the following paper (referred to as I and II) is to consider a simple dynamical model for the hadronic multiparticle discontinuity of the photon propagator in lowest order of QED. Specifically we have in mind a dynamical mechanism analogous to the multiperipheral picture⁷⁾ of hadronic processes; that is, we consider the production of pions in e^+e^- annihilation to proceed via a kind of "chain emission" or bremsstrahlung process: each step of the cascade is characterized by the "decay" of a spin-one system into a pion plus another similar spin-one system of opposite G-parity, where this system is a composite structure carrying a definite spin, isospin and G-parity, but having a continuous mass spectrum. Since these systems carry the quantum numbers of the ρ and ω mesons, we shall sometimes refer to them as such, even though large mass extrapolations are involved. The probability for the one-step decay of such a "meson" is characterized by an amplitude $F(s,s')$ which, for simplicity, we refer to as the transition form factor. Apart from these details, it should be noted, however, that the topology involved is the same as that of the Bardakci-Ruegg n -point function⁸⁾ in dual resonance models.

The model turns out to be explicitly solvable at high energies, and thus provides a very convenient dynamical framework for discussing a number of theoretical questions. We shall explore the full content of the model for a large class of form factors $F(s,s')$, whose asymptotic behaviour is of course unknown. However, in the low mass region, $F(s,s')$ can always be chosen, by construction, to describe well the results in the resonance region.

The work is divided into two principal parts: In Part I we study in detail the behaviour of the total and n -particle cross-sections, as well as the particle multiplicity distribution. A very brief summary of these results has already been presented elsewhere⁹⁾. In Part II we give a corresponding discussion of the once inclusive process.

2. INTEGRAL EQUATIONS

We consider here the cascade process shown in Fig. 1, restricting ourselves to (massive) spin-one systems carrying the quantum numbers of the ρ and ω mesons. For simplicity, we shall refer to it and think of it as a vector meson cascade. The primary ρ and ω "mesons" are coupled to the isovector ($I = 1$) and isoscalar ($I = 0$) part of the virtual photon according to a vector dominance type of prescription.

Neglecting crossed graphs¹⁰⁾, we may formally sum the n -pion contributions to the spectral function $\rho^{(I)}(q)_{\mu\nu}$ of the corresponding two-particle Green's function into a coupled set of integral equations for the $I = 0$ and $I = 1$ components, as shown in Fig. 2. The kernel of the integral equations involves the

" $\rho\omega\pi$ " vertex with the coupling $F(s,s') \varepsilon_{\mu\nu\lambda\rho} q^\lambda k^\rho$, where s, s' are the masses of the " ρ " and " ω " systems, and $F(s,s')$ is essentially the probability amplitude for such a system to decay into a pion plus a similar system. Within the vector dominance framework, one would have $F(s,s') = G(s,s')/(s - m_V^2)$ with the normalization $G(m_\omega^2, m_\rho^2) \equiv f_{\rho\omega\pi} [(f_{\rho\omega\pi}^2/4\pi) \approx 0.5 \mu^{-2}, \mu \equiv \text{pion mass}]$. To be specific we shall assume that the last link in the chain always decays into two pions, regardless of whether the total number of pions is even or odd. In a vector dominance framework this assumption would correspond to the Gell-Mann/Sharp/Wagner model of ω decay¹¹⁾. The driving term $\rho_0(s)$ is then given by just the 2π contribution, and the coupled set of integral equations may be written in the form

$$\rho^{(1)}(s) = \rho_0(s) + \frac{1}{24(2\pi)^2} \int_{(4\mu)^2}^{(\sqrt{s}-\mu)^2} \frac{ds'}{s} \frac{s'}{s} [(s-s'+\mu^2)^2 - 4s\mu^2]^{3/2} |F(s,s')|^2 \rho^{(0)}(s') \quad (1a)$$

$$\rho^{(0)}(s) = \frac{1}{24(2\pi)^2} \int_{(3\mu)^2}^{(\sqrt{s}-\mu)^2} \frac{ds'}{s} \frac{s'}{s} [(s-s'+\mu^2)^2 - 4s\mu^2]^{3/2} |F(s,s')|^2 \rho^{(1)}(s') \quad , \quad (1b)$$

where $\rho^{(I)}(q^2)$ is defined by

$$\rho^{(I)}(q)_\mu = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \rho^{(I)}(q^2) \quad (2)$$

and

$$\frac{1}{\gamma_\rho^2} \rho_0(s) = \frac{1}{44\pi} \left(1 - \frac{4\mu^2}{s}\right)^{3/2} |F_{\pi\pi}(s)|^2 \quad , \quad (3)$$

$F_{\pi\pi}$ being the pion electromagnetic form factor.

The e^+e^- total cross-section is assumed to be related via a vector dominance type prescription to $\rho^{(I)}(s)$:

$$\sigma(s) = \frac{8\pi^2\alpha^2}{s} \rho(s) \quad , \quad (4a)$$

where

$$\rho(s) = \frac{1}{\gamma_\rho^2} \rho^{(1)}(s) + \frac{1}{\gamma_\omega^2} \rho^{(0)}(s) \quad . \quad (4b)$$

The kernel of the integral equations (1) involves factors arising from the spin, as well as the form factor $F(s, s')$, about whose off-shell dependence little is known. The nature of the solutions depends, however, on the detailed off-shell behaviour of $F(s, s')$. Since we shall be primarily concerned with the asymptotic behaviour of the solutions, we shall make, for $s \gg s_0$, the replacement

$$s^2 |F(s, s')|^2 \longrightarrow 24(2\pi)^2 \lambda |h(\frac{s'}{s})|^2 \left(\frac{s}{s_0}\right)^{2\beta}, \quad (5)$$

where λ is a dimensionless "coupling constant", and s_0 is some appropriately chosen parameter which sets the scale for the asymptotic region. Clearly, the ansatz (5) for $s \gg s_0$, with $h(\eta)$ (so far) arbitrary, includes a large class of $\rho\omega\pi$ form factors.

Now, for $s \gg s_0$, the effects associated with the non-vanishing pion mass and a form factor behaviour deviating from that of Eq. (5) in the low mass region may in principle be incorporated as low-energy corrections into the driving term. For the purpose of studying the functional dependence of $\rho(s)$ for $s \gg s_0$, it is thus sufficient to set $\mu = 0$ in Eqs. (1), which thereby reduce to the more tractable form,

$$\rho^{(1)}(s) = \rho_0(s) + \left(\frac{s}{s_0}\right)^{2\beta} \lambda \int_0^1 d\eta \eta (1-\eta)^3 |h(\eta)|^2 \rho^{(0)}(\eta s) \quad (6a)$$

$$\rho^{(0)}(s) = \left(\frac{s}{s_0}\right)^{2\beta} \lambda \int_0^1 d\eta \eta (1-\eta)^3 |h(\eta)|^2 \rho^{(1)}(\eta s) . \quad (6b)$$

For the remainder of this section we shall restrict ourselves to the case $\beta = 0$, leaving the general case $\beta \neq 0$ for the following section. Then, introducing the Mellin transform

$$\tilde{F}(j) = \int_{s_0}^{\infty} \frac{ds}{s} \left(\frac{s}{s_0}\right)^{-j} F(s) ,$$

we may reduce the coupled set of equations (6) to an algebraic form having the solution

$$\tilde{\rho}^{(1)}(j) = \frac{\tilde{\rho}_0(j)}{1 - (\lambda H(j))^2} , \quad (7)$$

where

$$H(j) = \int_0^1 d\eta \, \eta^{j+1} (1-\eta)^3 |h(\eta)|^2. \quad (8)$$

The spectral function $\rho(s)$ is obtained by taking the inverse Mellin transform

$$\rho(s) = \int_{-i\infty+c}^{i\infty+c} \frac{dj}{2\pi i} \left(\frac{s}{s_0}\right)^j \frac{\tilde{\rho}_0(j)}{1 - (\lambda H(j))^2}, \quad (9)$$

where the contour is taken to lie to the right of all singularities in the j -plane.

The leading asymptotic behaviour of $\rho(s)$ is determined by the rightmost singularity of the integrand in Eq. (9). This singularity may either arise from the driving term -- where it would correspond to a simple pole at $j = -2$ -- from the denominator, or from both in the case of colliding singularities. Probably the most interesting case corresponds to the "nearest" singularity being given by a zero of the denominator at $j = \alpha(\lambda)$, where $\alpha(\lambda)$ satisfies the "eigenvalue" condition¹²⁾

$$1 - \lambda H(\alpha) = 0 \quad (10)$$

Noting that it follows from Eq. (10) that

$$\left[-\lambda \frac{\partial H(\alpha)}{\partial \alpha} \right]^{-1} = \lambda \frac{\partial \alpha(\lambda)}{\partial \lambda}, \quad (11)$$

we obtain from Eq. (9),

$$\rho''(s) \xrightarrow{s \rightarrow \infty} \frac{1}{2} \tilde{\rho}_0(\alpha) \lambda \frac{\partial \alpha(\lambda)}{\partial \lambda} \left(\frac{s}{s_0}\right)^{\alpha(\lambda)}. \quad (12)$$

It follows from the definition (8) of $H(j)$, that $\alpha(\lambda)$ is a monotonically increasing function of λ . Moreover, if $|h(\eta)|^2 \xrightarrow{\eta \rightarrow 0} \eta^\gamma$, then $\alpha(0) = -2 - \gamma$.

The multiplicity is simply given by the logarithmic derivative of $\rho(s)$ with respect to λ ,

$$\langle n(s) \rangle \xrightarrow{s \rightarrow \infty} \left(\lambda \frac{\partial \alpha}{\partial \lambda} \right) \log \left(\frac{s}{s_0} \right) + \text{const.} \quad (13)$$

and is seen to increase logarithmically. For the dispersion we find

$$D = \langle n^2 \rangle - \langle n \rangle^2 \longrightarrow \left[\lambda \frac{\partial \alpha}{\partial \lambda} + \lambda^2 \frac{\partial^2 \alpha}{\partial \lambda^2} \right] \log \left(\frac{s}{s_0} \right) + \text{const.} \quad (14)$$

Hence the multiplicity distribution is in general broader or narrower than a Poisson distribution, depending on the sign of $\partial^2 \alpha / \partial \lambda^2$.

The asymptotic behaviour (12) -- and, indeed, the "mechanism" leading to this behaviour -- is reminiscent of multiperipheralism in hadronic processes, in that it can be traced to a similar distribution of the energy among the final-state particles. The pions are not produced, however, in the forward direction, the angular spread relative to one another being determined by the relative energies. Correspondingly, the pions are in general not produced peripherally, so that one can speak only of an analogy with respect to multiperipheralism.

3. THE GENERAL CASE $\beta \neq 0$

In the case $\beta \neq 0$ the asymptotic behaviour of $\rho(s)$ is most easily calculated by summing over the $n + 2$ pion contributions $\rho_n(s)$, where $\rho_n(s)$ evidently satisfies the equation (in the case $\mu = 0$)

$$\rho_n(s) = \left(\frac{s}{s_0} \right)^{2\beta} \lambda \int_0^1 d\gamma \gamma (1-\gamma)^3 |h(\gamma)|^2 \rho_{n-1}(\gamma s) . \quad (15)$$

By taking the Mellin transform, this equation may be cast into the algebraic form,

$$\tilde{\rho}_n(j) = \lambda H(j-2\beta) \tilde{\rho}_{n-1}(j-2\beta) . \quad (16)$$

We have dropped the superscript (I), n even (odd) referring evidently to $I = 1$ ($I = 0$). Equation (16) may be solved by iteration,

$$\tilde{\rho}_n(j) = \lambda^n \tilde{\rho}_0(j-2n\beta) P_\beta^n(j) , \quad (17a)$$

where

$$P_\beta^n(j) = \prod_{r=1}^n H(j-2r\beta) . \quad (17b)$$

The basic character of the solution is determined by the value of β . We have already seen in Section 2 that the value $\beta = 0$ leads to a solution having the features of multiperipheral models, independent of the specific form of $h(\eta)$. Since we shall only be interested in the basic features of the solution in the general case, we shall simplify the discussion by considering the simplest ansatz for $h(\eta)$ consistent with a symmetric $\rho\omega\pi$ coupling: $|h(\eta)|^2 = \eta^{\beta-\frac{1}{2}}$. The basic conclusions of this section will not depend on the detailed form of $h(\eta)$.

For the above choice of $h(\eta)$ we may write down a closed expression for $P_\beta^n(j)$ as defined by Eq. (17b):

$$P_\beta^n(j) = \left(\frac{3}{8\beta^4}\right)^n \frac{\Gamma\left(\frac{1}{2} - \frac{2j+9}{4\beta}\right)\Gamma\left(\frac{1}{2} - \frac{2j+7}{4\beta}\right)\Gamma\left(\frac{1}{2} - \frac{2j+5}{4\beta}\right)\Gamma\left(\frac{1}{2} - \frac{2j+3}{4\beta}\right)}{\Gamma\left(n + \frac{1}{2} - \frac{2j+9}{4\beta}\right)\Gamma\left(n + \frac{1}{2} - \frac{2j+7}{4\beta}\right)\Gamma\left(n + \frac{1}{2} - \frac{2j+5}{4\beta}\right)\Gamma\left(n + \frac{1}{2} - \frac{2j+3}{4\beta}\right)} \quad (18a)$$

$\beta \neq 0$

and

$$P_0^n(j) = \left[\frac{6}{(j + \frac{9}{2})(j + \frac{7}{2})(j + \frac{5}{2})(j + \frac{3}{2})} \right]^n \quad (18b)$$

Substituting expressions (18) into Eq. (17a) and taking the inverse Mellin transform, we obtain, retaining only the contribution from the nearest singularity, the results summarized in Table 1, where we have omitted all proportionality coefficients independent of n . Note that we needed to distinguish five cases: $\beta < 0$, $\beta = 0$, $0 < \beta < \frac{1}{2}$, $\beta = \frac{1}{2}$, $\beta > \frac{1}{2}$, corresponding respectively to the nearest singularity being a simple pole at $j = -\frac{3}{2} + \beta$ with residue

$$R_n(\beta) = \frac{16\beta^4}{6} \frac{1}{(n-1)!} \left(\frac{\pi}{16\beta^4}\right)^n \frac{\Gamma\left(1 - \frac{3}{2\beta}\right)\Gamma\left(1 - \frac{2}{2\beta}\right)\Gamma\left(1 - \frac{1}{2\beta}\right)}{\Gamma\left(n - \frac{3}{2\beta}\right)\Gamma\left(n - \frac{2}{2\beta}\right)\Gamma\left(n - \frac{1}{2\beta}\right)} \quad ; \quad (19)$$

an n^{th} order pole of $P_\beta^n(j)$ at $j = -\frac{3}{2}$ with residue 1; a simple pole of $P_\beta^n(j)$ at $j = (2n-1)\beta - \frac{3}{2}$ with residue $R_n(-\beta)$; a double pole at $j = n-2$, corresponding to a collision of singularities between $P_\beta^n(j)$ and $\tilde{\rho}_0(j-2n\beta)$; and a simple pole of $\tilde{\rho}_0(j-2n\beta)$ at $j = 2n\beta - 2$.

Except for the case $\beta = 0$, $\rho(s)$ may be obtained by simply summing over ρ_n to leading order. In the case $\beta = 0$, one readily verifies that the leading "log s summation" fails to give the correct result, Eq. (12) -- a well-known phenomenon¹³⁾. For simplicity we have set here $\gamma_\rho = \gamma_\omega$ in Eq. (4b). The corresponding results for $\rho(s)$ obtained by summing the leading terms in the asymptotic expansion as shown in Table 1 are summarized in Table 2, where we have again omitted any proportionality coefficients.

As Tables 1 and 2 show, the asymptotic behaviour of $\rho(s)$ depends critically on the off-shell behaviour of the decay amplitude $F(s, s')$. This is exhibited most dramatically in the results for $\rho(s)$ as contained in Table 2, where the asymptotic solution changes from a power-type behaviour for $\beta \leq 0$, to an exponentially increasing (or runaway) solution for $\beta > 0$, as they are also characteristic of statistical bootstrap models⁵⁾. Physically they correspond to the production of energetic pions at large relative angles. Because of the slower exponential increase than that found in the statistical bootstrap model, the mean particle multiplicity increases more slowly than $\langle n \rangle \sim E$, and correspondingly the average energy per secondary does not tend to a constant, but increases with the incoming beam energy E . In fact, we have

$$\langle n(s) \rangle \longrightarrow \left(\frac{3}{8} \frac{\lambda}{\beta^4} \right)^{\frac{1}{4}} E^\beta, \quad (20)$$

that is, $\langle E_1 \rangle \sim E^{1-\beta}$.

For $\beta < -\frac{1}{2}$ the process is dominated by 2π production alone. For $\beta = -\frac{1}{2}$ the multiparticle contributions merely serve to modify the coefficient, but not the energy dependence, of the driving term. Physically we have the production of two jets moving in opposite directions in the c.m.s., one of the jets carrying only a single pion. For $-\frac{1}{2} < \beta < 0$ this two-jet structure evolves continuously into a picture in which the secondaries each carry, on the average, an energy proportional to the beam energy but are spread over increasingly large angles.

4. CONCLUSIONS

We have considered here a model for e^+e^- annihilation which, aside from the obvious virtue of simplicity, is in principle capable of accommodating the general features familiar from various kinds of models that have been discussed in the literature; in view of the present lack of understanding of the CEA results on e^+e^- annihilation, the cascade process considered here provides a simple but rather rich framework for discussing various theoretical questions. The principal aim of this paper was to emphasize the qualitative features that emerge from such a model, rather than try to fit the experimental data. In the general form in which

it has been formulated, there is of course much room for interpretation. In particular it may be viewed in a vector dominance framework as an alternative to the generalized vector dominance ideas as discussed by other authors¹⁴⁾, with due regard for the large mass extrapolations involved. In contrast to these models, the cascade provides however an exclusive picture of pion production.

In relating the asymptotic behaviour of $\rho^{(I)}(s)$ to that of the e^+e^- total cross-section $\sigma(s)$, we need to know the energy dependence of the photon-vector meson couplings γ_v^{-2} . In the case of the "runaway" solutions, the large q^2 behaviour of $\gamma_v^{-2}(q^2)$ would have to be exponentially damped, if these solutions are to be physically meaningful. In order to avoid an exponentially damped pion electromagnetic form factor, one would have to depart here from the vector dominance type picture, and associate the exponentially damped coupling only with the production of multipion final states. Since in the runaway case $\rho(s)$ is dominated by the production of a large number of pions, this does not affect the probability $\sigma_n(s)/\sigma(s)$ for producing a large number of particles, for which the model makes definite predictions depending on β (see Tables 1 and 2).

We found that the runaway solutions were able to accommodate multiplicities tending asymptotically to $\langle n(s) \rangle \sim (\sqrt{s})^\gamma$ with $0 < \gamma \leq 1$. The cascade thus offers an interesting alternative to the statistical bootstrap^{4,5)}, as it allows for power-behaved multiplicities and a rather broad, not necessarily statistical distribution of secondaries.

For $\beta = \frac{1}{2}$ the integral equation (6a) reduces essentially to the form encountered in the statistical bootstrap^{4,5)}, if one formally identifies $\rho(s)$ with the density of states⁴⁾: we have asymptotically a pure phase-space distribution, and hence a purely statistical decay of the "fireball". Accordingly one would naively expect to recover in this case the results of the statistical bootstrap⁵⁾ and in particular the results of Ref. 4. This is not the case; in fact, Eq. (20) shows that for $\beta = \frac{1}{2}$ the average multiplicity only grows like the square root of the energy, $\langle n \rangle \sim \sqrt{E}$, and correspondingly the average energy carried by the secondaries is large: $\langle E_1 \rangle \sim \sqrt{E}$. The reason is that we required every "fireball" in the decay chain to carry spin-one, whereas the usual statistical bootstrap models allow for any spin to be built up. The spin of the "fireball" effectively acts as an angular momentum barrier, resulting in a suppression of the pure phase-space contribution to $\rho_n(s)$ by two additional powers of $n!$ (see Table 1). The spin of the fireball thus plays here a crucial role and leads to observable consequences -- such as lower multiplicities, larger average energy of secondaries, and absence of a critical temperature -- which may turn out to be more compatible with future experimental data on e^+e^- annihilation.

In the case of a multiperipheral type of solution ($\beta = 0$), the assumed behaviour (5) of the form factor $F(s, s')$ in the limit s, s' large corresponds to the behaviour expected from a naive dimensional analysis (naive scaling). The corresponding solution $\rho(s)$ was shown to increase like $\tilde{\rho}_0(\alpha) s^{\alpha(\lambda)}$. However, whereas the power $\alpha(\lambda)$ is correctly given by the eigenvalue condition (10), independent of the behaviour of $F(s, s')$ in the resonance region, the value of the coefficient $\tilde{\rho}_0(\alpha)$ is seen to depend on the low-energy region as well. This is by no means surprising, since the full solution is known to be fully determined by the driving term, even though the $\rho^{\text{asym.}}(s)$ satisfies the homogeneous (eigenvalue) equation. Just to have an order of magnitude idea, we estimate the coefficient of the leading term by including the low-energy contribution shown in Fig. (3) in the driving term and choosing $h(\eta) = 1$ in Eq. (5). Assuming the CEA data to be fit by a $R(s)$ rising linearly with s , we require $\alpha(\lambda) = 1$, which gives $\lambda(\partial\alpha/\partial\lambda) \approx 1$. Noting further that for large s the $I = 0$ and $I = 1$ components give approximately the same contribution, we find $R(s) \approx 0.2 (s/s_0)$ as a very rough estimate. Choosing $s_0 = 1 \text{ GeV}^2$ as a typical scale, this is in surprisingly good agreement with the experimental data.

As for the logarithmic increase of the mean multiplicity, it reflects something like a plateau in rapidity space, if we ignore for the moment that the relative transverse momenta in the "multiperipheral" case will in general not all be small. Then, noting that in any *single* event the secondaries populate only half the rapidity axis, it is easily seen that a perfect plateau would correspond to pions spaced by two units of rapidity, which in turn would imply $\langle n(s) \rangle_p \sim 0.7 \ln s$. For $\alpha(\lambda) = 1$, we find $\partial\alpha/\partial\lambda \approx 1$ or $\langle n(s) \rangle \sim 1.0 \ln s$.

Finally, we remark that if $\alpha(\lambda) > 1$, then on account of unitarity we expect higher-order effects in QED eventually to play an important role at sufficiently high energies. This would be however at energies substantially higher than those considered at present.

Acknowledgements

We would like to thank R. Hagedorn, G. Kramer and F.M. Renard for useful discussions.

REFERENCES AND FOOTNOTES

- 1) A. Litke et al., Phys. Rev. Letters 30, 1189 (1973).
- 2) N. Cabibbo, G. Parisi and M. Testa, Nuovo Cimento Letters 4, 35 (1970).
G. Altarelli and M. Maiani, Phys. Letters 41 B, 480 (1972).
- 3) D. Everett, P. Grossman, P. Mason and H. Muirhead, Phys. Letters B47, 541 (1973).
- 4) J. Engels, H. Satz and K. Schilling, Nuovo Cimento 17 A, 535 (1973), and Bielefeld Preprint BI 74/07, Feb. 1974.
- 5) R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965);
R. Hagedorn and I. Montvay, Nuclear Phys. B59, 45 (1973);
S. Frautschi, Phys. Rev. D3, 2821 (1971).
- 6) J. Kogut, D.K. Sinclair and L. Susskind, Phys. Rev. D7, 3637 (1973);
N.S. Craigie, A.B. Kraemmer and K.D. Rothe, Z. Physik 259, 1 (1973), and
Nuovo Cimento 18 A, 188 (1973).
- 7) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26, 896 (1962).
- 8) K. Bardakci and H. Ruegg, Phys. Rev. 181, 1884 (1969).
- 9) N.S. Craigie and K.D. Rothe, CERN preprint (1973), TH. 1777, to be published in Phys. Letters.
- 10) It is well known that the contributions from the crossed graphs could in principle re-sum to change the leading asymptotic behaviour.
- 11) M. Gell-Mann, D. Sharp and W. Wagner, Phys. Rev. Letters 8, 261 (1962).
- 12) One can easily verify that the solution of $1 + \lambda H(\alpha) = 0$ does not correspond to the leading singularity.
- 13) This is also known from the hadronic multiperipheral model. The leading $\log s$ summation, however, tends to the correct solution in the weak coupling limit.
- 14) A. Bramon, E. Etim and M. Greco, Phys. Letters 41 B, 609 (1972).
M. Greco, Nuclear Phys. B63, 398 (1973).
M. Böhm, H. Joos and M. Kramer, DESY Preprint 72/62 (1972).
M. Böhm, H. Joos and M. Kramer, Acta Phys. Austriaca 38, 123 (1973).
H.D. Dahmen and F. Steiner, Phys. Letters 43 B, 521 (1973).
F.M. Renard, preprint (1973) MU-PM/73/10.

Table 1

β	$\rho_n(s)$
$\beta < 0$	$\sim \left(\frac{6\lambda}{\pi}\right)^n \tilde{\rho}_0[-\frac{3}{2} - (2n-1)\beta] R_n(\beta) s^{-\frac{3}{2}+\beta}$
$\beta = 0$	$\sim g^2 s^{-\frac{3}{2}} \frac{(\lambda \ln s)^{n-1}}{(n-1)!}$
$0 < \beta < \frac{1}{2}$	$\sim R_n(-\beta) s^{-\frac{3}{2}+\beta} \left(\frac{6\lambda}{\pi} s^{2\beta}\right)^{n-1}$
$\beta = \frac{1}{2}$	$\sim \frac{(6\lambda)^n}{(n+2)!(n+1)!n!(n-1)!} s^{n-2} \log s$
$\beta > \frac{1}{2}$	$\sim P_\beta^n(2n\beta-2) s^{-2} \left(\frac{6\lambda}{\pi} s^{2\beta}\right)^n$

Table 2

β	$\rho(s)$
$\beta < -\frac{1}{2}$	$\sim \rho_0(s)$
$\beta = -\frac{1}{2}$	$\sim s^{-2}$
$-\frac{1}{2} < \beta < 0$	$\sim s^{-\frac{3}{2}+\beta}$
$0 < \beta < \frac{1}{2}$	$\sim s^{-\frac{3}{2}+\beta} \left[\left(\frac{3}{8} \frac{g^2}{\beta^4} s^{2\beta}\right)^{1/4}\right]^{-\frac{3}{2}-\frac{3}{\beta}} e^{4[(3g^2/8\beta^4)s^{2\beta}]^{1/4}}$
$\beta = \frac{1}{2}$	$\sim (6g^2 s)^{-7/8} e^{4(6g^2 s)^{1/4}} \log s$

Figure captions

- Fig. 1 : Cascade contributing to the $I = 1$ part of $\sigma_n(s)$. The " ρ " and " ω " spin-one systems are denoted by a single and double line, respectively. The dotted line refers to the pion.
- Fig. 2 : Coupled integral equations for a) the $I = 1$ and b) the $I = 0$ part of the spectral function $\rho(s)$.
The pions are understood to be on the mass shell.
- Fig. 3 : Effective driving term in the integral equation (6a) taking account of low-energy corrections.

Fig. 1

Fig. 2

Fig. 3