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MULTI-HADRON PRODUCTION IN A CASCADE MODEL FOR e^+e^- ANNIHILATIONII. - MULTIPLICITY AND INCLUSIVE DISTRIBUTION

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A B S T R A C T

In Part I (see preceding paper) we studied the asymptotic behaviour of the n particle and total cross-sections for e^+e^- annihilation within the framework of a cascade model, allowing for a large class of transition form factors. We here extend the discussion to a study of the once inclusive process, restricting ourselves to the class of transition form factors leading to a multiplicity increasing logarithmically with the centre-of-mass energy. The corresponding set of two-dimensional, coupled integral equations is solved explicitly. We discuss the behaviour of the structure functions in the Bjorken limit, as well as the threshold behaviour and moment sum rules satisfied by the corresponding scaling functions. In addition, we draw some interesting conclusions which are expected to be valid beyond this model. We find that the limit $s \rightarrow \infty$, $\omega \rightarrow 0$ is reached non-uniformly, so that the asymptotic behaviour of $\langle n \rangle$ cannot be deduced from the scaling function $F(\omega)$ alone; Feynman scaling is broken for $\omega \rightarrow 0$; and for a smaller value of ω , the scaling limit is reached correspondingly slower.

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1. INTRODUCTION

We extend here our investigation of the cascade model¹⁾ to the study of the once inclusive process $\gamma_v \rightarrow \pi + X$. The behaviour of the structure functions in the Bjorken limit for $\omega > 0$ can already be guessed on the basis of the energy momentum sum rule and our knowledge of the high-energy behaviour of the spectral functions $\rho^{(I)}(s)$ as discussed in I. We show that the model implies a Feynman scaling law for $\omega > 0$, which is however broken as $\omega \rightarrow 0$. In fact, the cascade provides an explicit example of a non-uniform behaviour of the structure functions in the limit $s \rightarrow \infty$, $\omega \rightarrow 0$. The non-uniformity of such a limit may thus be expected to be also a characteristic feature of more sophisticated models, and implies that the computation of particle multiplicities from the knowledge of the scaling functions alone may give the incorrect result. We further discuss the possible corrections arising from non-leading terms, as well as the threshold behaviour of the scaling functions. In particular, we show that a transverse momentum cut-off would imply a Callan-Gross type relation. Moreover, the longitudinal structure function turns out to be suppressed relative to the transverse one in the threshold region, $\omega \rightarrow 1$. As an interesting by-product we find that even though $\langle n^+ \rangle = \langle n^- \rangle = \langle n^0 \rangle$, the major portion of the total c.m. energy is carried away by the neutral pions.

2. INTEGRAL EQUATION FOR INCLUSIVE STRUCTURE FUNCTIONS

As is evident from the results of I, for the purpose of studying the general features of the cascade model in question, we may ignore G parity. The integral equation for the inclusive process is then as shown in Fig. 1²⁾, where the inhomogeneous term is seen to be determined by the spectral function $\rho(s)$ calculated in I (we omit now the isospin label), and the transition form factor $F(s, s')$ (see I). The corresponding integral equation for the inclusive structure functions takes the form³⁾

$$W_{\mu\nu}(p, q) = |F(q^2, (q-p)^2)|^2 \rho((q-p)^2) \Gamma_{\mu}^{\lambda} ; \nu\lambda(p, q) \\ + \int \frac{d^4k}{(2\pi)^3} \delta_+(k^2 - \mu^2) |F(q^2, (q-k)^2)|^2 W^{\mu'\nu'}(p, q-k) \Gamma_{\mu\mu'} ; \nu\nu'(k, q), \quad (1)$$

where μ is the pion mass and

$$\Gamma_{\mu\lambda; \nu\varrho}(p, q) = \varepsilon_{\mu\lambda\alpha\beta} \varepsilon_{\nu\varrho\alpha'\beta'} q^\alpha p^\beta q^{\alpha'} p^{\beta'}. \quad (2)$$

We decompose $W_{\mu\nu}$ into covariants in the standard manner:

$$W_{\mu\nu}(p, q) = (g_{\mu\nu} q^2 - q_\mu q_\nu) V_1(q^2, \nu) + [(p_\mu q_\nu + q_\mu p_\nu) \nu - p_\mu p_\nu q^2 - g_{\mu\nu} \nu^2] V_2(q^2, \nu). \quad (3)$$

Projecting out the V_1 and V_2 components from Eq. (1) we arrive at

$$(\mu^2 s - \nu^2) (V_1(s, \nu) - \mu^2 V_2(s, \nu)) = \frac{1}{16\pi^2} \int \frac{d\nu'}{\nu} \int ds' |F(s, s')|^2 [s' V_1(s', \nu') - \nu'^2 V_2(s', \nu')] \xi(s, \nu; s', \nu') \quad (4a)$$

$$3s V_1(s, \nu) - (2\nu^2 + \mu^2 s) V_2(s, \nu) =$$

$$2|F(s, s-2\nu+\mu^2)|^2 \rho(s-2\nu+\mu^2)(\mu^2 s - \nu^2)(s-2\nu+\mu^2) + \frac{1}{16\pi^2} \int \frac{d\nu'}{\nu} \int ds' |F(s, s')|^2 \cdot \left\{ [s' V_1(s', \nu') - \nu'^2 V_2(s', \nu')] \frac{1}{2} [(s-s'+\mu^2)^2 - 4s\mu^2] - s' V_2(s', \nu') \xi(s, \nu; s', \nu') \right\}, \quad (4b)$$

where $\rho(s)$ is the spectral function determining the e^+e^- cross-section, as calculated in I; ν, ν', s, s' are defined by

$$\begin{aligned} \nu &= p \cdot q, \quad \nu' = p \cdot (q-k) \\ s &= q^2, \quad s' = (q-k)^2, \end{aligned} \quad (5)$$

and

$$\xi(s, \nu; s', \nu') = - \left\{ (\nu - \nu')(\nu' s - \nu s') + \mu^2 \left[(\mu^2 s - \nu^2) - \frac{(s - s' + \mu^2)^2}{4} \right] \right\}. \quad (6)$$

We make the change of variable

$$\omega = \frac{2\nu}{s}, \quad \omega' = \frac{2\nu'}{s'} \quad (7)$$

and define in the usual way,

$$s V_1(s, \nu) = F_L(s, \omega) \quad (8)$$

$$\nu^2 V_2(s, \nu) = \frac{\omega}{2} F_2(s, \omega),$$

where F_L is the structure function associated with the longitudinal polarization of the virtual photon. It is related to $F_{1,2}$ by $F_L = -F_1 + (\omega/2)F_2$. It is also convenient to introduce the combination

$$F(s, \omega) = F_1(s, \omega) - \frac{\omega}{6} F_2(s, \omega), \quad (9)$$

which determines the inclusive distribution for $\mu = 0$:

$$\frac{d\rho}{d\omega} = \frac{1}{8\pi} \sqrt{\omega^2 - \frac{4\mu^2}{s}} \left[F_1(s, \omega) - \frac{\omega}{6} \left(1 - \frac{4\mu^2}{\omega^2 s} \right) F_2(s, \omega) \right]. \quad (10)$$

In order to arrive at a more tractable set of equations we set, as in I, $\mu = 0$ everywhere in Eqs. (4). In the limit $s/s_0 \gg 1$ ($s_0 \simeq 1 \text{ GeV}^2$), Eqs. (4) then reduce to the form

$$F_L(s, \omega) = \frac{1}{32\pi^2} \int_{\omega}^1 \frac{d\omega'}{\omega} \int_0^{\frac{\omega}{\omega'} s} \frac{ds'}{s} s^3 |F(s, s')|^2 \phi\left(\frac{s'}{s}, \frac{\omega}{\omega'}\right) (F_L(s', \omega') + 3 F(s', \omega')) \quad (11a)$$

$$F(s, \omega) = \frac{1}{6} \omega^2 (1 - \omega) \rho(s(1 - \omega)) s^3 |F(s, (1 - \omega)s|^2$$

$$+ \frac{1}{32\pi^2} \int_{\omega}^1 \frac{d\omega'}{\omega} \int_0^{\frac{\omega}{\omega'} s} \frac{ds'}{s} s^3 |F(s, s')|^2. \quad (11b)$$

$$\cdot \left\{ \left(\frac{1}{2} \frac{s'}{s} \left(1 - \frac{s'}{s} \right)^2 - \left(\frac{\omega}{\omega'} \right)^2 \phi\left(\frac{s'}{s}, \frac{\omega}{\omega'}\right) \right) F(s', \omega') + \left(\frac{1}{6} \frac{s'}{s} \left(1 - \frac{s'}{s} \right)^2 - \left(\frac{\omega}{\omega'} \right)^2 \phi\left(\frac{s'}{s}, \frac{\omega}{\omega'}\right) \right) F_L(s', \omega') \right\}$$

where

$$\phi(\eta, z) = \frac{\eta}{z^2} (1-z)(z-\eta) . \quad (11c)$$

The function $\phi[(s'/s), (\omega/\omega')]$, Eq. (11c), appearing in the integrand has an interesting interpretation: we decompose phase space into a "longitudinal" and "transverse" part in a relativistically invariant manner, by introducing Sudakov variables⁴⁾

$$k = x p + y q + \kappa, \quad \kappa \cdot p = \kappa \cdot q = 0$$

where κ is a spacelike vector $\kappa^2 = -\tilde{\kappa}^2 < 0$.

It is then a simple matter to show that (for $\mu = 0$)

$$\frac{\kappa^2}{s} = [x y \omega + y^2] = \phi\left(\frac{s'}{s}, \frac{\omega}{\omega'}\right), \quad (12)$$

where ϕ has been defined in Eq. (11c). Hence Eq. (11a) shows that if one were to introduce a cut-off in $\tilde{\kappa}^2$ -- as has been fashionable in "softened field theoretic models"⁵⁾ -- this would imply a suppression of F_L relative to F , and a Callan-Gross type relation in the case of Bjorken scaling.

3. SOLUTION OF INTEGRAL EQUATIONS

In order to discuss the asymptotic form of the solution to Eqs. (11), we shall again make the replacement (I.5)¹⁾ in the transition form factor $F(s, s')$ for $s' = 0(s_0)$, $s_0 \gg 4\mu^2$, and $s \gg s_0$, where s_0 characterizes the region of integration giving the dominant contribution and thus sets the scale for the asymptotic domain. The general conclusions we shall draw are independent of this simplification. We shall, however, restrict ourselves here to the "scaling" case $\beta = 0$ which gave a Regge-type behaviour for $\rho(s)$ and a logarithmically increasing multiplicity.

With this substitution for $F(s, s')$ the equations may be written in the form

$$\begin{aligned} F_L(s, \omega) &= \lambda \int_{\omega}^1 \frac{d\omega'}{\omega'} \int_0^{\frac{\omega}{\omega'}} d\eta \Phi_1\left(\eta, \frac{\omega}{\omega'}\right) (F_L(\eta s, \omega') + 3 F(\eta s, \omega')) \\ F(s, \omega) &= F_0(s, \omega) + \lambda \int_{\omega}^1 \frac{d\omega'}{\omega'} \int_0^{\frac{\omega}{\omega'}} d\eta \left\{ \Phi_2\left(\eta, \frac{\omega}{\omega'}\right) F(\eta s, \omega') + \Phi_3\left(\eta, \frac{\omega}{\omega'}\right) F_L(\eta s, \omega') \right\}, \end{aligned} \quad (13)$$

where

$$\tilde{F}_0(s, \omega) = 16\pi^2 \lambda \omega^2 (1-\omega) \rho(s(1-\omega)) |h(1-\omega)|^2 \quad (14)$$

and

$$\begin{aligned} \Phi_1(\eta, z) &= 3\eta^2 |h(\eta)|^2 \frac{1}{z^3} (1-z)(\eta-z) \\ \Phi_2(\eta, z) &= \frac{\eta}{z} |h(\eta)|^2 \left[\frac{3}{2} (1-\eta)^2 - 3(1-z)(z-\eta) \right] \\ \Phi_3(\eta, z) &= \frac{\eta}{z} |h(\eta)|^2 \left[\frac{1}{2} (1-\eta)^2 - 3(1-z)(z-\eta) \right] \end{aligned} \quad (15)$$

Equations (13) are of the convolution type with respect to the s -plane and ω -plane Mellin transform, and may thus be reduced to an algebraic set of equations by introducing the double Mellin transform

$$\tilde{F}(j, \xi) = \int_0^1 \frac{d\omega}{\omega} \omega^\xi \int_{s_0}^\infty \frac{ds}{s} \left(\frac{s}{s_0} \right)^{-j} F(s, \omega) \quad (16)$$

We find

$$\begin{aligned} \tilde{F}_L(j, \xi) &= \lambda \tilde{\Phi}_1(j, \xi) [\tilde{F}_L(j, \xi) + 3 \tilde{F}(j, \xi)] \\ \tilde{F}(j, \xi) &= \tilde{F}_0(j, \xi) + \lambda [\tilde{\Phi}_2(j, \xi) \tilde{F}(j, \xi) + \tilde{\Phi}_3(j, \xi) \tilde{F}_L(j, \xi)] \end{aligned} \quad (17)$$

where

$$\tilde{F}_0(j, \xi) = 16\pi^2 \lambda \tilde{\rho}(j) \mathcal{H}(j, \xi) \quad , \quad \tilde{\rho}(j) = \frac{\tilde{\rho}_0(j)}{1 - \lambda H(j)} \quad (18)$$

$$\begin{aligned} \mathcal{H}(j, \xi) &= \int_0^1 d\eta \eta^{1+j} (1-\eta)^{\xi+1} |h(\eta)|^2 \\ \tilde{\Phi}_\alpha(j, \xi) &= \int_0^1 d\eta \eta^j \int_\eta^1 \frac{dz}{z} z^\xi \Phi_\alpha(\eta, z) \quad , \quad \alpha = 1, 2, 3 \end{aligned} \quad (19)$$

Note that $\mathcal{H}(j, 2) = H(j)$, where $H(j)$ is defined by (I.8). Solving the coupled set of equations (17), we recover the desired solution by taking the double inverse Mellin transform. The result for $F(s, \omega)$ is

$$F(s, \omega) = 16\pi^2 \lambda \int_{-i\infty + c'}^{i\infty + c'} \frac{d\mathbf{j}}{2\pi i} \left(\frac{s}{s_0}\right)^{\mathbf{j}} \tilde{\rho}(\mathbf{j}) G(\mathbf{j}, \omega) \quad , \quad (20a)$$

where

$$G(\mathbf{j}, \omega) = \int_{-i\infty + c}^{i\infty + c} \frac{d\xi}{2\pi i} \left(\frac{1}{\omega}\right)^{\xi} \frac{\mathcal{H}(\mathbf{j}, \xi)}{1 - \lambda \tilde{\Psi}(\mathbf{j}, \xi)} \quad (20b)$$

$$\tilde{\Psi}(\mathbf{j}, \xi) = \left[\tilde{\Phi}_2(\mathbf{j}, \xi) + \lambda \frac{\tilde{\Phi}_1(\mathbf{j}, \xi) \tilde{\Phi}_3(\mathbf{j}, \xi)}{1 - \lambda \tilde{\Phi}_1(\mathbf{j}, \xi)} \right]$$

with a corresponding expression for $F_L(s, \omega)$. The real constants c and c' must be chosen such that all singularities always lie to the left of the contours of integration.

4. BEHAVIOUR OF THE STRUCTURE FUNCTIONS

The explicit reconstruction of the function $F(s, \omega)$ from its double Mellin transform Eq. (20) evidently requires a very careful analysis of the singularity structure of $\tilde{F}(\mathbf{j}, \xi)$. One may, however, deduce some general properties of the structure function $F(s, \omega)$ [and $F_L(s, \omega)$] from the representation (20) without having to perform in detail the integrations for some specified form of $h(\eta)$. For the reader interested in details, we explicitly construct in the Appendix the solution to the case involving only spin-zero particles, which turns out to be completely solvable.

a) Bjorken scaling

The integrand of the integral representation (20) contains fixed poles at $j \leq -2$, $\xi \leq -2$ arising from the numerator; fixed poles at $j = \alpha_i(\lambda)$ corresponding to zeros of $1 - \lambda H(j)$ in $\tilde{\rho}(j)$; and moving singularities at $j = \beta_i(\lambda, \xi)$ corresponding to zeros of $1 - \lambda \Psi(j, \xi)$. Let $j = \alpha(\lambda)$ and $j = \beta(\lambda, \xi)$ be the "leading" fixed and moving j -plane singularities, respectively. They satisfy $1 - \lambda H(\alpha) = 0$ and $1 - \lambda \Psi(\beta, \xi) = 0$. Picking up the contribution from just these two singularities, we find

$$F(s, \omega) \sim F(\omega) \left(\frac{s}{s_0}\right)^{\alpha(\lambda)} - 16\pi^2 \int \frac{d\xi}{2\pi i} \left(\frac{1}{\omega}\right)^{\xi} \frac{\tilde{\rho}_0(\beta) \mathcal{H}(\beta, \xi)}{\left(\frac{\partial \tilde{\Psi}(\beta, \xi)}{\partial \beta}\right) (1 - \lambda H(\beta))} \left(\frac{s}{s_0}\right)^{\beta(\lambda, \xi)} \quad (21a)$$

where $\beta \equiv \beta(\lambda, \xi)$,

$$F(\omega) = 16\pi^2\lambda \tilde{\rho}_0(\alpha) \left(\lambda \frac{\partial \alpha}{\partial \lambda} \right) G(\alpha, \omega) , \quad (21b)$$

and $G(j, \omega)$ has been defined in Eq. (20b). In the Bjorken limit $s \rightarrow \infty$, ω fixed, only the first term survives, so that $F(s, \omega)$ satisfies a Feynman scaling law

$$F(s, \omega) \xrightarrow[s \rightarrow \infty]{\omega \text{ fixed}} F(\omega) \left(\frac{s}{s_0} \right)^{\alpha(\lambda)} . \quad (22a)$$

The longitudinal structure function $F_L(s, \omega)$ satisfies a similar scaling law, so that

$$\frac{F_L(s, \omega)}{F(s, \omega)} \xrightarrow[s \rightarrow \infty]{\omega \text{ fixed}} \frac{F_L(\omega)}{F(\omega)} \equiv R(\omega) . \quad (22b)$$

$F(\omega)$ and $R(\omega)$ can be shown to satisfy the coupled set of integral equations

$$R(\omega) = R_0(\omega) + \lambda \int_{\omega}^1 \frac{d\omega'}{\omega'} \left(\phi_1(\alpha, \frac{\omega}{\omega'}) \frac{F(\omega')}{F(\omega)} \right) R(\omega') \quad (23a)$$

$$F(\omega) = F_0(\omega) + \lambda \int_{\omega}^1 \frac{d\omega'}{\omega'} \left\{ \phi_2(\alpha, \frac{\omega}{\omega'}) + \phi_3(\alpha, \frac{\omega}{\omega'}) R(\omega') \right\} F(\omega') \quad (23b)$$

where

$$R_0(\omega) = 3\lambda \int_{\omega}^1 \frac{d\omega'}{\omega'} \phi_1(\alpha, \frac{\omega}{\omega'}) \frac{F(\omega')}{F(\omega)} \quad (24a)$$

and

$$F_0(\omega) = 16\pi^2\lambda \omega^2 (1-\omega)^{1+\alpha} |\mathcal{H}(1-\omega)|^2 \left(\lambda \frac{\partial \alpha}{\partial \lambda} \right) \tilde{\rho}_0(\alpha) . \quad (24b)$$

The functions $\phi_i(\alpha, z)$ are given in terms of the integrals

$$\phi_i(\alpha, z) = \int_0^z d\eta \eta^{\alpha} \Phi_i(\eta, z) , \quad (25)$$

where the functions $\Phi_i(\eta, z)$ have been defined in Eq. (15)

b) $\omega \rightarrow 0$ behaviour

As Eq. (21b) shows, the ω dependence of the scaling function $F(\omega)$ is entirely contained in $G(\alpha, \omega)$. In the limit $\omega \rightarrow 0$ the integral (20b) representing $G(\alpha, \omega)$ is dominated by the nearest singularity in the ξ -plane. It is a trivial matter to check that $1 - \lambda\Psi(\alpha, \xi)$ has a simple zero at $\xi = +2$: $\lambda\Psi(\alpha, 2) = 1$. Picking up the contribution from this singularity, and recalling that $\lambda\mathcal{H}(\alpha, 2) = 1$, we obtain

$$F(\omega) \xrightarrow{\omega \rightarrow 0} 16\pi^2 \tilde{\rho}_0(\alpha) \left(\frac{\partial \alpha}{\partial \lambda} \right) \gamma(\lambda) \omega^{-2}, \quad (26a)$$

where

$$\gamma(\alpha) = \left[- \left(\frac{\partial \Psi(\alpha, \xi)}{\partial \xi} \right)_{\xi=2} \right]^{-1}; \quad (26b)$$

that is, $F(\omega) \sim O(\omega^{-2})$ as $\omega \rightarrow 0$, just as one would expect on the basis of the known logarithmic increase of the mean multiplicity in this model (see I) and the well-known relation

$$\langle n(s) \rangle \rho(s) = \frac{1}{16\pi^2} \int_{\frac{24}{\sqrt{5}}}^1 d\omega \omega F(s, \omega). \quad (27)$$

The corresponding small ω behaviour of $R(\omega)$ may be deduced from the integral equation (23a). If $h(\eta)$ is not too singular at $\eta = 0$, the integral is dominated by $\omega' \simeq O(\omega)$ as $\omega \rightarrow 0$. Hence making the replacement $F(\omega')/F(\omega) \rightarrow \omega^2/\omega'^2$, we find upon taking the Mellin transform,

$$R(\omega) \xrightarrow{\omega \rightarrow 0} \int_{-i\infty+c}^{i\infty+c} \frac{d\xi}{2\pi i} \left(\frac{1}{\omega} \right)^\xi \frac{\tilde{R}_0(\xi)}{1 - \lambda \tilde{\Phi}_1(\alpha, \xi+2)},$$

where $\tilde{\Phi}(j, \xi)$ has been defined in Eq. (19). The $\omega \rightarrow 0$ behaviour is determined by the nearest singularity of $\tilde{R}_0(\xi)$ located at $\xi = 0$. Hence

$$R(\omega) \xrightarrow{\omega \rightarrow 0} \text{const.}$$

There are two important observations to be made with regard to the $\omega \rightarrow 0$ behaviour:

- 1) If we interchange the limit $s \rightarrow \infty$ with the integration in Eq. (27), we find, making use of Eqs. (22a) and (21b),

$$\langle n(s) \rangle \xrightarrow{s \rightarrow \infty} \gamma(\alpha) \log \left(\frac{s}{s_0} \right) . \quad (28)$$

One readily verifies, however, that $\gamma(\alpha) \neq \lambda(\partial\alpha/\partial\lambda)$, so that the result (28) disagrees with the correct result (I.12)⁶⁾.

- 2) The $\omega = 0$ singularity present in $F(\omega)$ is actually absent in $F(s, \omega)$, which means that in the limit $\omega \rightarrow 0$, the contribution (22a) to $F(s, \omega)$ must be cancelled. We have studied this question in detail in the context of the scalar model discussed in the Appendix. The basic cancellation mechanism is, however, of a more general nature, and also applies to the present case. It can best be understood by returning to Eqs. (21). In the limit $\omega \rightarrow 0$, s fixed we may close the ξ -integration contour on a semicircle of infinite radius on the left. Picking up the contribution from the leading singularity at $\xi = 2$, and noting that

$$\frac{\partial \Psi(\beta, \xi)}{\partial \beta} [1 - \lambda H(\beta(\lambda, \xi))] \xrightarrow{\xi \rightarrow 2} \left(\gamma(\alpha) \frac{\partial \alpha}{\partial \lambda} \right)^{-1} (\xi - 2) ,$$

we see that this contribution to the second term in Eqs. (21) exactly cancels the first term.

Both observations (1) and (2) show that the limit $s \rightarrow \infty$ is reached non-uniformly with respect to ω : The scaling function $F(\omega)$ exhibits a singularity at $\omega = 0$ which the actual structure function $F(s, \omega)$ does not contain; and observation (1) showed that the interchange of the limit $s \rightarrow \infty$ with the integration in Eq. (27) was not allowed. This should serve as a word of warning for the manipulations that are commonly done in deducing the high-energy behaviour of $\langle n \rangle$ from the scaling functions alone.

c) Threshold behaviour

From Eqs. (23) we readily deduce the threshold behaviour of the scaling functions:

$$R(\omega) \xrightarrow{\omega \rightarrow 1} R_0(\omega) \sim O((1-\omega)^2)$$

$$F(\omega) \xrightarrow{\omega \rightarrow 1} F_0(\omega) \sim O((1-\omega)^{1+\alpha} / |h(1-\omega)|^2) .$$

The longitudinal scaling function is thus suppressed relative to the transverse one in the threshold region.

d) Moment sum rules

To leading order in s the moment sum rules may be evaluated explicitly. From Eqs. (21b) and (20b) we have

$$s^{-\alpha} \int d\omega \omega^n F(\omega) \xrightarrow{s \rightarrow \infty} \int d\omega \omega^n F(\omega) \\ = 16\pi^2 \lambda \tilde{\rho}_0(\alpha) \left(\lambda \frac{\partial \alpha}{\partial \lambda} \right) \int_{-i\infty+2+0_+}^{i\infty+2+0_+} \frac{d\xi}{2\pi i} \frac{1}{n-\xi+1} \frac{\mathcal{H}(\alpha, \xi)}{1-\lambda \Psi(\alpha, \xi)} .$$

The integrand is found to vanish sufficiently fast, such as to allow closing the contour to the right at infinity. Since for $n \geq 2$ the only singularity we encounter is a pole at $\xi = n+1$, we obtain

$$\int d\omega \omega^n F(\omega) = 16\pi^2 \lambda \tilde{\rho}_0(\alpha) \left(\lambda \frac{\partial \alpha}{\partial \lambda} \right) \frac{\mathcal{H}(\alpha, n+1)}{1-\lambda \Psi(\alpha, n+1)}$$

As a final remark we should like to repeat that the general results (b) and (d) may be explicitly verified in the scalar model discussed in the Appendix. Barring "pathological" choices of $h(\eta)$, there is, however, no reason to doubt that they are also general properties of the model discussed here.

3. DISCUSSION

On the technical side, we have exhibited a powerful, although well-known, technique for reducing the solution of a certain class of integral equations to the solution of an algebraic problem. This technique could also have been applied, within the context of our model, to the discussion of higher-order inclusive e^+e^- annihilation, and hence to the study of correlation functions. The simplicity of the model allowed for a detailed study of some important theoretical questions within a gauge-invariant framework. For reasons that have been outlined in I, we have here restricted our attention to the case of a logarithmically increasing multiplicity.

Summarizing, the following interesting general features have evolved from our study of the inclusive e^+e^- annihilation:

- a) From the general structure of the integral equation (11a), and the identification (12) we conclude that if one was to introduce a cut-off in the transverse momentum $k_\perp^2$ [defined in a relativistically invariant manner by Eq. (12)], as is customary in so-called softened field theories⁵⁾, then we recover in a very natural manner a Callan-Gross type sum rule

$$\frac{F_L(s, \omega)}{F(s, \omega)} \xrightarrow{s \rightarrow \infty} O\left(\frac{1}{s}\right)$$

By enforcing such a transverse momentum cut-off in our model, we would thus recover this particular result of the naive parton model.

- b) The inclusive structure functions satisfy a Feynman scaling law $F(s, \omega) \sim s^\alpha F(\omega)$ for $\omega > 0$ fixed, as was to be expected on the basis of the energy momentum sum rule and the known asymptotic behaviour of $\rho(s)$ (see I).
- c) In the limit $\omega \rightarrow 1$ the scaling functions were found to exhibit the threshold behaviour $R(\omega) \sim (1 - \omega)^2$ and $F(\omega) \sim (1 - \omega)^{\alpha+1} |h(1 - \omega)|^2$, where $\alpha = \alpha(\lambda)$ determined the asymptotic behaviour of $\rho(s)$.

Hence, in the limit $\omega \rightarrow 1$, where the observed pion carries away closely to one-half of the total c.m. energy, the production is dominated by the transverse part of the structure functions.

- d) The scaling function $F(\omega)$ was found to behave like $F(\omega) \sim O(\omega^{-2})$ as $\omega \rightarrow 0$. This again had been expected on the basis of the sum rule for the mean particle multiplicity and the known logarithmic increase of $\langle n(s) \rangle$ (see I). However, $F(s, \omega)$ itself was non-singular at $\omega = 0$. Moreover, we found that

$$\int d\omega \omega F(s, \omega) \not\rightarrow s^\alpha \int d\omega \omega F(\omega)$$

This should serve as a word of warning. The asymptotic behaviour of the mean particle multiplicity is not determined by the scaling function $F(\omega)$ if the integral in the sum rule (27) is dominated by the lower endpoint of the integration. All this is summarized by the statement that the limit $s \rightarrow \infty$, $\omega \rightarrow 0$ is in general a non-uniform limit for the structure function $F(s, \omega)$. In fact, it follows from our discussion in Section 4.2 and Eq. (27) that:

- e) for decreasing values of ω , the scaling limit Eq. (21a) is reached more and more slowly; and, the scaling limit is reached from below.
- f) It follows from our observation (e) and the energy-momentum sum rule

$$\rho(s) = \frac{1}{32\pi^2} \int_{\frac{24}{\sqrt{5}}}^1 d\omega \omega^2 F(s, \omega)$$

that the faster the increase of the multiplicity, the slower the total e^+e^- cross-section reaches its asymptotic value. In particular, moment sum rules reach their asymptotic limit faster, the higher the moment.

- g) Finally, we observe that the full iteration of Eqs. (23) leads to a series expansion for the scaling function of the form

$$F(\omega) = a(1-\omega)^{\alpha+1} \left[\frac{1}{\omega^2} + \sum_{n=0}^{\infty} b_n \omega^n \right],$$

where the $1/\omega^2$ singularity is a result of an infinite number of iterations. (An explicit example is provided by the scalar model discussed in the Appendix, where the series terminates after the first two terms [see Eq. (A.17)]). It should be noted, however, that for any finite number of iterations, the resulting $F(\omega)$ vanishes at both endpoints, $\omega = 0$ and $\omega = 1$. Qualitatively the situation is as shown in Fig. 2: as the number of iterations increases, the maxima are seen to shift toward the left, eventually building up the ω^{-2} peak as $\omega \rightarrow 0$. The explicit solution of the scalar model, Eq. (A.17), provides a typical example for the resulting behaviour. Finally,

- h) one may readily verify for our model, that as the total number of secondaries grows large, π^+ , π^- , and π^0 mesons tend to be produced in equal quantities. Nevertheless, since the pion "radiated" first in the $I = 1$ cascade will always be neutral, a substantially larger fraction of the c.m. energy (a rough estimate suggests $\langle E_{\pi^0} \rangle / \langle E_{\pi^+} \rangle \simeq 2$) is carried away by the neutral pions.

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APPENDIX

THE SCALAR CASE: AN EXPLICITLY SOLUBLE MODEL

In order to have a simple basis for discussing the non-uniform convergence in the limit $s \rightarrow \infty$, $\omega \rightarrow 0$ encountered in Section 2, we discuss here a simple solvable model involving only scalar particles. The corresponding spectral function $\rho(s)$ and inclusive distribution $F(s, \omega) = 2p_0 d\rho(s)/d^3p$ satisfy the integral equations (shown in Fig. 1), which now take the form

$$\rho(s) = \rho_0(s) + \frac{1}{16\pi^2} \int \frac{ds'}{s} \left(1 - \frac{s'}{s}\right) s |\mathcal{F}(s, s')|^2 \rho(s') \quad (\text{A.1})$$

and

$$\mathcal{F}(s, \omega) = |\mathcal{F}(s, s')|^2 \rho(s(1-\omega)) + \frac{1}{16\pi^2} \int_{\omega}^1 \frac{d\omega'}{\omega} \int_0^{\frac{\omega}{\omega'} s} \frac{ds'}{s} \left(\frac{s'}{s}\right) s |\mathcal{F}(s, s')|^2 \mathcal{F}(s', \omega'). \quad (\text{A.2})$$

$F(s, \omega)$ and the mean particle multiplicity $\langle n(s) \rangle$ are then related as follows:

$$\langle n(s) \rangle \rho(s) = \frac{s}{16\pi^2} \int_{\frac{\mu}{\sqrt{s}}}^1 d\omega \omega \mathcal{F}(s, \omega). \quad (\text{A.3})$$

We shall consider only solutions of the multiperipheral type. Since we are interested here in an explicitly soluble model, we replace $s|\mathcal{F}(s, s')|^2$ by a constant $16\pi^2\lambda$.

Then, taking the single and double Mellin transforms of Eqs. (A.1) and (A.2) as in the text, respectively, we find, in the notation of Sections I.2 and II.2,

$$\tilde{\rho}(j) = \frac{\tilde{\rho}_0(j)}{1 - \lambda H(j)} \quad (\text{A.4})$$

$$\tilde{\mathcal{F}}(j, \xi) = 16\pi^2\lambda \frac{\tilde{\rho}_0(j+1)\mathcal{H}(j, \xi)}{(1 - \lambda H(j+1))(1 - \lambda \tilde{\psi}(j, \xi))}, \quad (\text{A.5})$$

where

$$H(j) = \frac{1}{(j+1)(j+2)} \quad (\text{A.6})$$

$$\psi(j, \xi) = \frac{1}{(j+2)(j+\xi+1)} \quad (\text{A.7})$$

and

$$\mathcal{H}(j, \xi) = \frac{\Gamma(\xi)\Gamma(j+2)}{\Gamma(\xi+j+2)} .$$

The leading asymptotic behaviour of $\rho(s)$ is thus determined by the nearest zero of the denominator in Eq. (A.4), located at

$$\alpha = -\frac{3}{2} + \frac{1}{2}\sqrt{1+4\lambda} \quad (\text{A.9})$$

and satisfying the eigenvalue condition $\lambda H(\alpha) = 1$.

We find

$$\rho(s) \xrightarrow{s \rightarrow \infty} \kappa(\lambda) \tilde{\rho}_0(\alpha) \left(\frac{s}{s_0}\right)^{\alpha(\lambda)} , \quad (\text{A.10})$$

where

$$\kappa(\lambda) = \lambda \frac{\partial \alpha}{\partial \lambda} = \frac{\lambda}{\sqrt{1+4\lambda}} . \quad (\text{A.11})$$

Correspondingly,

$$\langle n(s) \rangle \xrightarrow{s \rightarrow \infty} \kappa(\lambda) \log\left(\frac{s}{s_0}\right) . \quad (\text{A.12})$$

$F(s, \omega)$ in turn is obtained by taking the double inverse Mellin transform. We first compute $\tilde{F}(j, \omega)$ from

$$\tilde{F}(j, \omega) = 16\pi^2 \lambda \tilde{\rho}(j+1)(j+1) \int_{-i\infty+c}^{i\infty+c} \frac{d\xi}{2\pi i} \left(\frac{1}{\omega}\right)^\xi B(\xi, j+1) \frac{1}{\xi - \psi(j)} , \quad (\text{A.13a})$$

where

$$\psi(j) = \frac{\lambda - (j+1)(j+2)}{j+2}, \quad (\text{A.13b})$$

and where for fixed j , the contour of integration if chosen to lie to the right of all ξ -plane singularities. The integral may be evaluated by closing the contour to the left on a semicircle of infinite radius:

$$\tilde{F}(j, \omega) = 16\pi^2 \lambda \tilde{\rho}(j+1)(j+1) \left\{ \left(\frac{1}{\omega} \right)^{\psi(j)} B(\psi(j), j+1) - \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(j+1)}{\Gamma(j-n+1)} \frac{1}{\psi(j)+n} \frac{\omega^n}{n!} \right\} \quad (\text{A.14})$$

The infinite series is readily identified with the hypergeometric series associated with

$${}_2F_1(-j, \psi(j); \psi(j)+1; \omega) / \psi(j)$$

The result may thus be written in the form

$$\tilde{F}(j, \omega) = G(j) \left(\frac{1}{\omega} \right)^{\psi(j)} f(j, \omega) \quad (\text{A.15a})$$

where

$$G(j) = 16\pi^2 \lambda \tilde{\rho}(j+1)(j+1) \quad (\text{A.15b})$$

and

$$f(j, \omega) = \int_{\omega}^1 d\omega' \omega'^{\psi(j)-1} (1-\omega')^j. \quad (\text{A.15c})$$

The nearest j -plane singularity of $\tilde{F}(j, \omega)$ corresponds to a simple pole at $j = \alpha - 1$ arising from $\tilde{\rho}(j+1)$, provided $\alpha > 0$. We may thus write

$$F(s, \omega) = \int_{-i\infty+c}^{i\infty+c} \frac{dj}{2\pi i} \left(\frac{s}{s_0} \right)^j \tilde{F}(j, \omega),$$

where $\tilde{F}(j, \omega)$ is given by Eqs. (A.15) and where $c > \alpha$. In the limit $s \rightarrow \infty$, $\omega > 0$ fixed, we may close the contour to the left at infinity, the leading contribution arising from the pole at $j = \alpha$. Hence

$$F(s, \omega) \xrightarrow[s \rightarrow \infty]{\omega > 0} F(\omega) \left(\frac{s}{s_0} \right)^{\alpha-1} \quad (\text{A.16})$$

with

$$F(\omega) = 16\pi^2 \lambda \mathcal{L}(\lambda) \alpha \tilde{\rho}_0(\alpha) \frac{1}{\omega^2} (1-\omega)^\alpha (1+\alpha\omega), \quad (\text{A.17})$$

and where we have made use of the fact that $\psi(\alpha-1) = 2$. The limit (A.16) has been derived for $\omega > 0$, fixed. If we were to interchange the limit $s \rightarrow \infty$ with the integration in the sum rule (A.3) we would find

$$\frac{s}{16\pi^2} \int_{\frac{2k}{\sqrt{s}}}^1 d\omega \omega F(s, \omega) \xrightarrow[s \rightarrow \infty]{} \frac{\lambda \mathcal{L}(\lambda) \tilde{\rho}_0(\alpha)}{\alpha(\lambda) + 1} s^{\alpha(\lambda)} \log\left(\frac{s}{s_0}\right) \quad (\text{A.18})$$

which, comparing with (A.10) and (A.12), is seen to give the wrong result for $\langle n \rangle \rho(s)$. Since in the limit $s \rightarrow \infty$ the integral (A.18) is dominated by the $\omega \rightarrow 0$ region, this means that the limit $s \rightarrow \infty$, $\omega \rightarrow 0$ is reached non-uniformly, so that the exchange of the limit $s \rightarrow \infty$ with the integration in (A.18), to compute $\langle n(s) \rangle \rho(s)$, is in general not permissible if the integral is dominated by the $\omega \rightarrow 0$ region.

In fact, an exact evaluation yields

$$\int_0^1 d\omega \omega^n \tilde{F}(j, \omega) = \frac{G(j) B(n+1, j+1)}{n+1 - \psi(j)},$$

from where we obtain the correct limiting behaviour

$$\frac{s}{16\pi^2} \int d\omega \omega F(s, \omega) \xrightarrow[s \rightarrow \infty]{} [\mathcal{L}(\lambda)]^2 \tilde{\rho}_0(\alpha) \left(\frac{s}{s_0} \right)^\alpha \log\left(\frac{s}{s_0}\right)$$

and for $n \geq 2$

$$\frac{s}{16\pi^2} \int d\omega \omega^n F(s, \omega) \xrightarrow[s \rightarrow \infty]{} \lambda \mathcal{L}(\lambda) \tilde{\rho}_0(\alpha) \alpha \frac{B(n+1, \alpha)}{n-1} \left(\frac{s}{s_0} \right)^\alpha.$$

REFERENCES AND FOOTNOTES

- 1) We shall refer to the preceding paper in this volume as I. The corresponding equations will be prefixed by I.
- 2) We have disregarded here the possibility that the observed pion may correspond to either one of the two pions resulting from the decay of the last link in the chain, which merely corresponds to adding the constant 2 in the average multiplicity as calculated from this integral equation.
- 3) The usual structure functions are obtained by simply multiplying those of Eq. (1) by the coupling of the virtual photon to the "vector meson". To be specific, we shall assume this coupling to be independent of $q^2 = s$.
- 4) V.V. Sudakov, Soviet Phys. JETP 3, 65 (1956).
- 5) P.V. Landshoff and J.C. Polkinghorne, Phys. Reports 5C, 1 (1972).
- 6) Note that since we have not included G parity, we should get twice the answer of Eq. (I.12).

Figure captions

Fig. 1 : Integral equation for the inclusive distribution. The driving term is determined by the spectral function $\rho(s)$ calculated in I.

Fig. 2 : Plot of $F(\omega)$. The dotted lines show the qualitative result of the first four iterations of Eqs. (23). The maximum is seen to shift to the left as the number of iterations increases and eventually builds up the $1/\omega^2$ singularity in the full solution (unbroken curve).

Fig. 1

Fig. 2