BROKEN CONFORMAL WARD IDENTITIES IN NON-ABELIAN GAUGE THEORIES

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A B S T R A C T

Integrated broken conformal Ward identities for non-Abelian gauge theories are derived using a new method depending on dimensional regularization. Apart from terms very closely related to the anomalous terms in the corresponding Callan-Symanzik equations, there are two gauge dependent insertions. These terms, by power counting, are not negligible asymptotically relative to the other terms in the Ward identities for gauge variant Green functions. One of these terms can be predicted classically while the other involves Faddeev-Popov ghosts.

In view of the usefulness of broken scale invariance Ward identities ¹⁾, there has been much interest ²⁾ in the corresponding equations for infinitesimal special conformal transformations ³⁾. Derivations of conformal Ward identities have been given for quartically self-interacting scalar field theories and quantum electrodynamics ²⁾, but applying these methods to asymptotically free ⁴⁾ non-Abelian gauge theories ⁵⁾ is far from obvious. For this reason, we require a new method which is simple and works equally well for the non-Abelian and the ordinary field theories.

The method will consist of first writing conformal Ward identities for unrenormalized but regulated one-particle irreducible Green functions ⁶⁾. Dimensionally regularized ⁷⁾ field theories manifest ultraviolet infinities as poles at $n = 4$, where n is the dimension of space time. For values of n away from rational n the unrenormalized Green functions are finite. Having established these identities in n dimensions, we can invoke multiplicative renormalizability and so obtain Ward identities for the renormalized quantities.

We will collect together all fields into a vector $\{\phi_i\}$. The Ward identities for the dimensionally regulated Green functions are obtained by considering the generating functional ⁶⁾

$$Z[J] = \int \prod_i d\phi_i (\det M[\phi]) \exp i \left(S[\phi] - \frac{1}{2} F_\alpha(\phi) F^\alpha(\phi) + \phi^i J_i \right) \quad (1)$$

where $S[\phi]$ is the action, J_i is the source for ϕ_i and $F_\alpha(\phi)$ is the gauge fixing function. Under a co-ordinate transformation

$$\phi_i = \phi'_i + \left(f \left(x_i, \frac{\partial}{\partial x_i} \right) \phi' \right)_i \quad (2)$$

we have

$$Z[J] = \int \prod_i d\phi'_i \det M[\phi' + f\phi'] \det(1+f) \exp i \left(S[\phi' + f\phi'] - \frac{1}{2} (F_\alpha(\phi' + f\phi'))^2 + J^i (\phi' + f\phi')_i \right) \quad (3)$$

The generating functionals are defined up to possibly infinite factors which are independent of the sources and fields and represent vacuum graphs.

Equating Eqs. (1) and (3), we have, for infinitesimal f ,

$$\int \prod_{a', a, i} \mathcal{D}\phi_i \mathcal{D}c_a \mathcal{D}c_a^\dagger \exp i \left(S[\phi] - \frac{1}{2} (F_\alpha(\phi))^2 + J_i \phi^i + c_a^\dagger M_{ab} c_b \right) \cdot \left(\delta \int d^n x c_a^\dagger(x) M_{ab}(x) c_b(x) + \delta S[\phi] - \frac{1}{2} \delta (F_\alpha(\phi))^2 d^n x + J_i \left(f(x_i, \frac{\partial}{\partial x_i}) \phi_i \right) \right) \quad (4)$$

$$= 0$$

with the symbol δ denoting the effect of an infinitesimal transformation. The Faddeev-Popov-De-Witt ghost fields ⁸⁾, c^a , have been used to give the standard representation for $\det M$. Infinitesimal scale and conformal transformations are given by

$$\delta_s x^\mu = - \varepsilon x^\mu$$

and

$$\delta_c x^\mu = (2x^\mu x^\alpha - g^{\mu\alpha} x^2) \varepsilon_\alpha$$

respectively, where ε and ε_α are infinitesimal parameters. If we denote the variation of a Lagrangian $\mathcal{L}$ in n dimensions ⁹⁾ under infinitesimal scale and conformal transformations by $\varepsilon \delta_s \mathcal{L}$ and $\varepsilon_\alpha \delta_c^\alpha \mathcal{L}$ we have

$$\delta_s \mathcal{L} = \partial_\nu (x^\nu \mathcal{L}) + \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \left(\frac{1}{2}(n-2)+1 \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \phi} \frac{1}{2}(n-2) \phi - n \mathcal{L} \right)$$

$$\delta_c^\alpha \mathcal{L} = \partial_\nu \left((2x^\alpha x^\nu - g^{\alpha\nu} x^2) \mathcal{L} \right) + 2x^\alpha \left(\frac{\partial \mathcal{L}}{\partial \phi} \frac{1}{2}(n-2) \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \left(\frac{1}{2}(n-2)+1 \right) \partial_\mu \phi - n \mathcal{L} \right) \quad (5)$$

$$+ 2 \left(\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi)} \frac{1}{2}(n-2) \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \sum^{\nu\alpha} \phi \right)$$

where $\sum^{\nu\alpha}$ is the spin operator ²⁾. We will now specialize to the case of Yang-Mills fields, A_μ^a , for a compact semi-simple non-Abelian gauge group with gauge coupling g and structure constants f^{abc} . The bare Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \quad (6)$$

We will consider the class of Fermi type gauges given by

$$F_\alpha(A) = \xi^{-\frac{1}{2}} \partial \cdot A_\alpha \quad (7)$$

In order to identify the " β " and " γ " of standard terminology ²⁾, we first obtain the Callan-Symanzik equations. The scale invariance Ward identity in n dimensions for the regulated, one-particle irreducible Green functions is given by

$$\begin{aligned}
 & - \int d^n x \quad g \Gamma \left(\left(\frac{1}{2} (n-4) \frac{\partial}{\partial \mu} c_a^\dagger(x) A_d^\mu(x) c_c(x) f_{acd} \right) \prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \\
 & + \int d^n x \quad \frac{1}{2} g (n-4) \left(\frac{1}{2} F_{\mu\nu}^a(x) A_\mu^b(x) A_\nu^c(x) f_{abc} \prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \\
 & - \sum_{i=1}^l \left(\left(4 - \frac{1}{2} (n-2) \right) \Gamma \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) - \frac{\partial}{\partial x_i} (x_i \Gamma \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \right) \\
 & = 0
 \end{aligned} \tag{8}$$

We make the identification

$$\begin{aligned}
 \left. \frac{\partial}{\partial g} \Gamma \left(\prod_{i=1}^l A_{\mu_i}^{a_i}(x_i) \right) \right|_{\xi} &= i \int d^n x \Gamma \left(\left(\frac{1}{2} F_{\mu\nu}^a(x) A_\mu^b(x) A_\nu^c(x) f_{abc} \right. \right. \\
 & \quad \left. \left. - f_{abc} \frac{\partial}{\partial \mu} c_a^\dagger(x) A_\mu^c(x) c_b(x) \right) \prod_{i=1}^l A_{\mu_i}^{a_i}(x_i) \right)
 \end{aligned} \tag{9}$$

Multiplicative renormalizability of the theory is expressed by

$$\Gamma \left(\prod_{i=1}^l A_{\mu_i}^{a_i}(x_i) \right) = Z_3^{\frac{1}{2}l} \Gamma_R \left(\prod_{i=1}^l A_{\mu_i}^{a_i}(x_i) \right) \tag{10}$$

where Γ_R denotes Green functions of renormalized fields computed with the renormalized Lagrangian. Z_3 is a function of g , ξ , $(n-4)$ and μ where μ is the unit of mass required in dimensional regularization ⁹⁾. Using (10) we have

$$\begin{aligned}
 & Z_3^{-\frac{1}{2}l} g \frac{\partial}{\partial g} \left((n-4) Z_3^{\frac{1}{2}l} \Gamma_R \left(\prod_{i=1}^l A_{\nu_i}^{a_i}(x_i) \right) - i \sum_i \left(- \frac{\partial}{\partial x_i} (x_i \Gamma_R \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \right. \right. \right. \\
 & \quad \left. \left. + (5 - \frac{1}{2}n) \Gamma_R \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \right) \right) \\
 & = 0
 \end{aligned} \tag{11}$$

In the limit $n \rightarrow 4$, $\Gamma_R \left(\prod_{i=1}^l A_{\nu_i}^{a_i}(x_i) \right)$ is finite and so we have

$$\begin{aligned}
 & \left(\frac{1}{2} l \gamma + \beta \frac{\partial}{\partial g_R} + \rho \frac{\partial}{\partial \xi_R} \right) \bar{\Gamma}_R \left(\prod_{i=1}^l A_{\nu_i}^{a_i}(x_i) \right) \\
 & - i \sum_{i=1}^l \left(- \frac{\partial}{\partial x_i} (x_i \bar{\Gamma}_R \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \right) + 3 \bar{\Gamma}_R \left(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j) \right) \\
 & = 0
 \end{aligned} \tag{12}$$

with g_R and ξ_R denoting the renormalized gauge coupling and gauge fixing parameter,

$$\begin{aligned}\beta &= \lim_{n \rightarrow 4} \left(g \frac{\partial g_R}{\partial g} (n-4) \right) \Big|_{\xi, \mu} \\ \gamma &= \lim_{n \rightarrow 4} \left(g \frac{\partial}{\partial g} \log Z_3 (n-4) \right) \Big|_{\xi, \mu} \\ \rho &= \lim_{n \rightarrow 4} \left(g \frac{\partial \xi_R}{\partial g} (n-4) \right) \Big|_{\xi, \mu}\end{aligned}$$

and

$$\bar{\Gamma}_R \left(\prod_i A_{\nu_i}^{a_i}(x_i) \right) = \lim_{n \rightarrow 4} \Gamma_R \left(\prod_i A_{\nu_i}^{a_i}(x_i) \right)$$

This agrees with the results obtained by other procedures (which are usually written in momentum space). For later convenience, we note the relation

$$\frac{\partial}{\partial g_R} \bar{\Gamma}_R \left(\prod_i A_{\nu_i}^{a_i}(x_i) \right) = i \int d^4x \bar{\Gamma}_R \left(O(x) \prod_i A_{\nu_i}^{a_i}(x_i) \right) \quad (13)$$

with

$$\begin{aligned}O(x) &= \frac{1}{2} F_{\mu\nu}^a(x) A_\mu^b(x) A_\nu^c(x) f^{abc} - \frac{1}{4} \frac{\partial}{\partial g_R} Z_3 (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)^2 \\ &\quad + \frac{1}{2} \frac{\partial}{\partial g_R} (g_R(Z_1 - 1)) A_b^\mu A_c^\nu f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + \frac{\partial}{\partial g_R} \tilde{Z}_3 c_a^\dagger \partial^2 c_a \\ &\quad - \frac{1}{4} \frac{\partial}{\partial g_R} (g_R^2(Z_1^2/Z_3 - 1)) (f^{abc} A_b^\mu A_c^\nu)^2 - \frac{\partial}{\partial g_R} (g_R \tilde{Z}_3 Z_1/Z_3) \partial_\mu c_a^\dagger A_\mu^c c_b f^{abc} \\ Z_1 &= Z_3^{3/2} g/g_R\end{aligned}$$

and $\tilde{Z}_3$ as the wave function renormalization of the ghost fields ^{10).}

Similarly

$$\frac{\partial}{\partial \xi_R} \bar{\Gamma}_R \left(\prod_i A_{\mu_i}^{a_i}(x_i) \right) \Big|_{g_R, \mu} = i \int d^4x \Gamma_R (\hat{O}(x) \prod_i A_{\mu_i}^{a_i}(x_i)) \quad (14)$$

where

$$\begin{aligned}\hat{O}(x) &= \frac{1}{2} \xi_R^2 (\partial \cdot A_a)^2 - \frac{1}{4} \frac{\partial}{\partial \xi_R} Z_3 (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)^2 + \frac{1}{2} g_R \left(\frac{\partial}{\partial \xi_R} Z_1 \right) f^{abc} A_\mu^b A_\nu^c (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) \\ &\quad - \frac{1}{4} g_R^2 \frac{\partial}{\partial \xi_R} (Z_1^2/Z_3) (f^{abc} A_b^\mu A_c^\nu)^2 + \frac{\partial}{\partial \xi_R} \tilde{Z}_3 c_a^\dagger \partial^2 c_a - g_R \left(\frac{\partial}{\partial \xi_R} (\tilde{Z}_3 Z_1/Z_3) \right) \partial_\mu c_a^\dagger A_\mu^c c_b f^{abc}\end{aligned}$$

Performing similar analysis for the conformal transformations, we have, corresponding to (8),

$$\begin{aligned}
 & \sum_i (\partial_\nu ((2x_i^\alpha x_i^\nu - g^{\alpha\nu} x_i^2) \Gamma(\prod_j A_{\nu_j}^{a_j}(x_j)) - 2x_i^\delta \delta_{\alpha\nu_i} \Gamma(\prod_{j \neq i} A_{\nu_j}^{a_j}(x_j) A_\delta^{a_i}(x_i)) \\
 & + 2x_{\nu_i}^i \Gamma(\prod_{j \neq i} A_{\nu_j}^{a_j}(x_j) A_\alpha^{a_i}(x_i)) - (n-2)x_i^\alpha \Gamma(\prod_j A_{\nu_j}^{a_j}(x_j))) \\
 & + \int d^n x (2x^\alpha (\frac{1}{4}g(n-4) f^{abc} \Gamma(A_\mu^b(x) A_\nu^c(x) F_{\mu\nu}^a(x) \prod_j A_{\nu_j}^{a_j}(x_j)) \\
 & - \frac{1}{2}g(n-4) f^{abc} \Gamma(\partial_\mu c_a^\dagger(x) A_\mu^c(x) c_b(x) \prod_j A_{\nu_j}^{a_j}(x_j))) \\
 & + \int d^n x (-((n-4) - n/2\xi) \Gamma(A_\alpha^a(x) \partial \cdot A(x) \prod_i A_{\nu_i}^{a_i}(x_i)) - \frac{g(n-2)}{2} f^{abc} \Gamma(c_a^\dagger(x) A_\alpha^c(x) c_b(x) \prod_i A_{\nu_i}^{a_i}(x_i)))
 \end{aligned} \tag{15}$$

Except for the $2x^\alpha$ factor, the second set of terms in (15) is identical to those appearing in (8), the scale invariant case.

Hence we have

$$\begin{aligned}
 & (\sum_{i=1}^l \partial_\nu ((2x_i^\alpha x_i^\nu - g^{\alpha\nu} x_i^2) \bar{\Gamma}_R(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j)) - 2x_i^\delta (1 - \frac{1}{2}i\gamma) \bar{\Gamma}_R(\prod_{j=1}^l A_{\nu_j}^{a_j}(x_j)) \\
 & - 2x_i^\delta \delta_{\alpha\nu_i} \bar{\Gamma}_R(A_{\nu_1}^{a_1}(x_1) \dots A_\delta^{a_i}(x_i) \dots A_{\nu_l}^{a_l}(x_l)) \\
 & + 2x_{\nu_i}^i \bar{\Gamma}_R(A_{\nu_1}^{a_1}(x_1) \dots A_\alpha^{a_i}(x_i) \dots A_{\nu_l}^{a_l}(x_l))) \\
 & + \beta \int d^4 x 2x_\alpha \bar{\Gamma}_R(O'(x) \prod_{i=1}^l A_{\nu_i}^{a_i}(x_i)) \\
 & + \rho \int d^4 x 2x_\alpha \bar{\Gamma}_R((\hat{O}'(x) \prod_{i=1}^l A_{\nu_i}^{a_i}(x_i)) \\
 & + \lim_{n \rightarrow 4} (\int d^n x \Gamma_R(O''(x) \prod_{i=1}^l A_{\nu_i}^{a_i}(x_i))) \\
 & = 0
 \end{aligned} \tag{16}$$

The operator $O''(x)$ is defined as

$$O''(x) = O_1(x) + O_2(x)$$

$$O_1(x) = Z_3 \left(-\frac{1}{2}(n-4) + \frac{n}{2\epsilon} \right) A_a^\alpha(x) \partial \cdot A_a^\alpha(x)$$

$$O_2(x) = -\frac{1}{2} g(n-2) f^{abc} Z_3^{1/2} c_a^\dagger(x) A_\alpha^c(x) c_b(x)$$

and O' and $\hat{O}'$ are the same as O and $\hat{O}$ except for terms which differ from those already in O and $\hat{O}$ by partial integrations ¹¹⁾.

Equation (16) is the broken conformal Ward identity for the theory. We see that, in addition to the classical terms, we have terms dependent on β , γ and ρ . There is, however, an additional piece given by $O''(x)$ which is finite because the scale Ward identity allows us to conclude that all the other terms appearing in (16) are finite. At a non-zero eigenvalue of β and ρ conformal invariance is broken by $O''(x)$. The operator $O_1(x)$ is expected classically and is analogous to the term found in quantum electrodynamics by Nielsen ²⁾. The O_2 term is new, and is a particular characteristic of non-Abelian gauge field theories in the usual Fermi type gauges. It follows from power counting that the gauge variant Green functions with an $O''(x)$ insertion are not asymptotically negligible when compared to the other terms in (16). Consequently, for these Green functions asymptotic scale invariance does not imply conformal invariance. We have found no non-trivial case in which $O''(x)$ annihilates the Green functions into which it is inserted. For example, for the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + h \bar{\psi} \psi \phi + \bar{\psi} i \gamma^\mu D_\mu \psi$$

$$- \bar{\psi} m \psi - \frac{1}{4!} \lambda \phi^4 - \lambda' \phi^3 / 3! - \frac{1}{2} \mu^2 \phi^2$$

$$D_\mu = \partial_\mu - ig \sigma^a A_\mu^a$$

(where in addition to the gauge fields we have a scalar gauge invariant field ϕ and Fermions ψ transforming under a representation σ of the gauge group) $\Gamma(O''(x)\phi(y)\phi(y'))$ is non-zero in low orders of perturbation theory although ϕ is a gauge invariant quantity. This is perhaps not too surprising since we are not working at an eigenvalue of β .

Many details will be given elsewhere ¹¹⁾.

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