SUPER-GAUGE TRANSFORMATIONS ON THE SIX-DIMENSIONAL HYPERCONE

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A B S T R A C T

Super-gauge transformations on physical space-time are defined on the projective six-dimensional hypercone. The basic observation which motivates the present approach is that the product of any two super-gauges is a general conformal $(+ \gamma_5)$ transformation and indeed these latter transformations are simply realized on the six-dimensional hypercone. It turns out that also super-gauge transformations have a natural action on this projective space where the parameters of such transformations are (totally anticommuting) spinors which transform according to the fundamental (eight-dimensional) representation of the spinor group $SU(2,2)$, isomorphic to the conformal group. The enlarged algebra of super-gauges, conformal and chiral transformations and its most relevant representations are discussed in the present formalism.

1- INTRODUCTION

Recently super-gauge transformations have been introduced in physical (four-dimensional) space time ¹⁾ and non-trivial Lagrangian field theories, invariant under this symmetry, have been constructed ²⁾. The resulting Lagrangian interaction appears, in the massless case, as a superposition of renormalizable conformal invariant interactions (Yukawa scalar and pseudoscalar and ϕ^4) where a unique coupling constant is needed, as a consequence of super-gauge invariance. It is well known, on the other hand, that conformal invariant field theories are naturally described making use of a projective space of Minkowski space, the hypercone of a pseudo-Euclidean space in six dimensions. This was observed a long time ago by Dirac ³⁾ in the context of Q.E.D. and was reconsidered recently with the renewed interest of this asymptotic symmetry to exploit properties of short distance phenomena ⁴⁾. Moreover, in the context of renormalization group, this formalism has been proved to be useful in order to investigate properties of the theory at the Gell-Mann-Low eigenvalue ⁵⁾. Motivated by the previous reasons we apply this formalism, in the present paper, to strictly super-gauge invariant (massless) field theories. The interest for this kind of theories should be not only academic, since such theories could be candidate to describe asymptotic Lagrangian for strong interacting particles as well as generalized massless Q.E.D. We consider, in this work, super-gauge transformations on the six-dimensional projective space of Minkowski space and we find that super-gauges are equally simply described on this space as conformal and γ_5 transformations. This seems quite natural because the product of any two super-gauges is in fact a general conformal (rotation on the six-dimensional cone) plus γ_5 transformation. Super-gauges transformations turn out to be generated by a spinor generator which transforms according to the fundamental (eight-dimensional) self-conjugate representation of the spinor group $SU(2,2)$, locally isomorphic to $O(4,2)$, i.e., the pseudorotations in six dimensions. As a consequence they are 8-parameter transformations which are the components of a covariant spinor. The commutator of a super-gauge with a conformal charge expresses nothing but the covariance property of the super gauge charge with respect to $O(4,2)$ rotations. The paper is organized as follows : in section 2, we give the properties of the super-gauge parameters in six dimensions and express conformal $+\gamma_5$ transformations as product of super-gauges. The 24 - parameter extended Lie algebra is given and the correspondence with the usual formulation in Minkowski space is obtained ¹⁾. In section 3, we give the transformation law on the six-dimensional cone of the "scalar representation" (for arbitrary weight) and of the "vector representation" for canonical weight. The

reason why we restrict the weight to be canonical in the second case is because vector multiplets should describe gauge theories where the anomalous dimension is not a useful concept. Of course our formulae, when projected back in Minkowski space are equivalent to those derived by Wess and Zumino ¹⁾ but one hopes that the six-dimensional formalism will provide a better technique in order to find and classify more general representations. Surprisingly enough we find that "the scalar representation" is a representation of the super-gauge algebra in six dimensions up to a geometrical gauge transformation on the spinor field. This resembles the pseudo-conservation of the electromagnetic current in the Adler covariant formulation of massless Q.E.D. ⁵⁾. Section 4 is concerned with super-gauge invariant two-point functions. Finally in two Appendices we report some results on spinor calculus for the group $SU(2,2)$ and some needed technicalities.

2 - THE SUPER-GAUGE ALGEBRA

Let us consider the hypercone $\eta^A \eta_A = 0$ ($\eta^A = g^{AB} \eta_B$) on the six-dimensional pseudo-Euclidean space with metric tensor

$$g_{AB}: g_{AB} = 0 \quad A \neq B, \quad g_{11} = g_{22} = g_{33} = g_{55} = -g_{66} = -g_{66} = 1 \quad (2.1)$$

It is well known that the action of the conformal group in Minkowski space can be identified with the action of the pseudo-orthogonal group $O(4,2)$ on the hypercone ^{3),6)}, provided one identifies the Minkowski coordinates with the points on the hypercone according to the definition :

$$x_\mu = \eta_\mu / \eta_5 + \eta_6 \quad (2.2)$$

It is in fact straightforward to show that an infinitesimal rotation on the points of the cone

$$\delta \eta_A = 2 \Omega_A{}^B \eta_B \quad \Omega_{AB} = -\Omega_{BA} \quad (2.3)$$

induces an infinitesimal conformal transformation on (2.2)

$$\delta x_\mu = \omega_\mu{}^\nu x_\nu + c_\mu + \varepsilon x_\mu + a_\mu x^2 - 2 x_\mu a \cdot x \quad (2.4)$$

where

$$\omega_{\mu\nu} = 2 \Omega_{\mu\nu}, \quad \varepsilon = 2 \Omega_{56}, \quad a_\mu = \Omega_{5\mu} + \Omega_{6\mu}, \quad c_\mu = \Omega_{6\mu} - \Omega_{5\mu} \quad (2.5)$$

The above derivation will be reported, for convenience of the reader, in Appendix A.

In order to introduce super-gauge parameters let us define coordinate independent spinors χ , transforming according to the fundamental self-conjugate representation of the group $SU(2,2)$, locally isomorphic to $O(4,2)$. This representation is eight-dimensional and the generators γ_{AB} are 8×8 matrices in the product space of Dirac and Pauli matrices ^{3),6)}. It is possible and in fact more convenient ¹⁾ to introduce a Majorana representation for such matrices. Moreover a Clifford algebra exists in this representation, i.e., a vector matrix can be defined $\beta_A, ([\beta_A, \gamma_{BC}] = g_{AB}\beta_C - g_{AC}\beta_B)$, such that

$$\{\beta_A, \beta_B\} = 2g_{AB}, \quad \frac{1}{4} [\beta_A, \beta_B] = \gamma_{AB} \quad (2.6)$$

An additional (pseudo-scalar) invariant exists

$$\beta_7 = \frac{1}{6!} \epsilon_{ABCDEF} \beta^A \beta^B \beta^C \beta^D \beta^E \beta^F = \beta_0 \beta_1 \beta_2 \beta_3 \beta_5 \beta_6 \quad (2.7)$$

with the property

$$[\gamma_{AB}, \beta_7] = \{\beta_A, \beta_7\} = 0 \quad (2.8)$$

The explicit expression of matrices $\beta_A, \beta_7, \gamma_{AB}$, which is needed for the rest of the paper, can be found in Appendix A.

The adjoint spinor is given by $\bar{\chi} = \chi A$ where

$$A = \gamma_{60} \quad (2.9)$$

In the chosen representation. We define as parameters of a super-gauge transformation the set of totally anticommuting (real eight-dimensional) spinors

$$\{\chi_\alpha, \chi'_\beta\} = 0 \quad (2.10)$$

transforming according to the above $SU(2,2)$ representation.

Let us now consider two of such spinors

$$\chi_1 = \begin{pmatrix} -\alpha_1^1 \\ \alpha_1^0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} -\alpha_2^1 \\ \alpha_2^0 \end{pmatrix}$$

where α^1, α^0 are the upper and lower components (four-dimensional spinors). Then it is possible to generate an orthogonal $O(4,2)$ transformation with parameters

$$\Omega_{AB} = -2i \bar{\chi}_1 \gamma_{AB} \chi_2 \quad (2.11)$$

when γ_{AB} are the previously defined generators of the spinor representation. In terms of the four-dimensional components, using the known expression of γ_{AB} given by formula (A-8) one gets

$$\begin{aligned} \Omega_{\mu\nu} &= \frac{i}{2} (\bar{\alpha}_2^0 [\gamma_\mu, \gamma_\nu] \alpha_2^1 - \bar{\alpha}_2^1 [\gamma_\mu, \gamma_\nu] \alpha_2^0) \\ \Omega_{56} &= i (\bar{\alpha}_1^0 \alpha_2^1 - \bar{\alpha}_2^0 \alpha_1^1) \\ \Omega_{6\mu} + \Omega_{5\mu} &= 2i \bar{\alpha}_1^1 \gamma_\mu \alpha_2^1 \\ \Omega_{6\mu} - \Omega_{5\mu} &= 2i \bar{\alpha}_1^0 \gamma_\mu \alpha_2^0 \end{aligned} \quad (2.12)$$

Moreover, one can define a pseudoscalar as follows

$$\eta = 4i \bar{\chi}_1 \beta_7 \chi_2 = 4i (\bar{\alpha}_1^1 \gamma_5 \alpha_2^0 - \bar{\alpha}_2^1 \gamma_5 \alpha_1^0) \quad (2.13)$$

The corresponding conformal transformation in Minkowski space (2.4) can be expressed as

$$\delta \chi_\mu = 2i \bar{\alpha}_1(x) \gamma_\mu \alpha_2(x) \quad (2.14)$$

where $\alpha_1(x), \alpha_2(x)$ are the lower components of the spinors χ_1, χ_2 rotated by the matrix $V(x) = (\gamma_{6\mu} + \gamma_{5\mu}) x^\mu$

$$\begin{aligned} \alpha_1(x) &= e^{V(x)} \chi_1 = \alpha_1^0 + \not{x} \alpha_1^1 \\ \alpha_2(x) &= e^{V(x)} \chi_2 = \alpha_2^0 + \not{x} \alpha_2^1 \end{aligned} \quad (2.15)$$

where $\not{x} = x^\mu \gamma_\mu$ and we used the explicit expression for the matrix $\gamma_{6\mu} + \gamma_{5\mu} = \gamma_\mu \sigma_-$ and its nilpotent property $(\gamma_{6\mu} + \gamma_{5\mu})^2 = 0$.

Having this composition rule the abstract super-gauge algebra can be written down. It is a 24-generator extended Lie algebra with the following commutation and anticommutation relations.

$$\begin{aligned}
 [J_{AB}, J_{CD}] &= g_{AD} J_{BC} + g_{BC} J_{AD} - g_{AC} J_{BD} - g_{BD} J_{AC} \\
 [\pi, J_{AB}] &= 0 \\
 [Q_\alpha, \pi] &= \beta_{\gamma\alpha}{}^\beta Q_\beta \\
 [Q_\alpha, J_{AB}] &= \gamma_{AB\alpha}{}^\beta Q_\beta \\
 \{Q_\alpha, Q_\beta\} &= 2(\gamma_{\alpha\gamma}^{AB} J_{AB} + 2\beta_{\gamma\alpha\gamma} \pi) A^\gamma{}_\beta
 \end{aligned} \tag{2.16}$$

when J_{AB} , π , Q_α are the generators of conformal, γ_5 and super-gauge transformations respectively. Note in particular that the last anticommutator reflects the fact that the product of two super-gauges is just a conformal + chiral transformation while the first four commutators have an obvious meaning. In particular the second and third commutator express the covariance property of the spinor generator under conformal + γ_5 transformations.

Having the covariant super-gauge algebra it is easy to write the corresponding formulae in Minkowski space. At this purpose we call $Q_\alpha^0, -Q_\alpha^1$ the upper and lower components of Q_α ($\alpha = 0, 1, 2, 3$). Recalling that

$$J_{6\mu} - J_{5\mu} = K_\mu, J_{6\mu} + J_{5\mu} = P_\mu, I_{56} = D, J_{\mu\nu} = M_{\mu\nu} \tag{2.17}$$

we get

$$\begin{aligned}
 [Q_\alpha^0, P_\mu] &= 0 & [Q_\alpha^0, K_\mu] &= \gamma_{\mu\alpha}{}^\beta Q_\beta^1 \\
 [Q_\alpha^1, P_\mu] &= \gamma_{\mu\alpha}{}^\beta Q_\beta^0 & [Q_\alpha^1, K_\mu] &= 0 \\
 [Q_\alpha^0, \pi] &= \gamma_{5\alpha}{}^\beta Q_\beta^0 & [Q_\alpha^1, \pi] &= -\gamma_{5\alpha}{}^\beta Q_\beta^1 \\
 \{Q_\alpha^0, Q_\beta^0\} &= 2\gamma \cdot P \gamma^\alpha{}_\beta & \{Q_\alpha^1, Q_\beta^1\} &= 2\gamma \cdot K \gamma^\alpha{}_\beta \\
 \{Q_\alpha^0, Q_\beta^1\} &= -2(\gamma^{\mu\nu} M_{\mu\nu} - D + 2\gamma_5 \pi) \gamma^\alpha{}_\beta \\
 \{Q_\alpha^1, Q_\beta^0\} &= -2(\gamma^{\mu\nu} M_{\mu\nu} + D - 2\gamma_5 \pi) \gamma^\alpha{}_\beta
 \end{aligned} \tag{2.18}$$

From (2.18) it follows that the conformal and γ_5 generators can be expressed as bilinear forms in the super-gauges charges :

$$\begin{aligned}
 -\frac{1}{32} \text{Tr} [(Q^0 Q^1 - Q^1 Q^0) \gamma^0 \gamma_5] &= \pi \\
 -\frac{1}{16} \text{Tr} [(Q^0 Q^1 - Q^1 Q^0) \gamma^0] &= D \\
 -\frac{1}{8} \text{Tr} [(Q^0 Q^0 + Q^0 Q^0) \gamma^0 \gamma_\mu] &= P_\mu \\
 -\frac{1}{8} \text{Tr} [(Q^1 Q^1 + Q^1 Q^1) \gamma^0 \gamma_\mu] &= K_\mu \\
 \frac{1}{16} \text{Tr} [(Q^0 Q^1 + Q^1 Q^0) \gamma^0 \gamma_\mu \gamma_\nu] &= M_{\mu\nu}
 \end{aligned} \tag{2.19}$$

And moreover, one has the identities

$$\begin{aligned}
 \frac{1}{4} \gamma^\mu [Q^0, K_\mu] &= Q^1 \\
 \frac{1}{4} \gamma^\mu [Q^1, P_\mu] &= Q^0
 \end{aligned} \tag{2.20}$$

Note that among the commutators (2.16) there are some which have not been reported in (2.18). They are the well-known commutators for the conformal charges and the trivial transformation properties of the chiral and super-gauge charges under dilatation and Lorentz transformations.

3. TRANSFORMATION LAW OF FIELDS

In order to write the transformation law under super-gauge transformations of fields defined on the six-dimensional cone we need some preliminaries. We recall, according to Ref. 6) that an irreducible Lorentz field in Minkowski space $\phi(x)$ is described by a field $\phi(\eta)$ on the cone $\eta^A \eta_A = 0$, transforming according to an irreducible representation of $SU(2,2)$, i.e.,

$$[\Phi(\eta), J_{AB}] = (L_{AB} + S_{AB}) \Phi(\eta) \tag{3.1}$$

where $L_{AB} = \eta_A \partial_B - \eta_B \partial_A$ and S_{AB} are the orbital and spin part of the rotation, respectively. Moreover $\phi(\eta)$ must satisfy a homogeneity property on the cone ⁸⁾

$$(\eta \cdot \partial - \lambda) \Phi(\eta) = 0 \tag{3.2}$$

Correspondence with fields in Minkowski space is obtained as follows

$$\hat{\Phi}(x) = (\eta_5 + \eta_6)^{-\lambda} e^{(S_{6\mu} + S_{5\mu})x^\mu} \Phi(\eta)$$

irreducibility is ensured considering only those components of $\hat{\Phi}(x)$ satisfying

$$K_\mu \Phi(0) = 0 \quad K_\mu = S_{6\mu} - S_{5\mu} \quad (3.3)$$

Note that (3.3) implies that $\hat{\Phi}(0)$ is the lowest dimension Lorentz representation contained in the multicomponent field $\hat{\Phi}(0)$ as $[D, K_\mu] = K_\mu$. Let us consider the case of interest, i.e., a spinor field $\psi(\eta)$. In this case the physical four-dimensional Dirac field is identified with the lower four components of the $SU(2,2)$ spinor. In this representation

$K_\mu = \gamma_{6\mu} - \gamma_{5\mu} = -\gamma_\mu \sigma_+$ then

$$(K_\mu \psi(0))_{\text{lower}} = 0 \quad (3.4)$$

$$\begin{aligned} \psi_{\text{phys.}}(x) &= (\eta_5 + \eta_6)^{-\lambda} e^{(\delta_{6\mu} + \delta_{5\mu})x^\mu} \psi(\eta) = \\ &= e^{-\not{x} \sigma_-} \psi(\eta) (\eta_5 + \eta_6)^{-\lambda} = (1 - \not{x} \sigma_-) \psi(\eta) (\eta_5 + \eta_6)^{-\lambda} \end{aligned} \quad (3.5)$$

The scalar representation of the super-gauge algebra¹⁾ is given by a multiplet of field defined over the six-dimensional cone having the following properties: an eight-dimensional spinor field $\psi(\eta)$ of homogeneity degree⁸⁾ λ , a scalar F and a pseudo-scalar G of homogeneity degree λ , a scalar A and a pseudo-scalar B of homogeneity degree $\lambda + 1$. They have the following transformation law

$$\begin{aligned} \delta A &= -i \bar{\chi} \eta \cdot \beta \psi \\ \delta B &= -i \bar{\chi} \beta_7 \eta \cdot \beta \psi \\ \delta F &= i \bar{\chi} (\gamma^{AB} L_{AB} - 4 - \eta \cdot \partial) \psi \\ \delta G &= i \bar{\chi} \beta_7 (\gamma^{AB} L_{AB} - 4 - \eta \cdot \partial) \psi \\ \delta \psi &= -\beta \cdot \partial (A - \beta_7 B) \chi + (F - \beta_7 G) \chi \end{aligned} \quad (3.6)$$

Note that the transformation law has to be properly chosen in order to respect the homogeneity degree of fields and in such a way that the unphysical (four-

dimensional upper spinor) components of the spinor field are not involved. This is not a trivial fact and will be shown in Appendix B. Performing the product of two super-gauges one gets (for details in the calculations see Appendix B.)

$$\begin{aligned}
 (\delta_1 \delta_2 - \delta_2 \delta_1) A &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} A - 2i \bar{\chi}_2 \beta_7 \chi_1 (\lambda+1) B \\
 (\delta_1 \delta_2 - \delta_2 \delta_1) B &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} B + 2i \bar{\chi}_2 \beta_7 \chi_1 (\lambda+1) A \\
 (\delta_1 \delta_2 - \delta_2 \delta_1) F &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} F + 2i \bar{\chi}_2 \beta_7 \chi_1 (\lambda+4) G \quad (3.7) \\
 (\delta_1 \delta_2 - \delta_2 \delta_1) G &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} G - 2i \bar{\chi}_2 \beta_7 \chi_1 (\lambda+4) F \\
 (\delta_1 \delta_2 - \delta_2 \delta_1) \Psi &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 (L_{AB} + \gamma_{AB}) \Psi + i \bar{\chi}_2 \beta_7 \chi_1 (2\lambda+5) \beta_7 \Psi \\
 &\quad + \text{Gauge terms}
 \end{aligned}$$

where gauge terms :

$$-i [\bar{\chi}_2 \beta_7 \chi_1 \beta_7 \eta \cdot \beta \cdot \partial \Psi - \bar{\chi}_2 \gamma^{AB} \chi_1 \eta \cdot \beta (\beta_A \partial_B - \beta_B \partial_A) \Psi]$$

Note that we called these terms gauge terms as they do not contribute to the lower physical components of the spinor field. In fact physical quantities must be invariant under a geometrical gauge on the spinor field $\psi'(\eta) \rightarrow \psi(\eta) + \eta \beta \chi(\eta)$ because, under this gauge, $\psi'_{\text{lower}}(x) = \psi_{\text{lower}}(x)$.

Let us consider now the vector representation ¹⁾. For canonical dimensions the super-gauge multiplet is described by a scalar, a (anti) symmetric tensor and a spinor field having the same homogeneity degree $\lambda = -2$. Moreover the tensor field G_{AB} must satisfy the supplementary condition ⁶⁾ $\eta^A G_{AB} = 0$.

The super gauge transformation laws are

$$\begin{aligned}
 \delta D &= i \bar{\chi} \beta_7 (\gamma^{AB} L_{AB} - 2) \Psi \\
 \delta \Psi &= -\gamma^{AB} G_{AB} \chi - D \beta_7 \chi \\
 \delta G_{AB} &= i \bar{\chi} (2\gamma_{AB} + \eta^C \gamma_{CB} \partial_A - \eta^C \gamma_{CA} \partial_B) \Psi
 \end{aligned} \quad (3.8)$$

where

$$(2 + \eta \cdot \partial) \Psi = (2 + \eta \cdot \partial) D = (2 + \eta \cdot \partial) G_{AB} = 0$$

Also in this case the transformation laws (3.8) are fixed by the requirements to preserve the homogeneity degree and not to couple physical components to unphysical ones. In particular we observe the property $\eta^{A\delta} G_{AB} = 0$.

4. SUPER-GAUGE INVARIANT PROPAGATORS

In order to construct super-gauge invariant propagators the Ward identities for such transformations are needed together with the form of the propagators on the six-dimensional cone. This is already fixed by $O(4,2)$ covariance.

Let us start with the scalar representation, then

$$\begin{aligned} \langle A(\eta_1) A(\eta_2) \rangle &= C_A (\eta_1 \cdot \eta_2)^{2+1} \\ \langle B(\eta_1) B(\eta_2) \rangle &= C_B (\eta_1 \cdot \eta_2)^{2+1} \\ \langle F(\eta_1) F(\eta_2) \rangle &= C_F (\eta_1 \cdot \eta_2)^2 \\ \langle G(\eta_1) G(\eta_2) \rangle &= C_G (\eta_1 \cdot \eta_2)^2 \\ \langle \psi_\alpha(\eta_1) \psi_\beta(\eta_2) \rangle &= i C_\psi (\eta_1 \cdot \eta_2)^2 \delta_{\alpha\beta} \end{aligned} \quad (4.1)$$

Using the identity

$$\{ \psi S, Q \} = \psi [S, Q] + \{ \psi, Q \} S \quad (4.2)$$

where S is one of A, B, F, G and the commutators

$$\begin{aligned} [A, Q_\alpha] &= -i \bar{\Psi} \eta \cdot \beta \\ [B, Q_\alpha] &= -i \bar{\Psi} \beta \cdot \eta \\ [F, Q_\alpha] &= -i \bar{\Psi} (\delta^{AB} L_{AB} + 4 + \eta \cdot \partial) \\ [G, Q_\alpha] &= i \bar{\Psi} \beta \cdot \eta (\delta^{AB} L_{AB} + 4 + \eta \cdot \partial) \\ [\psi, Q_\alpha] &= -\beta \cdot \partial (A - \beta \cdot B) + (F - \beta \cdot G) \end{aligned} \quad (4.3)$$

One easily gets

$$C_A = C_B = \frac{1}{1+\lambda} C_\psi \quad C_F = C_G = -2(2+\lambda) C_\psi \quad (4.4)$$

Note that if $\lambda = -2$, i.e., canonical dimension : $C_F = C_G = 0$, i.e., the two point functions $\langle FF \rangle$ $\langle GG \rangle$ vanish identically. This implies, by positivity $F = G = 0$, i.e., the free field solutions ¹⁾.

Finally let us consider the vector multiplet. In this case the $O(4,2)$ invariant propagators are

$$\begin{aligned} \langle \psi_\alpha(\eta_1) \psi_\beta(\eta_2) \rangle &= i c_\psi (\eta_1 \eta_2)^{-2} \delta_{\alpha\beta} \\ \langle D(\eta_1) D(\eta_2) \rangle &= c_D (\eta_1 \eta_2)^{-2} \\ \langle G_{AB}(\eta_1) G_{CD}(\eta_2) \rangle &= c_G (\eta_1 \eta_2)^{-4} \{ [\eta'_A \eta_D - (\eta \cdot \eta') g_{AD}] \cdot (4.5) \\ &\quad \cdot [\eta'_B \eta_C - (\eta \cdot \eta') g_{BC}] - [\eta'_A \eta_C - (\eta \cdot \eta') g_{AC}] [\eta'_B \eta_D - (\eta \cdot \eta') g_{BD}] \} \end{aligned}$$

The corresponding expressions in Minkowski space are

$$\begin{aligned} \langle \psi_\alpha(x_1) \psi_\beta(x_2) \rangle &= (-i c_\psi) \not{x}_{12} (x_{12}^2)^{-2} \\ \langle D(x_1) D(x_2) \rangle &= c_D (x_{12}^2)^{-2} \\ \langle G_{\mu\nu}(x_1) G_{\rho\sigma}(x_2) \rangle &= (M_{\mu\rho}(x_{12}) M_{\nu\sigma}(x_{12}) - M_{\mu\sigma}(x_{12}) M_{\nu\rho}(x_{12})) (x_{12}^2)^{-4} c_G \end{aligned} \quad \begin{aligned} x_{12} &= x_1 - x_2 \\ (4.6) \end{aligned}$$

Where

$$M_{\mu\rho}(x_{12}) = 2 x_{12\mu} x_{12\rho} - g_{\mu\rho} x_{12}^2$$

Super-gauge invariance gives in this case

$$c_G = -2 c_\psi \quad c_D = 0 \quad (4.7)$$

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APPENDIX A

Let us consider a pseudo-orthogonal $[0(4,2)]$ transformation on the cone $\eta^A \eta_A = 0$ in six dimensions. We make the following choice of coordinates

$$\eta_5 + \eta_6 = K \quad \eta_\mu / K = x_\mu, \text{ then } \eta_6 - \eta_5 = K x^2 \quad (\text{A.1})$$

Under an infinitesimal change we have

$$\delta \eta_A = \Omega^{BC} (g_{BA} g_{CE} - g_{BE} g_{CA}) \eta^E = 2 \Omega_A{}^B \eta_B \quad (\text{A.2})$$

more explicitly

$$\begin{aligned} \delta \eta_\mu &= 2 \Omega_\mu{}^B \eta_B = 2 \Omega_\mu{}^\nu \eta_\nu + (\Omega_\mu{}^5 + \Omega_\mu{}^6) (\eta_5 + \eta_6) + \\ &+ (\Omega_\mu{}^5 - \Omega_\mu{}^6) (\eta_5 - \eta_6) = 2 \Omega_\mu{}^\nu K x_\nu + \\ &+ (\Omega_\mu{}^5 - \Omega_\mu{}^6) K - (\Omega_\mu{}^5 + \Omega_\mu{}^6) K x^2 \\ \delta \eta_5 &= 2 \Omega_5{}^B \eta_B = 2 \Omega_5{}^6 \eta_6 + 2 \Omega_5{}^\mu \eta_\mu \\ \delta \eta_6 &= 2 \Omega_6{}^B \eta_B = 2 \Omega_6{}^5 \eta_5 + 2 \Omega_6{}^\mu \eta_\mu \\ \delta (\eta_5 + \eta_6) &= \delta K = 2 \Omega_{65} K - 2 (\Omega_{\mu 5} + \Omega_{\mu 6}) \eta^\mu \\ \delta K / K &= -\delta \frac{1}{K} = 2 \Omega_{65} - 2 (\Omega_{\mu 5} + \Omega_{\mu 6}) x^\mu \end{aligned} \quad (\text{A.3})$$

then it follows that

$$\begin{aligned} \delta x_\mu &= \delta (\eta_\mu / K) = \delta \eta_\mu / K - x_\mu \frac{\delta K}{K} = \omega_\mu{}^\nu x_\nu + \\ &+ c_\mu + \epsilon x_\mu + a_\mu x^2 - 2 a \cdot x x_\mu \end{aligned}$$

having defined the parameters of the conformal transformation (A.4) as in Eq. (2.5)

We consider now the eight-dimensional spinor representation of $SU(2,2)$. Its properties were already widely discussed in Ref. 6). However, for our purposes it is convenient to use a Majorana representation for the matrices β_A of the Clifford algebra. We define the components of the six-vector matrix β_A as (A.5) $\beta_A = (\gamma_\mu \sigma_3, -\sigma_1, \sigma_2)$

$$\beta_A = (\gamma_\mu \sigma_3, -\sigma_1, \sigma_2) \quad (\text{A.5})$$

where γ_μ are the Dirac matrices in the Majorana representation and $\sigma_1 = \tau_1, \sigma_2 = i\tau_2, \sigma_3 = \tau_3$ where τ_1, τ_2, τ_3 are the usual Pauli matrices. Note that β_A is now a real matrix. Moreover

$$\{\beta_A, \beta_B\} = 2g_{AB} \quad (A.6)$$

We have the generator matrix

$$\gamma_{AB} = \frac{1}{4} [\beta_A, \beta_B] \quad (A.7)$$

Its 15 components are given by

$$\gamma_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu], \quad \gamma_{\mu 5} = -\frac{1}{2} \gamma_\mu \sigma_2, \quad \gamma_{\mu 6} = \frac{1}{2} \gamma_\mu \sigma_1, \quad \gamma_{56} = \frac{1}{2} \sigma_3 \quad (A.8)$$

of course they verify the following covariance properties (A.9)

$$[\gamma_{AB}, \gamma_{CD}] = g_{AD} \gamma_{BC} + g_{BC} \gamma_{AD} - g_{AC} \gamma_{BD} - g_{BD} \gamma_{AC}, \quad [\beta_A, \gamma_{BC}] = g_{AB} \beta_C - g_{AC} \beta_B$$

The pseudoscalar $\beta_7 = \beta_0 \beta_1 \beta_2 \beta_3 \beta_5 \beta_6$ is given by

$$\beta_7 = \gamma_5 \sigma_3 \quad (A.10)$$

Note in addition that

$$\gamma_{\mu 6} \pm \gamma_{\mu 5} = \gamma_\mu \sigma_\mp \quad (\sigma_\mp = \frac{1}{2} (\sigma_1 \mp \sigma_2)) \quad (A.11)$$

are nilpotent matrices, i.e., their square vanishes. The conjugate spinor $\bar{\chi}$ of χ is given by

$$\bar{\chi} = \chi A \quad (A.12)$$

where $A = \gamma^0 \sigma_1 \quad (A^T = -A \quad A^2 = -1)$

we remind that the spinors are real in the Majorana representation.

A complete set of matrices in the representation space is provided by the following 64 matrices

$$\Gamma_a = I, \beta_7, \beta_A, \beta_A \beta_7, \beta_A \beta_B, \beta_A \beta_B \beta_7, \beta_A \beta_B \beta_C \quad (A.13)$$

$\underset{A < B}{}, \quad \underset{A < B}{}, \quad \underset{A < B < C}{}$

They are all traceless except the identity. Moreover they have the following property :

$$\begin{aligned} A \Gamma_a A &= \Gamma_a^T \quad \text{for } \Gamma_a = \beta_A \beta_B, \beta_7, \beta_A \beta_B \beta_C \\ A \Gamma_a A &= -\Gamma_a^T \quad \text{for } \Gamma_a = I, \beta_A, \beta_A \beta_7, \beta_A \beta_B \beta_7 \end{aligned} \quad (\text{A.14})$$

Define now

$$\Gamma^a = I, \beta_7, \beta^A, -\beta^A \beta_7, -\beta^A \beta^B, -\beta^A \beta^B \beta_7, -\beta^A \beta^B \beta^C \quad (\text{A.15})$$

in such a way that, for fixed a $\Gamma_a^a = 1$.

The following completeness theorem holds for any triple or totally anticommuting spinors χ_1, χ_2, χ_3

$$(\bar{\chi}_1 \chi_2) \chi_3 = -\frac{1}{8} \sum_{a=1}^{64} (\bar{\chi}_1 \Gamma_a \chi_3) \Gamma^a \chi_2 \quad (\text{A.16})$$

This generalizes a well-known property of four component Dirac spinors ¹⁾.

Let us now consider the super-gauge algebra defined by the commutation and anticommutation relations (2.16). We verify some non-trivial Jacobi identities for the algebra. Anticommuting the third commutator in Eq. (2.16) with the super-gauge charge Q_α we get

$$\{[Q_\alpha, J_{AB}], Q_\rho\} = [\{Q_\alpha, Q_\rho\}, J_{AB}] - \{Q_\alpha, [Q_\rho, J_{AB}]\}$$

Using the other commutators one just get an identity as a consequence of the following relations

$$\gamma_{AB} \beta_7 A + \beta_7 A \gamma_{AB}^T = 0$$

$$[\gamma_{AB}, \gamma^{EF}] J_{EF} = \gamma^{EF} [J_{EF}, J_{AB}]$$

Moreover, commuting with the γ_5 generator the last anticommutator in (2.16) one gets

$$[\{Q_\alpha, Q_\rho\}, \pi] = 0 = \{[Q_\alpha, \pi], Q_\rho\} + \{Q_\alpha, [Q_\rho, \pi]\}$$

which is verified as a consequence of the following identity

$$\beta_7 \gamma^{AB} A \bar{J}_{AB} + \gamma^{AB} A \beta_7^T \bar{J}_{AB} = 0$$

Finally, in order to obtain the algebra (2.18) in terms of the usual generators in Minkowski space one has simply to use the correspondence relations given by (2.17) and formulae (A.8) and (A.11).

Let us write the explicit computation for the most interesting part of the super-gauge algebra, i.e., the anticommutators, of the spinor-gauge generators. One has

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2 [\gamma^{\mu\nu} J_{\mu\nu} + (\gamma_5 \gamma_\mu - \gamma_6 \gamma_\mu)(J_6^\mu + J_5^\mu) - \\ &- (\gamma_5^\mu + \gamma_6^\mu)(J_6^\mu - J_5^\mu) - 2\gamma_5 \gamma_6 J_5 + 2\beta_7 \pi] A = \\ &= 2(\gamma^{\mu\nu} M_{\mu\nu} \sigma_1 + \gamma \cdot P \sigma^+ + \gamma \cdot K \sigma^- - D \sigma_2 + 2\gamma_5 \pi \sigma_2) \delta^\circ \end{aligned}$$

where $\sigma^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
are projection operators.

If we call $Q_{\alpha_1}^0 - Q_\alpha^1$ the upper and lower (four-dimensional) components of the eight dimensional spinor charge Q_α one finally gets

$$\begin{aligned} \{Q_\alpha^0, Q_\beta^0\} &= 2 \gamma \cdot P \delta^\circ & \{Q_\alpha^\pm, Q_\beta^\pm\} &= 2 \gamma \cdot K \delta^\circ \\ \{Q_\alpha^0, Q_\beta^\pm\} + \{Q_\alpha^\pm, Q_\beta^0\} &= -[\gamma^\mu, \gamma^\nu] M_{\mu\nu} \delta^\circ \\ \{Q_\alpha^0, Q_\beta^\pm\} - \{Q_\alpha^\pm, Q_\beta^0\} &= 4(D - 2\gamma_5 \pi) \delta^\circ \end{aligned}$$

APPENDIX B

In this last appendix we explicitly evaluate the product of two super-gauge transformations on the "scalar" representation. As a preliminary remark we observe that the covariants which appear in the transformation law (3.6) are uniquely fixed by the homogeneity condition on the fields, i.e.,

$$(\eta \cdot \partial - \lambda - 1)A = (\eta \cdot \partial - \lambda - 1)B = (\eta \cdot \partial - \lambda)F = (\eta \cdot \partial - \lambda)G = (\eta \cdot \partial - \lambda)\psi = \alpha \quad (B.1)$$

together with the requirement that, in the transformation law, only the lower component of the spinor field ψ are involved. This is not trivial as in the transformation law of the scalar fields A, B, F, G $SU(2,2)$ scalar products are involved so the requirement that only the lower components of ψ appear is very stringent. To make an explicit example let us consider the transformation law of the field F. As F and ψ have the same homogeneity degree the covariant which appears in the super-gauge transformation must have a zero homogeneity degree. There are in principle three possible covariants, they are $\gamma^{AB} L_{AB}$, $\eta \cdot \partial$ and a constant. However, we will show in a moment that only the combination which appears in (3.6) has the desired properties.

Let us compute the transformation law at $x = 0$. We remember that

$$\partial_A = \partial/\partial \eta^A = \frac{1}{K} [g_A^\mu + (g_A^6 - g_A^5)x^\mu] \partial_\mu - [g_A^6 - g_A^5] \frac{\partial}{\partial K} + 2\eta_A \frac{\partial}{\partial \eta^\mu} \quad (B.2)$$

As $L_{AB} = \eta_A \partial_B - \eta_B \partial_A$

one also finds

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad L_{56} = x \cdot \partial - K \frac{\partial}{\partial K} \quad (B.3)$$

$$L_{6\mu} + L_{5\mu} = \partial_\mu, \quad L_{6\mu} - L_{5\mu} = x^2 \partial_\mu - 2x_\mu x \cdot \partial + 2\lambda x_\mu$$

$$\gamma^{AB} L_{AB} \psi(\eta) \big|_{x=0} = [(\gamma^{15} + \gamma^{16})(L_{\mu 5} + L_{\mu 6}) + 2\gamma^{56} L_{56}] \psi$$

where

$$\psi(\eta) = K^\lambda (1 + \lambda \sigma^-) \psi(x) \quad (L_{\mu\nu} = 0 = L_{6\mu} + L_{5\mu} \text{ at } x=0)$$

then we get

$$\begin{aligned} & \sigma_+ \gamma \cdot \partial \psi(x) + 4\sigma^+ \psi(x) + \sigma_3 \lambda \psi(x) \big|_{x=0} = \\ & = \begin{pmatrix} \gamma \cdot \partial \psi_{\text{lower}}^{(0)} + 4\psi_{\text{upper}}^{(0)} + \lambda \psi_{\text{upper}}^{(0)} \\ -\lambda \psi_{\text{lower}}^{(0)} \end{pmatrix} \quad (B.4) \end{aligned}$$

where ψ_{upper} , ψ_{lower} are the components of the rotated spinor $\psi(x)$. So we see that in order to eliminate $\psi_{\text{upper}}(0)$ we must use the combination

$$(\gamma^{AB} L_{AB} - 4 - \eta \cdot \partial) \psi = \begin{pmatrix} \gamma \cdot \partial \psi_{\text{lower}} \\ -2(2+\lambda) \psi_{\text{lower}} \end{pmatrix} \quad (\text{B.5})$$

In particular for $\lambda = -2$, i.e., canonical dimensions, the additional covariant becomes (-2) and one just recovers the Dirac operator

$$(\gamma^{AB} L_{AB} - 2) \psi = \begin{pmatrix} \gamma \cdot \partial \psi_{\text{lower}} \\ 0 \end{pmatrix} \quad \text{at } x = 0$$

Finally consider the transformation law of A . In this case one obtains

$$\eta \cdot \beta \psi = -\sigma_+ \psi = \begin{pmatrix} -\psi_{\text{lower}} \\ 0 \end{pmatrix} \quad \text{at } x = 0$$

as required. The transformation law of G and B have of course the same structure as the previous ones.

We explicitly evaluate now the product of the super-gauge transformations on the scalar representation. Using (3.6) we obtain

$$\begin{aligned} (\delta_1 \delta_2 - \delta_2 \delta_1) A &= -i \bar{\chi}_2 \eta \cdot \beta \delta_1 \psi - (1 \leftrightarrow 2) = \\ &= i \bar{\chi}_2 \eta \cdot \beta \beta \cdot \partial \chi_1 A - i \bar{\chi}_2 \eta \cdot \beta \beta \cdot \partial \beta_7 \chi_1 A - \\ &\quad - i \bar{\chi}_2 \eta \cdot \beta \chi_1 F + i \bar{\chi}_2 \eta \cdot \beta \beta_7 \chi_1 G - (1 \leftrightarrow 2) = \\ &= 2i \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} A - 2i \bar{\chi}_2 \beta_7 \chi_1 (A+1) B \end{aligned} \quad (\text{B.6})$$

where we used the identity

$$\eta \cdot \beta \beta \cdot \partial = \gamma^{AB} L_{AB} + \eta \cdot \partial$$

and the properties

$$\begin{aligned} \bar{\chi}_2 \chi_1 &= \bar{\chi}_1 \chi_2, \quad \bar{\chi}_2 \beta_A \chi_1 = \bar{\chi}_1 \beta_A \chi_2, \quad \bar{\chi}_2 \beta_A \beta_7 \chi_1 = \bar{\chi}_1 \beta_A \beta_7 \chi_2, \\ \bar{\chi}_2 \gamma_{AB} \beta_7 \chi_1 &= \bar{\chi}_1 \gamma_{AB} \beta_7 \chi_2, \quad \bar{\chi}_2 \beta_7 \chi_1 = -\bar{\chi}_1 \beta_7 \chi_2, \quad \bar{\chi}_2 \gamma_{AB} \chi_1 = -\bar{\chi}_1 \gamma_{AB} \chi_2 \end{aligned} \quad (\text{B.7})$$

which are consequences of (A.14) and of the anticommuting properties of the spinors χ . Analogously

$$(\delta_1 \delta_2 - \delta_2 \delta_1) B = -i \bar{\chi}_2 \beta_7 \eta \cdot \beta \delta_1 \psi - (1 \leftrightarrow 2) =$$

$$\begin{aligned}
 &= -i\bar{\chi}_2 \beta_7 \eta \cdot \beta \beta \cdot \partial \beta_7 \chi_1 B + i\bar{\chi}_2 \beta_7 \eta \cdot \beta \beta \cdot \partial \chi_1 A - \\
 &- i\bar{\chi}_2 \beta_7 \eta \cdot \beta \chi_1 F + i\bar{\chi}_2 \beta_7 \eta \cdot \beta \beta_7 \chi_1 G - (1 \leftrightarrow 2) = \\
 &= 2i\bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} B + 2i\bar{\chi}_2 \beta_7 \chi_1 (2+1) A \quad (B.8)
 \end{aligned}$$

$$\begin{aligned}
 (\delta_1 \delta_2 - \delta_2 \delta_1) F &= i\bar{\chi}_2 (\gamma^{AB} L_{AB} - 4 - \lambda) \delta_1 \psi - (1 \leftrightarrow 2) = \\
 &= i\bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} F - i\bar{\chi}_2 (4 + \lambda) \chi_1 F - i\bar{\chi}_2 \gamma^{AB} L_{AB} \beta_7 \chi_1 G \\
 &+ i\bar{\chi}_2 \beta_7 \chi_1 (4 + \lambda) G - i\bar{\chi}_2 (\gamma^{AB} L_{AB} - 4 - \eta \cdot \partial) \beta \cdot \partial \chi_1 A + \\
 &+ i\bar{\chi}_2 (\gamma^{AB} L_{AB} - 4 - \eta \cdot \partial) \beta \cdot \partial \beta_7 \chi_1 B - (1 \leftrightarrow 2) = 2i\bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} F + 2i(4 + \lambda) \bar{\chi}_2 \beta_7 \chi_1 G
 \end{aligned}$$

Note that terms like $\gamma^{AB} L_{AB} \beta \cdot \partial A - (1 \leftrightarrow 2)$ do not contribute to (B.8) as two derivatives can saturate only a term containing one β matrix which is even under the interchange $1 \leftrightarrow 2$.

Finally in order to get the transformation on the spinor field we have to use the completeness relation (A.16) together with properties (A.14). Moreover, we observe that under β_7 the matrices Γ_a have the following property

$$\begin{aligned}
 \beta_7 \Gamma_a \beta_7 &= -\Gamma_a \quad \text{for } \Gamma_a = I, \beta_7, \beta_A \beta_B, \beta_A \beta_B \beta_7 \\
 \beta_7 \Gamma_a \beta_7 &= \Gamma_a \quad \text{for } \Gamma_a = \beta_A, \beta_A \beta_7, \beta_A \beta_B \beta_C
 \end{aligned} \quad (B.9)$$

As in the super-gauge transformation of ψ , the scalar plus its pseudo scalar counterpart always appear, (A.14) and (B.9) imply that, performing the commutation $(\delta_1 \delta_2 - \delta_2 \delta_1) \psi$ only those Γ_a survive which are odd both under A and β_7 , i.e., $\Gamma_a = \beta_7, \beta_A \beta_B$. These matrices count four times because they are doubled both from the β_7 operation and from the interchange $(1 \leftrightarrow 2)$.

Performing the calculation we get

$$\begin{aligned}
 (\delta_1 \delta_2 - \delta_2 \delta_1) \psi &= -\beta \cdot \partial (\delta_1 A - \beta_7 \delta_1 B) \chi_2 + \\
 &+ (\delta_1 F - \beta_7 \delta_1 G) \chi_2 - (1 \leftrightarrow 2) = \quad (B.10)
 \end{aligned}$$

$$= i [\beta \cdot \partial \bar{\chi}_1 \eta \cdot \beta \psi \chi_2 - \beta \cdot \partial \bar{\chi}_1 \beta_7 \eta \cdot \beta \psi \beta_7 \chi_2 + \\ + \bar{\chi}_1 (\gamma^{AB} L_{AB} - 4 - \lambda) \psi \chi_2 - \bar{\chi}_1 \beta_7 (\gamma^{AB} L_{AB} - 4 - \lambda) \psi \beta_7 \chi_2] \quad (B.10) \\ - (1 \leftrightarrow 2)$$

We first compute the last two pieces. According to the previous discussion, using the completeness relation (A.16), one has

$$\bar{\chi}_1 (\gamma^{AB} L_{AB} - 4 - \lambda) \psi \chi_2 - \bar{\chi}_1 \beta_7 (\gamma^{AB} L_{AB} - 4 - \lambda) \psi \beta_7 \chi_2 - \\ - (1 \leftrightarrow 2) = -\frac{1}{2} \sum_a \bar{\chi}_1 \Gamma_a \chi_2 \Gamma^a (\gamma^{AB} L_{AB} - 4 - \lambda) \psi = \quad (B.11) \\ = -\frac{1}{2} \bar{\chi}_1 \beta_7 \chi_2 \beta^7 (\gamma^{AB} L_{AB} - 4 - \lambda) \psi + \\ + \frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^A \beta^B (\gamma^{CD} L_{CD} - 4 - \lambda) \psi$$

The first two pieces in (B.10) give

$$\beta \cdot \partial \bar{\chi}_1 \eta \cdot \beta \psi \chi_2 - \beta \cdot \partial \bar{\chi}_1 \beta_7 \eta \cdot \beta \psi \beta_7 \chi_2 - (1 \leftrightarrow 2) = \\ = -\frac{1}{2} \bar{\chi}_1 \beta_7 \chi_2 \beta^A \beta^7 \beta_A \psi - \frac{1}{2} \bar{\chi}_1 \beta_7 \chi_2 \beta^A \beta^7 \eta \cdot \beta \partial_A \psi + \quad (B.12) \\ + \frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^C \beta^A \beta^B \beta_C \psi + \frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^C \beta^A \beta^B \eta \cdot \beta \partial_C \psi$$

Collecting together the terms in $\bar{\chi}_1 \beta_7 \chi_2$ of (B.11) and (B.12) one gets

$$\bar{\chi}_1 \beta_7 \chi_2 \left(-\frac{1}{2} \beta^7 \beta \cdot \eta \eta \cdot \partial \psi + (2 + \lambda) \beta^7 \psi \right) + 3 \bar{\chi}_1 \beta_7 \chi_2 \beta^7 \psi \\ - \frac{1}{2} \bar{\chi}_1 \beta_7 \chi_2 \eta \cdot \beta \beta \cdot \partial \psi + \bar{\chi}_1 \beta_7 \chi_2 \lambda \beta^7 \psi = \quad (B.13) \\ = \bar{\chi}_1 \beta_7 \chi_2 \beta^7 \psi (5 + 2\lambda) - \bar{\chi}_1 \beta_7 \chi_2 \beta^7 \eta \cdot \beta \beta \cdot \partial \psi$$

Let us compute now the contribution to the spin part of the conformal transformation. The remaining term in (B.11) is

$$\frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^A \beta^B [\eta \cdot \beta \beta \cdot \partial - 4 - 2\lambda] \psi \quad (B.14) \\ A < B$$

and the only piece which can give a contribution to the spin part is

$$\frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^A \beta^B (-4) \psi = -4 \bar{\chi}_1 \gamma_{AB} \chi_2 \gamma^{AB} \psi \quad (\text{B.15})$$

$A < B$

The piece which contributes from (B.12) is

$$\frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^C \beta^A \beta^B \beta_C \psi = \frac{1}{4} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^C \beta^A \beta^B \beta_C \psi \quad (\text{B.16})$$

but $A < B$ $A \neq B$

$$\beta^C \beta^A \beta^B \beta_C = (-\beta^A \beta^C + 2g^{AC})(-\beta_C \beta^B + 2g_C^B) = 2\beta^A \beta^B$$

then

$$\frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^A \beta^B \psi = 2 \bar{\chi}_1 \gamma_{AB} \chi_2 \gamma^{AB} \psi$$

collecting together (B.15) and (B.16) we obtain

$$-2 \bar{\chi}_1 \gamma_{AB} \chi_2 \gamma^{AB} \psi = 2 \bar{\chi}_2 \gamma_{AB} \chi_1 \gamma^{AB} \psi \quad (\text{B.17})$$

Finally, we compute the orbital part of the conformal transformation. The remaining terms in (B.12) and (B.13) are

$$\begin{aligned} & \frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^A \beta^B [\eta \cdot \beta \beta \cdot \partial - 2\lambda] \psi + \\ & + \frac{1}{2} \bar{\chi}_1 \beta_A \beta_B \chi_2 \beta^C \beta^A \beta^B \eta \cdot \beta \partial_C \psi \end{aligned} \quad (\text{B.18})$$

$A < B$ $A < B$

Using the identity

$$\begin{aligned} \beta^C \beta^A \beta^B \eta \cdot \beta \partial_C &= -\beta^A \beta^B \eta \cdot \beta \beta \cdot \partial + 2\lambda \beta^A \beta^B + 2(\beta^B \eta \cdot \beta \partial_A - \\ &- \beta^A \eta \cdot \beta \partial_B) = -\beta^A \beta^B \eta \cdot \beta \beta \cdot \partial + 2\lambda \beta^A \beta^B - 2\eta \cdot \beta (\beta_B \partial_A - \beta_A \partial_B) + \\ &+ 4(\eta_B \partial_A - \eta_A \partial_B) \end{aligned}$$

one finally gets from (B.16)

$$2 \bar{\chi}_2 \gamma^{AB} \chi_1 L_{AB} \psi + \bar{\chi}_2 \gamma^{AB} \chi_1 \eta \cdot \beta (\beta_A \partial_B - \beta_B \partial_A) \quad (\text{B.19})$$

Summing up (B.13), (B.17) and (B.19) one recovers the final result.

$$(\delta_1 \delta_2 - \delta_2 \delta_1) \psi = 2i \bar{\chi}_2 \gamma^{AB} \chi_1 (\gamma_{AB} + L_{AB}) \psi + \\ + i \bar{\chi}_2 \beta_7 \chi_1 (5 + 2\lambda) \beta_7 \psi + \text{Gauge terms} \quad (\text{B.20})$$

when the gauge terms are given by

$$i \left[-\bar{\chi}_2 \beta_7 \chi_1 \beta_7 \eta \cdot \beta \beta \cdot \partial \psi + \bar{\chi}_2 \gamma^{AB} \chi_1 \eta \cdot \beta (\beta_A \partial_B - \beta_B \partial_A) \right] \psi \quad (\text{B.21})$$

Note that they do not affect the transformation law of the lower physical components (four-dimensional) spinor field, in fact

$$(\eta \cdot \beta \psi)_{x=0} = -(\sigma_7 \psi)_{x=0} = 0$$

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