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UNIFIED DUAL MODEL FOR INTERACTING OPEN AND CLOSED STRINGS

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A B S T R A C T

We construct the vertices which describe the emission of closed string particles (Pomerons) out of open string states (Reggeons). Using those vertices we can construct a ghost-free unified dual model for the amplitude of an arbitrary number of Reggeons and Pomerons which contains both the conventional Veneziano model and the Shapiro-Virasoro model. The factorization properties of this model are studied in detail. In particular, we extract the Reggeon-Pomeron direct coupling from the channels where both Reggeons and Pomerons can be exchanged.

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1. - INTRODUCTION

Among the ghost-free dual models ¹⁾ the most simple ones are the generalized Veneziano model (GVM) and the Shapiro-Virasoro model (SVM) which have very strong similarities. Both of them have in fact in the spectrum a zero mass particle which is a "photon" in the GVM and a "graviton" in the SVM. This fact fixes the intercept of the Regge trajectory to be $\alpha_0 = 1$ in the case of the first model and $\alpha_0 = 2$ in the SVM. In terms of the vertex operators associated with these zero mass particles, one can construct a new set of harmonic oscillators ²⁾ having only transverse degrees of freedom which generate an infinite number of physical states. In particular, for a critical value of the space-time dimension ($D = 26$ for both models) this new set of oscillators is sufficient to span the entire physical space which is therefore purely transverse ³⁾.

The scattering amplitudes among physical particles can be constructed starting from the vertices associated with the corresponding states which have definite commutation relations with the generators of the Virasoro group ⁴⁾.

The strong connection between these two models is even more evident in the quantum mechanical treatment of a relativistic string ⁵⁾ where the same Lagrangian can be used to describe the physical spectrum of both models. In analogy with the case of a point-like particle, the motion of a string is described by an action which is proportional to the area of the world sheet swept by the string. In this way the oscillators which appear in the dual resonance model (DRM) can be identified with the modes of vibration of a string. In particular it has been shown that the quantization of an open string, i.e., a string with free ends, reproduces all the features of the spectrum of GVM ; the same procedure on a closed string gives back the spectrum of SVM.

The aim of this paper is to build up a unified dual model which describes, besides the usual amplitudes of the GVM and SVM, more general processes like the scattering among closed and open strings.

In order to achieve our goal, we have to introduce an interaction among strings. Now the most relevant feature of the string model is that the physical properties of the free string are uniquely determined by the intrinsic geometry of its world surface. We assume the same property also

for interacting strings, that is we demand that the couplings among strings only depend on the geometrical properties of their world surface. We are then led to a sort of relativity principle for the strings which amounts to be a geometrical formulation of duality : all the physical laws which describe the dynamics of the string must be covariant under the reparametrization group of their world surface. This implies some specific commutation relations of the vertices with the generators of the Virasoro group, as we will see in Section 2.

Two different ways for describing a covariant interaction among strings have been proposed.

The first one describes the splitting of strings using the path integral formalism without the addition of any interaction term to the free string Lagrangian ^{6),7)}. This approach has the good feature of treating all the interacting strings in a quite symmetrical way ; however, the system is described in a Wick rotated frame where τ is imaginary. The path integral procedure which has been successful for reproducing the couplings among physical particles of the conventional models ⁷⁾ (GVM and SMV) has been also so powerful to allow Mandelstam to compute the four-fermion amplitude ⁸⁾.

The second approach introduces the interaction adding to the free Lagrangian an additional term which describes the coupling of a string with an external field ⁹⁾. In this way the dual amplitudes can be easily reproduced in their most manageable form.

In spite of the fact that some particles are treated as external field, and others as string states, there is no real asymmetry between them. In fact it is always possible to change their rôle using the dual properties of the amplitude.

As we have shown in I, the term describing the interaction of a string with an external field is constrained to satisfy the following requirements :

- i) it must preserve the reparametrization invariance of the free Lagrangian ;
- ii) the quantum expression of the interaction Lagrangian must not give rise to any infinity when sandwiched between physical states.

These requirements are in fact so constraining that they seem to fix almost uniquely the form of the possible interaction terms. Actually, we have found in I two kinds of covariant vertices :

- a) a vertex which represents the coupling of an external field (say, for instance, a "strong photon" field) to a current generated by a "charge" located at the ends of an open string ; this kind of vertex describes the coupling of any number of open strings reproducing of course the scattering amplitude of GVM ; the reparametrization invariance of the integration Lagrangian forces the mass spectrum of the external fields to coincide with the one of the open string states ;
- b) a vertex which represents the coupling of an external field with the internal points of the closed string ; this vertex describes the coupling of closed strings reproducing the scattering amplitude of SVM. Again the invariance of the action under reparametrizations forces the mass spectrum of these external fields to coincide with that of a closed string.

The most prominent example of such external fields is given by the field of the "strong graviton" which is coupled to the string as a true graviton ; in I we have shown in fact that the "strong graviton" couples universally with the energy momentum tensor of the string. Taking advantage of this universal character of the graviton, it is easy to write down the coupling of a graviton (state of a closed string) also with an open string ; we have then the first example of a mixed coupling between open and closed strings. Because in the dual model there is no particle with a privileged rôle, it is conceivable that such a kind of coupling does exist for any excited state of a closed string.

In this paper we use the external field approach for constructing a dual scattering amplitude for any number of excited states of closed and open strings.

In our approach an open string particle, denoted by $\mathbb{R}$, is described by a vertex operator $\mathcal{V}_\alpha(x,p)$ which is the usual vertex of GVM associated with a physical state $|\alpha, p\rangle$ of momentum p . The invariance of the action under reparametrization forces $\mathcal{V}_\alpha$ to transform covariantly under the Virasoro operators [see Eq. (2)] with a restriction on the masses of the states : $1 - \alpha' p^2 = n$ (n is an integer ≥ 0).

A closed string state, denoted by $\mathbb{P}$, is described by a vertex operator $\psi_{\alpha}(z, \bar{z}, q) = \psi_{\alpha}(z, q/2) \psi_{\bar{\alpha}}(\bar{z}, q/2)$ (β denotes the two indices α and $\bar{\alpha}$) which satisfies the right commutation relations with the Virasoro operators [see Eq. (2)] with a restriction on the masses of the closed string particles : $2 - (\alpha'/2)q^2 = 2n$ (n is again an integer ≥ 0). The invariance under reparametrization forces the $\mathbb{P}$ particles to lie on a trajectory with twice the intercept and half of the slope of that of the $\mathbb{R}$ particles.

Starting from our amplitude depicted in Fig. 2 and factorizing in various channels, we can prove :

- i) the amplitude can be factorized in all the channels where it is supposed to resonate according to the dual rules and the poles are associated with both open and closed string states ;
- ii) if we factorize in a configuration where one side contains only $\mathbb{P}$ particles, we find both $\mathbb{R}$ and $\mathbb{P}$ poles ; in particular, the residue at the pole of the $\mathbb{P}$ particle factorizes in a finite number of terms containing a product of amplitudes with only $\mathbb{P}$ particles, which coincide with the corresponding amplitude extracted from SVM times again amplitudes containing $\mathbb{P}$ and $\mathbb{R}$ particles ;
- iii) if the value of the mass at the pole allows the $\mathbb{R}$ and $\mathbb{P}$ poles to occur simultaneously then the amplitude shows a double pole which is due to the direct closed-open string transition.

Our amplitude provides then a unified model which includes the GVM and SVM.

The topological structure of our mixed amplitudes is just the same as in the dual loop theory at the critical value of space-time dimension. Indeed in such a case, besides the ordinary Regge trajectory, a new trajectory of factorizable poles with twice the intercept and half of the slope of the ordinary one is present. This has been identified with the Pomeron trajectory. Thus at the critical dimension there are two different sets of asymptotic states : the resonances belonging to the ordinary trajectory, called Reggeons, and the ones belonging to the new trajectory, called Pomerons. From duality it follows that the scattering amplitudes for particles belonging to these two trajectories must have the same location of poles of our mixed amplitudes. Actually the similarity of these two kinds of amplitudes is not only at the level of the pole structure ¹⁰⁾ ; indeed we will

prove in a forthcoming paper ¹¹⁾ that they do coincide everywhere. For this reason, in the following, we call the closed string states Pomerons and the open string states Reggeons.

The paper is organized as follows. In Section 2 we construct the physical vertices associated with the external fields of the closed string. The scattering amplitude of the unified model with Reggeons and Pomerons is obtained in Section 3. Section 4 and 5 are devoted to the factorization, Section 6 to the direct Pomeron-Reggeon transition and finally in Section 7 we discuss amplitudes with excited Pomerons.

2. - PHYSICAL VERTICES

In this section we derive the expression for the physical vertex describing the emission of a closed string from an open one, corresponding to the process depicted in Fig. 1. In order to induce only transitions between physical states of the open string, the vertex has to transform in a covariant way under the fundamental gauge operators of the open string reparametrization group. In order to derive the proper transformation law and the explicit form of the vertex, it is useful to work at first in the classical Lagrangian formalism.

We follow the approach of I and treat the closed string interaction as an external field adding to the free string action S an interaction term of the form

$$S_{int} = \iint d\xi_+ d\xi_- \mathcal{L}_{int}(\xi_+, \xi_-) \quad (2.1)$$

where the light-cone surface co-ordinate $\xi_{\pm} = \tau \pm \sigma$ are used for convenience and the integration domain is the entire world surface of the open string : $0 \leq \sigma \leq \pi$, $\tau_i \leq \tau \leq \tau_f$.

In order to preserve the invariance of the action under reparametrization $\mathcal{L}_{int}$ must transform in a covariant way under the gauge operators. For the sake of simplicity we work in a gauge where

$$\frac{\partial}{\partial \xi_+} x^\mu \frac{\partial}{\partial \xi_-} x_\mu = 0 \quad (2.2)$$

In addition we limit ourselves to transformations which leave invariant the end points of the string ($\sigma = 0, \pi$). As we have already shown in I, the most general transformation which preserves the gauges and leaves invariant the points $\sigma = 0, \pi$ can be written as

$$\delta \xi_{\pm} = \epsilon f(\xi_{\pm}) \quad (2.3)$$

where $f(\xi)$ is periodic of period 2π .

The invariance of the action (2.1) under (2.3) is assured if the Lagrangian density $\mathcal{L}_{int}$ transforms in the following way :

$$\{L_f, \mathcal{L}_{int}(\xi_+, \xi_-)\} = \partial_+ [f(\xi_+) \mathcal{L}_{int}(\xi_+, \xi_-)] + \partial_- [f(\xi_-) \mathcal{L}_{int}(\xi_+, \xi_-)] \quad (2.4)$$

where the gauge operators L_f and the canonical Poisson brackets have been introduced.

The simplest way to satisfy the condition (2.4) is by taking a Lagrangian factorized into a product of two functions of ξ_+ and ξ_- , separately :

$$\mathcal{L}_{int}(\xi_+, \xi_-) = \mathcal{L}^+(\xi_+) \mathcal{L}^-(\xi_-) \quad (2.5)$$

each of them being covariant in its own variable :

$$\{L_f, \mathcal{L}^{\pm}(\xi_{\pm})\} = \partial_{\pm} [f(\xi_{\pm}) \mathcal{L}^{\pm}(\xi_{\pm})] \quad (2.6)$$

As an example, let us consider the "graviton" interaction Lagrangian

$$\mathcal{L}_{grav.} = \partial_+ x_{closed}^{\mu} \partial_- x_{closed}^{\nu} \epsilon_{\mu\nu} e^{iK \cdot x_{closed}} \quad (2.7)$$

which has been used in I for describing a closed string interaction with an external gravitational field. $x_{\text{closed}}(\xi_+, \xi_-)$ represents the free closed string solution and k and ϵ the momentum and the polarization of the graviton. $\mathcal{L}_{\text{grav}}$ transforms covariantly under the conformal closed string group.

If we want now to describe a gravitational field acting on an open string we have to add to the free open string Lagrangian a term of the same form (2.7) but with x_{closed} replaced by the open string free solution x_{open} . Since, however,

$$x_{\text{open}}^{\mu}(\tau, \sigma) = \frac{1}{2} \left[x_{\text{open}}^{\mu}(\xi_+, 0) + x_{\text{open}}^{\mu}(\xi_-, 0) \right] \quad [\text{See I}] \quad (2.8)$$

the Lagrangian (2.7) for the open string factorizes into a product of two photon-like interaction Lagrangians

$$\epsilon_{\mu} \epsilon_{\nu} \partial_+ x^{\mu}(\xi_+, 0) e^{i \frac{k \cdot x}{2}(\xi_+, 0)} \partial_- x^{\nu}(\xi_-, 0) e^{i \frac{k \cdot x}{2}(\xi_-, 0)} \quad (2.9)$$

each of them being covariant under the open string gauge operators and representing the interaction with an external "photon" field of momentum $k/2$ - [$x_{\text{open}}^{\mu} \equiv x^{\mu}$]. As a consequence the Lagrangian (2.9) which describes the interaction of an open string with an external gravitational field, is covariant under the gauge operators. Note that $\partial_+ x^{\mu} \partial_- x^{\nu}$ is the energy-momentum tensor $\theta^{\mu\nu}$ for both closed and open strings ; thus, as expected, the coupling of the graviton to the string is universal.

All the above considerations have been made in the framework of the classical string theory. However, they can be easily generalized to the quantum case. In correspondence with the classical interaction Lagrangian we will have a quantum operator acting on the Fock space of the open string excitations which will depend on the string variables $\xi_{\pm}$ and the harmonic oscillators of the conventional model :

$$\mathcal{L}_{\text{int}}(\xi_+, \xi_-) \rightarrow \mathcal{W}(a, a^+, \xi_+, \xi_-) \quad (2.10)$$

The vertex $\mathcal{W}_{(a,a^+, \xi_+, \xi_-)}$ describes the emission of a closed string from an open one, i.e., in the dual language, the Pomeron vertex. The quantization procedure consists simply in the correspondence between Poisson brackets and commutators. It is now immediate to recognize that, in close analogy with the classical case, a simple way to construct an operator $\mathcal{W}$ satisfying Eq. (2.4) is by taking

$$\mathcal{W}_\beta(z, \bar{z}; \kappa) = \mathcal{V}_\alpha(z, \frac{\kappa}{2}) \mathcal{V}_{\bar{\alpha}}(\bar{z}, \frac{\kappa}{2}) \quad (2.11)$$

where $z = e^{\xi_+}$, $\bar{z} = e^{\xi_-}$ and $\mathcal{V}_\alpha$, $\mathcal{V}_{\bar{\alpha}}$ are the usual on-mass shell Reggeon vertices (with internal quantum numbers α and $\bar{\alpha}$), which are separately covariant under the Virasoro group :

$$[L_f, \mathcal{V}_\alpha(z, \kappa)] = z \frac{d}{dz} (f(z) \mathcal{V}_\alpha(z, \kappa)) \quad (2.12)$$

Notice that $\mathcal{W}$ is not an ordered operator ; the ordering prescription would destroy its covariance. As it will be shown in the following the reordering of $\mathcal{W}$ will give rise to the typical quantum effect of the Pomeron-Reggeon transition. We remark that the on-mass shell physical vertices (2.11) can be obtained starting from the Shapiro-Virasoro vertices and identifying the two sets of oscillators.

It is worth while to notice that the intercept and the slope of the Pomeron trajectory α_{oP} and α'_P are related to the analogous Reggeon quantities α_{oR} and α'_{IR} by

$$\begin{aligned} \alpha_{oP} &= 2 \alpha_{oR} \\ \alpha'_P &= \frac{1}{2} \alpha'_{IR} \end{aligned} \quad (2.13)$$

We have in fact that the momentum k_P associated with an on-mass shell Pomeron vertex is twice the momentum k_R of the two constituent Reggeons

$$k_P = 2 k_R \quad (2.14)$$

On the other hand, looking at the spin n of the Pomeron and Reggeon leading trajectory, we have obviously :

$$n_P = 2 n_R \quad (2.15)$$

Comparing the two trajectories when the two constituent Reggeons are at the level n :

$$\begin{aligned} n &= \alpha_{0R} + \alpha'_{R} \left(\frac{k}{2}\right)^2 \\ 2n &= \alpha_{0P} + \alpha'_{P} k^2 \end{aligned} \quad (2.16)$$

the relations (2.13) are immediately recovered.

Note that Eqs. (2.13) are just the same that relate the parameters of the open and closed string states in the theory of the free string, as it was shown in I.

We conclude then that the spectrum of the external fields which act covariantly inside the string coincides with the spectrum of the closed string states ^{*)}.

3. - CONSTRUCTION OF THE UNIFIED MODEL

The Pomeron vertex obtained in the previous section will be now used to obtain a dual amplitude involving an arbitrary number of Reggeons and Pomerons.

It is useful to make use again of the string picture and to write down the amplitude for the process depicted in Fig. 2, where an initial open string state $|\alpha_{1,p_1}\rangle$ emits at the time τ_i (τ_j) a Reggeon α_i (a Pomeron β_j) of momentum p_i (k_j), and finally, after the emission of $N-1$ Reggeons and M Pomerons jumps in the open string state $|\alpha_N, p_N\rangle$. The total amplitude for this process is

^{*)} Actually one could generalize in a covariant way the Pomeron vertex given in (2.11) by choosing two different momenta for the two component on-shell Reggeons. This more general vertex may be interpreted as a covariant off-shell extension of the Pomeron vertex.

$$\prod_{i=2}^{N-1} \prod_{j=1}^M \int_0^\pi d\sigma_j \int_{-\infty}^\infty d\tau_i d\tau_j \langle \alpha_i p_i | T(\mathcal{H}_{\alpha_i}(\tau_i, p_i) \mathcal{H}_{\beta_j}(\tau_j, \sigma_j; \kappa_j)) | \alpha_N p_N \rangle, \quad (3.1)$$

where T indicates the ordering product with respect to the time τ , and the two interaction Hamiltonians $\mathcal{H}_\alpha, \mathcal{H}_\beta$ are the Reggeon and Pomeron vertices which act respectively at the ends and inside the open string

$$\begin{aligned} \mathcal{H}_\alpha(\tau; p) &= \mathcal{V}_\alpha(e^{-i\tau}, p) \\ \mathcal{H}_\beta(\tau, \sigma; \kappa) &= \mathcal{W}(e^{-i(\tau+\sigma)}, e^{-i(\tau-\sigma)}; \kappa) \end{aligned} \quad (3.2)$$

As a consequence of the invariance of the amplitude with respect to translations in τ we can eliminate one integration by fixing the last interaction time (say, τ_{N-1}) at zero. Furthermore, we can limit ourselves to consider only a set of terms of the time-ordered product (3.1), namely the set in which the Reggeons appear in a given cyclic order (say, 1, 2, ..., N). Indeed this decomposition has a topological character (see Fig. 3), then it is invariant with respect to the Virasoro group. On the contrary, the ordering of the Pomerons, which are coupled inside the string, is not invariant under topological transformations, then we have to keep a time-ordering prescription T^* of the Pomerons among themselves and with respect to the Reggeons.

Our amplitude becomes then

$$\begin{aligned} A(MP, NR) &= \prod_{i=2}^{N-2} \prod_{j=1}^M \int_0^\pi d\sigma_j \int_{-\infty}^0 d\tau_i d\tau_j \mathcal{J}(\tau_{i+1} - \tau_i) \cdot \\ &\cdot \langle \alpha_i p_i | T^* (\mathcal{H}_{\alpha_i}(\tau_i; p_i) \mathcal{H}_{\beta_j}(\tau_j, \sigma_j; \kappa_j) \mathcal{H}_{\alpha_{N-1}}(0; p_{N-1})) | \alpha_N p_N \rangle \end{aligned} \quad (3.3)$$

The full factorization properties of this amplitude will be studied in Section 5. Here we will show the most transparent factorization corresponding to the chain of Reggeon channels drawn in Fig. 2.

Using the time translation invariance, we put

$$\mathcal{H}(\tau) = e^{i\tau L_0} \mathcal{H}(0) e^{-i\tau L_0} \quad (3.4)$$

Then, taking into account the mass-shell condition $(L_0+1)|\alpha_1, p_1\rangle = 0$, we have

$$\begin{aligned} A(MP, NR) &= \prod_{i=2}^{N-2} \prod_{j=1}^M \int_0^\pi d\sigma_j \int_{-\infty}^0 d\tau_i d\tau_j \mathcal{O}(\tau_{i+1} - \tau_i) \cdot \\ &\cdot \langle \alpha_1, p_1 | T^* (\mathcal{H}_{\alpha_2}(0; p_2) e^{-i\tau_i(L_0+1) + i\tau_i L_0} \mathcal{H}_{\alpha_3}(0; p_3) \dots \\ &\dots e^{-i\tau_{N-2} L_0} \mathcal{H}_{\alpha_{N-1}}(0; p_{N-1})) | \alpha_N, p_N \rangle \end{aligned} \quad (3.5)$$

Now the τ integrations can be easily performed. To make the integrals well defined, we can switch adiabatically the interaction multiplying the $\mathcal{H}$'s by a factor $e^{\epsilon\tau}$ and, after the integration, we let $\epsilon \rightarrow 0$. The result is, apart from unessential numerical factors,

$$\begin{aligned} A(MP, NR) &= \langle \alpha_1, p_1 | T^* \left(\mathcal{H}_{\alpha_2}(0; p_2) \frac{1}{L_0+1} \mathcal{H}_{\alpha_3}(0; p_3) \dots \right. \\ &\dots \left. \frac{1}{L_0+1} \int_0^\pi d\sigma \mathcal{H}_{\beta_1}(0, \sigma; p_1) \frac{1}{L_0+1} \dots \frac{1}{L_0+1} \mathcal{H}_{\alpha_{N-1}}(0; p_{N-1}) \right) | \alpha_N, p_N \rangle \end{aligned} \quad (3.6)$$

The factorization properties of the amplitude in the multiperipheral chain of Reggeon channels are now manifest. Clearly the masses of the intermediate states are just the same as the open string resonances. It is also easy to see, as will be shown in Appendix A, that these intermediate states $|\alpha, p\rangle$ satisfy the usual physical conditions $L_n|\alpha, p\rangle = 0$ of the resonances of GVM.

We conclude this section by writing down a different representation of the mixed amplitude (3.3) which is more suitable for comparison with the amplitudes of the conventional dual models.

This new representation can be formally obtained from (3.3) by putting $\tau \rightarrow i\tau$. In terms of the new variables

$$x_i = e^{\tau_i} \quad z_j = e^{\tau_j} e^{i\sigma_j} \quad (3.7)$$

we have

$$A(MP, NR) = \langle \alpha_1, p_1 | \prod_{i=2}^{N-2} \int_0^1 \frac{dx_i}{x_i} \prod_{j=1}^M \int \frac{d^2 z_j}{|z_j|^2} \mathcal{I}(x_{i+1} - x_i) \cdot$$

$$T^* \left(\mathcal{V}_{\alpha_i}(x_i, p_i) \mathcal{W}_{\beta_j}(z_j, \bar{z}_j; k_j) \mathcal{V}_{\alpha_{N-1}}(1; p_{N-1}) \right) | \alpha_N p_N \rangle \quad (3.8)$$

$\bar{z}$ is the complex conjugate of z . The open string variables x_i are integrated over the real axis with $\alpha_{N-1} = 1$, while the closed string variables z_j are integrated over the upper half part of the complex plane (see Fig. 4). The T^* prescription refers now to the ordering of the Pomerons among themselves and with respect to the Reggeons according to the moduli of their variables z and x .

The equivalence of the two representations (3.3) and (3.8) can be simply tested by factorizing, by means of (3.4), the amplitude (3.8). The result is again the factorized amplitude (3.6). In (3.8) the Reggeons 1, $N-1$ and N are treated on a different footing because the corresponding variables are kept fixed to 0, 1, ∞ . However, it is easy to show (Appendix A) that the invariance of the theory under the three-parameter projective group allows one to fix any three real variables. In general, one can write

$$A(MP, NR) = \int \frac{dV}{dV_{abc}} \langle 0 | T^* \left(\prod_{i=1}^N \prod_{j=1}^M \mathcal{V}_{\alpha_i}(x_i, p_i) \mathcal{W}_{\beta_j}(z_j, \bar{z}_j; k_j) \right) | 0 \rangle \quad (3.9)$$

where

$$dV = \prod_{i=1}^N \prod_{j=1}^M \frac{dx_i}{x_i} \frac{d^2 z_j}{|z_j|^2} \tilde{g}(x_{i+1} - x_i) \quad (3.10)$$

The x_i are integrated over the real axis ; the $\prod \tilde{g}$ means that the x_i are ordered in the projective sense (the real axis must be considered as a circle closed at the infinity point). The volume dV_{dbc} in (3.9) depends on what variables are kept fixed and is given for different cases in Appendix A.

4. - ELEMENTARY FACTORIZATION OF THE UNIFIED MODEL

A complete discussion of the factorization properties of the amplitude (3.8) will be the object of Sections 5 and 6.

In this section the same problem is treated in a very simple case, without using the operatorial formalism, in order to give a first intuitive approach to the matter.

When all the external particles are Reggeon or Pomeron ground states, it is very easy to calculate the expectation value in the amplitude (3.9) ; one gets

$$A(MP, NR) = \int \frac{dV}{dV_{abc}} \prod_{N \geq i > j \geq 1} |x_i - x_j|^{2p_i \cdot p_j} \prod_{i=1}^M |\bar{z}_i - z_i|^2 \cdot \prod_{M \geq i > j \geq 1} |(z_i - z_j)(\bar{z}_i - \bar{z}_j)|^{k_i \cdot k_j} \prod_{i=1}^N \prod_{j=1}^M |x_i - z_j|^{2p_i \cdot k_j} \quad (4.1)$$

where the Reggeons have momenta $p_i (p_i^2 = 1)$ and the Pomerons have momenta $k_i (k_i^2 = 4)$. From the expression (4.1) the peculiar pole structure of the mixed amplitude can be directly guessed. Simple poles in the Reggeon channel of momentum $p_{ij} = p_i + p_{i+1} + \dots + p_N + p_1 + \dots + p_j$ (only Reggeons in the right-hand side of Fig. 5) are obtained in the familiar way to the conventional dual model : they are generated by the end point singularity due to the approaching of the variables $x_i, x_{i+1} \dots x_N, x_1 \dots x_j$ to a common value.

In the channel of momentum k equal to the total momentum of the Reggeons (or of the Pomerons)

$$\kappa = - \sum_{i=1}^N p_i = \sum_{j=1}^M \kappa_j \quad (\text{fig. 6})$$

we can produce a pole not only when all the x_i ($i = 1, 2, \dots, N$) collapse together with a fixed x_a but also when all the z_j ($j = 1, 2, \dots, M$) go to a common value z_b ^{*)}. Therefore, using suitable variables, one gets something like

$$\int_0^1 du \int_0^1 dv \, u^{q^2} v^{q^2/2} f(u, v)$$

which explains how a double pole can enter into the game. It is easy to see that an analogous situation occurs in the channel depicted in Fig. 7. In this section we will give a qualitative discussion of the main features of the factorization of Fig. 6. It is here convenient to keep fixed a variable of a Reggeon, say x_1 , and a variable of a Pomeron, say z_1 .

The corresponding dV_{abc} is

$$dV_{abc} = \frac{d^2 z_1 \, dx_1}{|z_1 - x_1|^2 |z_1 - \bar{z}_1|} \quad (4.2)$$

Choosing then $z_1 = i$ and $x_1 = 0$, the effect of dV_{abc} in (4.1) is only the elimination of the integration over z_1 and x_1 . In order to see at first the Pomeron pole, it is convenient to exploit the freedom left by the conformal invariance and perform the projective transformation

*) This is the consequence of the fact that we can fix a complex variable z_b and a real one x_a . This is in fact equivalent to fix three real variables x_a, x_b, x_c (see Appendix A).

$$\begin{aligned} x \rightarrow x' = T(x) &= -\frac{x-i}{x+i} \\ z \rightarrow z' = T(z) &= -\frac{z-i}{z+i} \end{aligned} \quad (4.3)$$

which maps the real axis into the unit circle and the upper half plane into the unit disk. In such a way the variables z_1 and $\bar{z}_1$ corresponding to the Pomeron state of momentum k_1 are transformed respectively into $z'_1 = T(z_1) = 0$ and $\bar{z}'_1 = T(\bar{z}_1) = \infty$; in general, one has

$$T(\bar{z}) = \frac{1}{T(z)} \quad (4.4)$$

After the transformation (4.3) the amplitude (4.1) becomes

$$\begin{aligned} A(MP, NR) &= \int_0^{2\pi} \prod_{i=2}^N d\varphi_i \mathcal{G}(\varphi_i - \varphi_{i-1}) \prod_{N \geq i > j \geq 1} |e^{i\varphi_i} - e^{i\varphi_j}|^{2p_i \cdot p_j} \\ &\cdot \int \prod_{i=2}^M d^2 z_i \prod_{i=2}^M (1 - |z_i|^2)^2 \prod_{M \geq i > j \geq 1} |(z_i - z_j)(1 - z_i \bar{z}_j)|^{k_i \cdot k_j} \\ &\cdot \prod_{i=1}^N \prod_{j=2}^M |1 - z_j e^{i\varphi_i}|^{2p_i \cdot k_j} \end{aligned} \quad (4.5)$$

inside the unit disk

where

$$x'_i = e^{i\varphi_i} \quad i=1, 2, \dots, N \quad \varphi_1 = 0 \quad (4.6)$$

and the primes in the z'_1 have been suppressed. The poles in the variables q^2 can be seen by putting

$$\begin{aligned} z_M = z &\equiv \rho e^{i\varphi} \\ z_i &= \xi_i z \quad i=1, 2, \dots, M \quad (\xi_1=0, \xi_M=1) \end{aligned} \quad (4.7)$$

then one gets :

$$\begin{aligned}
 A(MP, NR) &= \int_0^{2\pi} d\varphi \int_0^1 d\rho \rho^{2M-3+\sum_{M \geq i > j \geq 1} k_i \cdot k_j} \int_0^{2\pi} \prod_{i=2}^N d\varphi_i g(\varphi_i - \varphi_{i-1}) \cdot \\
 &\cdot \int \prod_{i=2}^{M-1} d^2 \xi_i \prod_{i=2}^M (1 - |\xi_i z|^2)^2 \prod_{M \geq i > j \geq 1} |(\xi_i - \xi_j)(1 - \rho^2 \xi_i \bar{\xi}_j)|^{k_i \cdot k_j} \cdot \\
 &\cdot \prod_{i=1}^N \prod_{j=2}^M |1 - z \xi_j e^{i\varphi_i}|^{2 p_i \cdot k_j}
 \end{aligned} \quad (4.8)$$

where the integration on the ξ_i is now extended over the entire complex plane.

In (4.8) the integration over ρ is convergent ^{*)} if

$$\alpha_P(k^2) \stackrel{\text{def}}{=} 2 - \frac{1}{2} k^2 = 2 - 2M - \sum_{M \geq i > j \geq 1} k_i \cdot k_j < 0 \quad (4.9)$$

Moreover by expanding the integrand in powers of z and $\bar{z}$, one sees that poles in q^2 can occur when

$$\alpha_P(k^2) = 2n \quad n = 0, 1, 2, \dots \quad (4.10)$$

[because of the φ integration, only the terms $(z\bar{z})^n = \rho^{2n}$ survive].

In this section we discuss only the term of zeroth order in ρ .
In this case the amplitude has a very simple factorized form :

*) The inequality (4.9) (that can be read $k^2 > 4$) shows that the integral representation (3.9) of the mixed amplitude is well defined only when at least two Reggeons are present. In fact when there is only one Reggeon (on its own mass shell), k^2 is forced to be $-1, 0, 1, 2, \dots$, according to the excitation state which is considered. However, if the Pomeron pole is not actually present ($k^2 \neq -4n$, $n = -1, 0, 1, 2, \dots$) the amplitude can be equally defined by an analytic continuation. If no Reggeon at all appears in the amplitude (3.9), the momentum conservation forces k to vanish ; hence the amplitude is actually divergent. Let us note that $A(MP, OR)$ is topologically different from the SV amplitude for M Pomerons. The latter amplitude is represented by a world surface without boundaries ; the former corresponds to a surface with one boundary. The amplitude $A(MP, OR)$ corresponds topologically to the insertion of a tadpole in the SV amplitude.

$$A^{(0)} = \frac{2\pi}{\alpha_P(k^2)} \mathcal{A} \mathcal{B} \quad (4.11)$$

where

$$\mathcal{A} = \int \prod_{i=2}^{M-1} d^2 \xi_i \prod_{M \geq i > j \geq 1} |\xi_i - \xi_j|^{k_i \cdot k_j} \quad \xi_1 = 0 \quad \xi_M = 1 \quad (4.12)$$

is the SV amplitude for the decay of an off-mass shell ground state Pomeron of momentum k in M on-mass shell ground state Pomerons of momenta k_i ($i = 1, 2, \dots, M$), and

$$\mathcal{B} = \int \prod_{i=2}^N d\varphi_i \delta(\varphi_i - \varphi_{i-1}) \prod_{N \geq i > j \geq 1} |e^{i\varphi_i} - e^{i\varphi_j}|^2 p_i \cdot p_j \quad \varphi_1 = 0 \quad (4.13)$$

is the decay amplitude of the same off-mass shell ground state Pomeron in N ground state Reggeons [we notice that the expression (4.13) coincides with the form (4.5) of the unified model when only one Pomeron is present].

It is amusing to observe that putting $\varphi_i = 2\theta_i$ the amplitude (4.13) becomes

$$\mathcal{B} \propto \int_0^\pi d\theta_N \int_0^{\theta_N} d\theta_{N-1} \dots \int_0^{\theta_3} d\theta_2 \prod_{N \geq i > j \geq 1} [\sin(\theta_i - \theta_j)]^{2 p_i \cdot p_j} \quad (4.14)$$

that coincides with the (non-planar) amplitude for one scalar current that Neveu and Scherk¹²⁾ obtained factorizing the orientable non-planar one-loop dual graph. We expect that the amplitude (4.13) has still poles in k^2 , due to the end point singularity in $\varphi_i = 0$ ($i = 1, 2, \dots, N$). For the sake of simplicity, we discuss only the Pomeron decay in three ground state Reggeons (Fig. 8).

In such a case the integration can be explicitly performed and one gets¹²⁾

$$\mathcal{B} \propto \frac{\Gamma(-\frac{1}{2}\alpha(s)) \Gamma(-\frac{1}{2}\alpha(t)) \Gamma(-\frac{1}{2}\alpha(u))}{\Gamma(-\frac{1}{2}\alpha(s) - \frac{1}{2}\alpha(t)) \Gamma(-\frac{1}{2}\alpha(t) - \frac{1}{2}\alpha(u)) \Gamma(-\frac{1}{2}\alpha(u) - \frac{1}{2}\alpha(s))} \quad (4.15)$$

where

$$\alpha_R = 1 - k^2$$

$$\alpha(s) = 1 - (p_1 + p_2)^2 \quad \alpha(t) = 1 - (p_2 + p_3)^2 \quad \alpha(u) = 1 - (p_3 + p_1)^2 \quad (4.16)$$

In (4.15) the poles in q^2 due to the Pomeron-Reggeon transition appear in the $\Gamma \left[-\frac{1}{2} \alpha_R(k^2) \right]$; moreover the residue of (4.15) at $\alpha_R(q^2) = 0$ is the sum of the Veneziano amplitudes corresponding to the various possible insertions of the Pomeron line between the Reggeon ones (Fig. 9). It is instructive, however, to work out explicitly the k^2 poles directly from the integral representation (4.13). If we go back to the original variables x_i and we invert the projective transformation (4.3), we get

$$B = \left(\int_0^\infty dx_3 \int_0^{x_3} dx_2 + \int_{-\infty}^0 dx_3 \int_0^\infty dx_2 + \int_{-\infty}^0 dx_3 \int_{-\infty}^0 dx_2 \right) |x_2|^{2p_1 p_2} |x_3|^{2p_1 p_3} \cdot |x_2 - x_3|^{2p_2 p_3} |x_2 - i|^{2k \cdot p_2} |x_3 - i|^{2k \cdot p_3} \quad (4.17)$$

Since we are interested in the poles in k^2 (due to the simultaneous vanishing of x_2 and x_3), we limit the integration domain of x_3 to the interval $|x_3| \leq 1$ neglecting in such a way terms regular in k^2 . Then the change of variables

$$|x_3| = x \quad |x_2| = xy \quad (4.18)$$

gives

$$\int_0^1 dx x^{k^2-2} \int_{-\infty}^\infty dy |y|^{2p_1 p_2} |1-y|^{2p_2 p_3} |xy-i|^{2k \cdot p_2} |x-i|^{2k \cdot p_3} \quad (4.19)$$

This expression is convergent if ^{*})

$$\alpha_R(k^2) \equiv 1 - k^2 < 0 \quad (4.20)$$

^{*}) Due to the inequality (4.20), the integral representation (3.9) of the mixed amplitude is well defined only if all the Pomeron momenta are such that $k_i^2 > 1$; hence for on-mass shell Pomerons only, the ground state $k_1^2 = 4$ is allowed. However, the amplitude can be defined also in the general case (see Section 7).

and has poles for *)

$$\alpha_{\mathbb{R}}(\kappa^2) = 0, 2, 4, \dots \quad (4.21)$$

as one can see by expanding the integrand in a power series of x .

Taking into account only the lowest pole, one gets :

$$\begin{aligned} \frac{1}{\alpha_{\mathbb{R}}(\kappa^2)} \left[\int_{-\infty}^0 + \int_0^1 + \int_1^{\infty} \right] dy |y|^{2P_1 \cdot P_2} |1-y|^{2P_2 \cdot P_3} = \\ = \frac{1}{\alpha_{\mathbb{R}}(\kappa^2)} \left[B(-\alpha(u), -\alpha(s)) + B(-\alpha(s), -\alpha(t)) + B(-\alpha(t), -\alpha(u)) \right] \end{aligned} \quad (4.22)$$

that reproduces the beta functions corresponding to the graphs of Figs. 9a, 9b, 9c, respectively.

We have so proved that, at least for ground states, the unified model has good factorization properties ; the factors that enter into the residue reproduce in the various cases the conventional, or the Shapiro-Virasoro or the mixed amplitude. In Sections 5 and 6, this proof will be extended to the most general case.

5. - OPERATORIAL FACTORIZATION OF THE UNIFIED MODEL IN A POMERONIC CHANNEL

The factorization of the mixed amplitudes is quite natural in our approach when all the intermediate states are open strings (Reggeons) as in the multiperipheral configuration of Fig. 2. This kind of factorization has been performed in Section 3. In order to prove the consistency of our amplitudes, we have to factorize them also in the channels where closed strings are exchanged, and that seems a more difficult task, since a double set of oscillators should be needed in such a factorization. A first step in this direction has been done in the last section, where the amplitude of N Reggeon and M Pomeron ground states has been shown to factorize, on the ground state pole of a multi-Pomeron channel, into a SV amplitude and a mixed amplitude.

*) Poles at odd integers are forbidden because in (4.19) the function that multiplies x^{k^2-2} is even under the reflection $x \rightarrow -x$.

In this section, using the operatorial formalism, we shall prove that the amplitude (3.9) can be factorized on each level of the multi-Pomeron channel of Figs. 6 and 7, in terms of closed string states, and that SV amplitudes with one excited leg appear in the residue. Moreover, a double pole is found whenever the Pomeron-Reggeon transition is allowed, and the Pomeron-Reggeon vertex turns out to be a particular case of the three Reggeon vertex.

Since the unified amplitude with on-mass shell excited Pomeron states is actually divergent when the direct Pomeron-Reggeon transition is allowed (see note on p. 18 of Section 4), we discuss at first the amplitude among N Reggeons (in their ground or excited states) and M ground state Pomerons.

Following the suggestions of Section 4, we fix $z_1 = i$ and $x_1 = 0$ in the expression (3.9) of the amplitude.

Then by means of the projective transformation

$$z \rightarrow z' = - \frac{z-i}{z+i} \quad (5.1)$$

we send the real axis into the unit circle and the upper half plane into the unit disk. After the projective transformation (5.1), the two variables $\bar{z}$ and $\bar{z}'$ associated with a Pomeron are no longer complex conjugate of each other but are connected by the relation (see Fig. 10)

$$\bar{z}' = \frac{1}{\bar{z}} \quad (5.2)$$

Hence the modulus ordering requires that each Pomeron vertex $\mathcal{W}(z', \frac{1}{\bar{z}'}; k)$ be split into its constituents $\mathcal{V}(z', \frac{k}{2})$ and $\mathcal{V}(\frac{1}{\bar{z}'}, \frac{k}{2})$; as $|z'_1| \leq |x_1| = 1 \leq |\frac{1}{\bar{z}'_1}|$ all the $\mathcal{V}(z'_i, \frac{k_i}{2})$ must be put on the left of the Reggeon vertices $\mathcal{V}(x'_1, p_1)$ and all the $\mathcal{V}(\frac{1}{\bar{z}'_1}, \frac{k_1}{2})$ on the right. Then the amplitude takes the following form :

$$A(NR, MP) = e^{i \frac{\pi}{2} \kappa^2} \int \prod_{i=2}^M \frac{d^2 z_i}{|z_i|^2} \int_0^{2\pi} \prod_{j=2}^N d\vartheta_j \, \theta(\vartheta_j - \vartheta_{j-1}).$$

$$\cdot \langle -\frac{\kappa}{2} | T(\prod_{i=2}^N \mathcal{V}_{\frac{\kappa}{2}}(z_i)) \mathcal{V}_{p_1} \mathcal{V}_{p_j}(e^{i\vartheta_j}) T(\prod_{i=2}^M \mathcal{V}_{\frac{\kappa}{2}}(\frac{1}{\bar{z}_i})) | \frac{\kappa}{2} \rangle \quad (5.3)$$

where the primes labelling the new variables have been suppressed, z_1 and x_1 have been taken 0 and 1, respectively, and $x_i = e^{i\theta_i}$. k is the total momentum of the Pomerons : $k = -\sum_{i=1}^N p_i = \sum_{i=1}^N k_i$; the internal excitation indices of the Reggeon vertices are understood. Let us remark that the Pomeron of momentum k_1 has gone into the two states at the ends of the brackets, which are finally equal to tachyon states of open strings with momentum $k_1/2$. In fact, since $z_1 = 0$ and consequently $1/\bar{z}_1 = \infty$, we have

$$\lim_{z_1 \rightarrow 0} \langle 0 | \frac{\mathcal{V}(z_1, k_1/2)}{z_1} = \langle -\frac{k_1}{2} | , \quad \lim_{\bar{z}_1 \rightarrow 0} \frac{\mathcal{V}(1/\bar{z}_1, k_1/2)}{\bar{z}_1} | 0 \rangle = | \frac{k_1}{2} \rangle$$

The phase factor in front of the amplitude comes from the reordering of the vertices. We are now ready to factorize in the channel of momentum k . In order to achieve this goal, a complete set of states is inserted on both sides of the set of Reggeon vertices in Eq. (5.3) and, by using the transformation property of the vertices

$$\mathcal{V}(z_i) = z_M^{L_0} \mathcal{V}\left(\frac{z_i}{z_M}\right) z_M^{-L_0} \quad (5.4)$$

the arguments z_i (correspondingly $1/\bar{z}_i$) of all Pomeron vertices are divided by z_M (correspondingly by $1/\bar{z}_M$). We get :

$$\begin{aligned} A(NR, MP) &= e^{i\frac{\pi}{2}q^2} \sum_{\lambda, \mu} \int \prod_{i=2}^M \frac{d^2 z_i}{|z_i|^2} \int_0^{2\pi} \prod_{j=2}^N d\theta_j \theta(\theta_j - \theta_{j-1}) \cdot \\ &\cdot \langle -\frac{k_1}{2} | T\left(\prod_{i=2}^M \mathcal{V}\left(\frac{z_i}{z_M}, \frac{k_i}{2}\right)\right) | \lambda, \frac{q}{2} \rangle \langle \lambda, \frac{q}{2} | z_M^{-L_0} | \lambda, \frac{q}{2} \rangle \cdot \\ &\cdot \langle \lambda, \frac{q}{2} | \mathcal{V}(1, p_1) \mathcal{V}(e^{i\theta_1}, p_1) | \mu, -\frac{q}{2} \rangle \langle \mu, -\frac{q}{2} | \bar{z}_M^{-L_0} | \mu, -\frac{q}{2} \rangle \cdot \\ &\cdot \langle \mu, -\frac{q}{2} | T\left(\prod_{i=2}^M \mathcal{V}\left(\frac{\bar{z}_i}{\bar{z}_M}, \frac{k_i}{2}\right)\right) | k_1/2 \rangle | z_M |^{-2} \quad (5.5) \end{aligned}$$

Let us now make the change of variables

$$\xi_i = \frac{z_i}{z_M} \quad (i=2, 3, \dots, M-1) \quad , \quad \xi_M = 1 \quad (5.6)$$

The amplitude splits then into three pieces

$$\begin{aligned}
 A(NR, MP) &= e^{i\frac{\pi}{2}q^2} \sum_{\lambda, \mu} \int \prod_{i=2}^{M-1} \frac{d^2 \xi_i}{|\xi_i|^2} \langle -\frac{k_1}{2} | T(\prod_{i=2}^M \mathcal{V}(\xi_i, \frac{k_i}{2})) | \lambda, \frac{q}{2} \rangle \cdot \\
 &\cdot \langle \mu, -\frac{q}{2} | T(\prod_{i=2}^M \mathcal{V}(1/\bar{\xi}_i, \frac{k_i}{2})) | \frac{k_1}{2} \rangle \int \frac{d^2 z_M}{|z_M|^2} |z_M|^{-2} \langle \lambda, \frac{q}{2} | \bar{z}_M^{-L_0} | \frac{q}{2}, \lambda \rangle \langle \mu, -\frac{q}{2} | \bar{z}_M^{-L_0} | \mu, -\frac{q}{2} \rangle \cdot \\
 &\cdot \int_0^{2\pi} \prod_{j=2}^N d\theta_j \Theta(\theta_j - \theta_{j-1}) \langle \lambda, \frac{q}{2} | \mathcal{V}(1, p_1) \mathcal{V}(e^{i\theta_j}, p_j) | \mu, -\frac{q}{2} \rangle \quad (5.7)
 \end{aligned}$$

where, because of their definition, the integration on the variables ξ_i 's goes through the whole complex plane.

The product of the two brackets in the first integral can be written as a single scalar product in a space spanned by two commuting sets of oscillators a_n and $\bar{a}_n$. So, by taking into account the relation

$$\langle \mu, -q/2 | T(\prod_{i=2}^M \mathcal{V}(\frac{1}{\xi_i}, \frac{k_i}{2})) | \frac{k_1}{2} \rangle = \langle -\frac{k_1}{2} | T(\prod_{i=2}^M \mathcal{V}(\bar{\xi}_i, \frac{k_i}{2})) | \mu, q/2 \rangle \quad (5.8)$$

that follows from

$$\mathcal{V}_p(z)^\dagger = \mathcal{V}_{-p}(\frac{1}{\bar{z}}) \quad , \quad \mathcal{V}_p(z)^* = \mathcal{V}_{-p}(\bar{z})$$

the first piece of the amplitude becomes

$$\int \prod_{i=2}^{M-1} \frac{d^2 \xi_i}{|\xi_i|^2} \langle -\frac{k_1}{2} | \langle -\frac{k_1}{2} | T(\prod_{i=2}^M \mathcal{V}_{(\xi_i, \frac{k_i}{2})}^{(a)} \mathcal{V}_{(\bar{\xi}_i, \frac{k_i}{2})}^{(\bar{a})})) | \lambda, \frac{q}{2} \rangle_a | \mu, \frac{q}{2} \rangle_{\bar{a}} \quad (5.9)$$

The structure of the SVM is now apparent ; in fact, the above expression is the scalar product of the state $|\lambda, q/2\rangle_a |\mu, q/2\rangle_{\bar{a}}$ with the coherent state of the M interacting Pomerons of momenta k_i 's. The form of that coherent state is the one characteristic of the SVM :

$$\langle -k_1, -k_2, \dots, -k_M | = \int \prod_{i=2}^{M-1} \frac{d^2 \xi_i}{|\xi_i|^2} \langle -\frac{k_1}{2} | \langle -\frac{k_1}{2} | T(\prod_{i=2}^M \mathcal{V}_{(\xi_i, \frac{k_i}{2})}^{(a)} \mathcal{V}_{(\bar{\xi}_i, \frac{k_i}{2})}^{(\bar{a})})) \quad (5.10)$$

and the quantity (5.9) coincides, when $|\lambda, q/2\rangle_a |\mu, q/2\rangle_{\bar{a}}$ is a physical on-shell state, with a SV amplitude with one excited leg.

The second piece of Eq. (5.7) can be treated in the same way and the integration on z_M can be done explicitly. The result is :

$$\int_{|z_M| < 1} \frac{d^2 z_M}{|z_M|^2} |z_M|^{-2} \langle \lambda, \frac{q}{2} | z_M^{-L_0} | \lambda, \frac{q}{2} \rangle \langle \mu, -\frac{q}{2} | \bar{z}_M^{-\bar{L}_0} | \mu, -\frac{q}{2} \rangle =$$

$$= - \langle \lambda, \frac{q}{2} | \langle \mu, \frac{q}{2} | \frac{2i\pi(L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} \frac{1}{L_0 + \bar{L}_0 + 2} | \lambda, \frac{q}{2} \rangle_a | \mu, \frac{q}{2} \rangle_{\bar{a}} \quad (5.11)$$

which is just the propagator of SV states ¹³⁾ (L_0 and $\bar{L}_0$ act respectively in the space of the oscillators a and $\bar{a}$). It requires that $\langle \lambda |$ and $\langle \mu |$ have the same eigenvalue of $R = \sum_n n a_n^+ a_n$ [otherwise the factor $(\sin \pi(L_0 - \bar{L}_0))/(L_0 - \bar{L}_0)$ vanishes], and it shows poles at

$$2 - \frac{1}{2} R^2 = 2m \quad (5.12)$$

where m is the eigenvalue of R , acting on $|\lambda\rangle$ and $|\mu\rangle$.

Summarizing, the amplitude (5.7) can be rewritten in the form :

$$A(NR, MP) = \sum_{\lambda, \mu} \langle -k_1, -k_2, \dots, -k_N | \lambda, \mu; q \rangle_{a\bar{a}} \langle \lambda, \mu, q | \frac{2i\pi(L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} \frac{1}{L_0 + \bar{L}_0 + 2} | \lambda, \mu, q \rangle_{a\bar{a}}$$

$$\cdot \int \prod_{j=2}^{2\pi N} d\vartheta_j \theta(\vartheta_j - \vartheta_{j-1}) e^{i\frac{\pi}{2} q^2} \langle \lambda, \frac{q}{2} | V(1, P_1) V(e^{i\vartheta_j}, P_j) | \mu, -\frac{q}{2} \rangle \quad (5.13)$$

where $\langle k_1, \dots, k_N |$ is given by Eq. (5.10), and $|\lambda, \mu; q\rangle_{a\bar{a}}$ is a shortened notation for $|\lambda, q/2\rangle_a |\mu, q/2\rangle_{\bar{a}}$.

The last term in (5.13)

$$\langle \lambda, \frac{q}{2} | B(1, 2 \dots N) | \mu, -\frac{q}{2} \rangle = e^{i\frac{\pi}{2} q^2} \int \prod_{i=2}^{2\pi N} d\vartheta_i \theta(\vartheta_i - \vartheta_{i-1}) \langle \lambda, \frac{q}{2} | V(1, P_1) V(e^{i\vartheta_i}, P_i) | \mu, -\frac{q}{2} \rangle \quad (5.14)$$

can be interpreted as the decay amplitude of an excited off-mass shell Pomeron into N Reggeons (Fig. 11). The formal consistency of the unified amplitude (3.9) can be seen observing that, when the Pomeron is on its mass shell (and hence the fictitious Reggeon states $\langle \lambda, k/2 |, | \mu, -k/2 \rangle$ that represent the Pomeron are on their own mass shell), the amplitude (5.14) can be written in the form

$$\langle \lambda, \frac{k}{2} | B(1,2,\dots,N) | \mu, -\frac{k}{2} \rangle = e^{i\frac{\pi}{2}k^2} \lim_{z_0 \rightarrow 0} \int_0^{2\pi} \prod_{i=2}^N d\vartheta_i \theta(\vartheta_i - \vartheta_{i-1}) \cdot$$

$$\cdot \langle 0 | \frac{V}{z_0} (z_0, \frac{k}{2}) V(1, p_1) V(e^{i\vartheta_1}, p_1) \frac{V}{z_0} (\frac{1}{z_0}, \frac{k}{2}) | 0 \rangle \quad (5.15)$$

By inverting the projective transformation (5.1) and coming back to the old variables x_i and z_i (with the Reggeon variables on the real axis), one gets

$$\langle \lambda, \frac{k}{2} | B(1,2,\dots,N) | \mu, -\frac{k}{2} \rangle =$$

$$= \lim_{x_1 \rightarrow 0} \int_{-\infty}^{+\infty} \prod_{i=2}^{N-1} \frac{dx_i}{z_i} \prod_{i=1}^N \tilde{\theta}(x_{i+1} - x_i) \langle 0 | \frac{V(x_1, p_1)}{x_1} T_{\lambda\mu}^* (V(x_{i-1}, p_{i-1}) V(x_2, p_2) \dots V(x_N, p_N)) | 0 \rangle \quad (5.16)$$

that coincides with the amplitude (3.9) with N Reggeons and 1 Pomeron.

However, the expressions (5.15) and (5.16) are only formal because the on-mass shell amplitude (5.16) is generally divergent (except when the Pomeron is on the ground state).

In fact the Pomeron $|\lambda, \mu, -k\rangle$ can decay into N Reggeons through a direct Pomeron-Reggeon transition that produces a new set of poles in k^2 in the amplitude (5.16). The lowest of these poles, due to the transition to a Reggeon tachyon, limits the convergence of the integral in the right-hand side of Eq. (5.16) to the region $k^2 > 1$.

In Section 6, such phenomenon will be discussed in a general way and the poles due to the Pomeron-Reggeon transition will be explicitly displayed.

We end this section by observing that the factorization in the channel of momentum k represented in Fig. 7 can be performed in a way strictly analogous to that one used for factorizing in the configuration of Fig. 6, and again gives consistent results.

6. - THE POMERON-REGGEON TRANSITION

The decay amplitude (5.14) of an off-mass shell Pomeron into N Reggeons has poles in q^2 , due to the end point singularity that arises when all the Reggeon variables $e^{i\theta_i}$ approach a common value.

In order to explicitly exhibit such poles it is convenient to bring the Reggeon variables back on the real axis. However, the relevant projective transformation cannot be easily performed directly on the expression (5.14) because the off-mass shell states are not covariant under the projective group. Such a difficulty can be overcome starting from the whole amplitude (5.3), slightly modified in order to keep the Pomeron poles explicitly exhibited.

The form of the amplitude which is most convenient for our purpose is the one reproduced below, which is derived in Appendix B. Here we shall consider only the case of an amplitude with two Pomerons, in order to avoid unessential complications in the formulas ; in Appendix B, the general case is treated. We get :

$$A(NR, 2P) = \sum_{n=0}^{\infty} \frac{e^{i\frac{\pi}{2}q^2}}{2 - \frac{q^2}{2} - 2n} \oint_{|z|=0} \frac{dz}{|z|^2} |z|^{2 - \frac{q^2}{2} - 2n} \int_0^{2\pi} \prod_{j=2}^N d\theta_j \Theta(\theta_j - \theta_{j-1}).$$

$$\cdot \langle k_1/2 | \mathcal{V}(z, k_2/2) \mathcal{V}(1, p_1) \mathcal{V}(e^{i\theta_j}, p_j) \mathcal{V}(k_2/2, 1/\bar{z}) | \frac{k_1}{2} \rangle$$

(6.1)

with

$$\oint_{|z|=0} \frac{dz}{|z|^2} \equiv \oint \frac{d\varphi}{\varphi} \int_0^{2\pi} d(\arg z)$$

(6.2)

where the cyclic integral over $\oint$ is performed in the complex plane $\oint$ around the origin $\oint = 0$. The meaning of (6.1) is the following : the cyclic integral projects the coherent state of two Pomerons into the subspace with $R = \bar{R} = n$; in this subspace, the propagator is a number and may be taken off the sandwich ; the remaining expectation value describes the coupling of that projected Pomeron state with the Reggeons.

In each term of the sum in (6.1), the projective invariance is, of course, lost, but in a way easy to handle ; in fact the only piece that does not transform in a covariant way under a projective transformation is the factor $|z|^{2-q/2-2n}$. Let us now invert the transformation (5.1), performing in (6.1) the substitution

$$z = t(z') = - \frac{z' - i}{z' + i} \quad (6.3)$$

that maps the unit circle into the real axis and, in particular, 1 into 0. Equation (6.1) then becomes :

$$A(NR, 2P) = \sum_{n=0}^{\infty} \frac{1}{2 - \frac{k^2}{2} - 2n} \int \prod_i \frac{dx_i}{x_i} \oint \frac{d^2 z'}{|z'|^2} [t(z')]^{2 - \frac{k^2}{2} - 2n} \\ e^{i\psi} \langle p_1 | T^* \left(\prod_{j=1}^N \mathcal{V}(x_j, p_j) \mathcal{V}(i, \frac{k_1}{2}) \mathcal{V}(-i, \frac{k_1}{2}) \mathcal{V}(z', \frac{k_2}{2}) \mathcal{V}(\bar{z}', \frac{k_2}{2}) | 0 \rangle \right. \\ \left. \right) \quad (6.4)$$

where ψ is a phase that depends on the ordering of the vertices in a way that we need not to specify, and where the x_i 's are integrated on the real axis in the ordering

$$x_{i+1} < x_{i+2} < \dots < x_N < x_1 = 0 < x_2 < \dots < x_i \\ (i = 1, 2, \dots, N)$$

It is the same ordering of the θ_i 's in the unit circle, being real axis closed through the infinite point. We notice that (6.4) is very similar to (3.8) ; the only difference being that the z integration in the upper half plane is substituted with the projector on the subspace of fixed R . The k^2 singularities come out in Eq. (6.4) as an end point singularities when all

the α_i 's approach together the point $\alpha_1 = 0$. That may happen in N different ways according to the variables that approach zero from positive values and those which approach zero from negative ones. The meaning of that is rather transparent : as a consequence of the topological properties of the Pomeron interaction (see Section 3), the Pomeron-Reggeon transition may take place everywhere on the border of the string's world sheet, so that the internal Reggeon line of momentum k may be inserted between any two external Reggeon lines. Therefore, the sum over the N possible insertions must appear in the residue of the Reggeon pole. That can be easily checked, starting from Eq. (6.4) by factorizing in the channel of N Reggeons with the usual techniques. We get :

$$A(NR, 2P) = \sum_{n=0}^{\infty} \frac{1}{2 - \frac{k^2}{2} - 2n} \left\{ \sum_{m=0}^{\infty} \sum_{\lambda} \frac{1}{1 - k^2 - m} \sum_{i=1}^N \langle -p_i, -p_{i+1}, \dots, p_{i-1} | \lambda, m, k \rangle \cdot \right. \\ \left. \langle \lambda, m, k | \mathcal{V}(i, \frac{k}{2}) \mathcal{V}(-i, \frac{k}{2}) \oint \frac{dz}{|z|^2} (t(z))^{2 - \frac{k^2}{2} - 2n} \mathcal{V}(z, \frac{k}{2}) \mathcal{V}(\bar{z}, \frac{k}{2}) | 0 \rangle + \right. \\ \left. + \text{terms regular in } k^2 \right\} \quad (6.5)$$

where $|\lambda, m, k\rangle$ is a state of momentum k , with $R = m$ and with the other quantum numbers labelled by λ . The state $\langle p_1 \dots p_{i-1} |$ is a coherent state of N Reggeons, and it is given by

$$\langle -p_i, -p_{i+1}, \dots, p_{i-1} | = \langle -p_i | \int \prod_{j \neq i-1}^N \frac{dx_j}{x_j} T(\mathcal{V}(x_{i-1}, p_{i-1}) \mathcal{V}(x_j, p_j)) \quad (6.6)$$

with

$$0 = \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_N = 1 \quad \text{if } i=1 \\ \alpha_i \leq \alpha_{i+1} \leq \dots \leq \alpha_N = -1 < \alpha_1 = 0 \leq \alpha_2 \leq \dots \leq \alpha_{i-1} \quad \text{if } i \neq 1$$

The scalar product $\langle -p_i \dots -p_{i-1} | \lambda, m, k \rangle$ is therefore the amplitude for the N Reggeons of momenta p_j and the Reggeon $|\lambda\rangle$, with $|\lambda\rangle$ inserted between the lines of momenta p_i and p_{i-1} .

The structure of the poles in k^2 is explicitly exhibited in Eq. (6.5).

A Reggeon pole occurs whenever

$$\alpha_{\text{TR}}(\kappa^2) = m, \text{ i.e. } \kappa^2 = 1 - m \quad (m = 1, 2 \dots) \quad (6.7a)$$

A Pomeron pole occurs whenever

$$\alpha_{\text{P}}(\kappa^2) = 2n, \text{ i.e. } \kappa^2 = 4(1 - n) \quad (n = 1, 2 \dots) \quad (6.7b)$$

A more detailed analysis follows.

(i) - A double pole occurs whenever both the intermediate Pomeron and Reggeon of momentum k are on their own mass shell. This happens at $k^2 = -4\ell$ with ℓ non-negative integer; in fact, in that case, Eqs. (6.7a) and (6.7b) are both satisfied, with $m = 1 + 4\ell$ and $n = 1 + \ell$. The coefficient of the double pole at $k^2 = -4\ell$ (which we shall denote by $R_{\ell}^{(2)}$) is:

$$R_{\ell}^{(2)} = \sum_{\lambda} \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, m, \kappa \rangle \langle \lambda, m, \kappa | \mathcal{W}(i, -i, \kappa) \oint_{z=i} \frac{d^2 z}{|z|^2} \mathcal{W}(z, \bar{z}; \kappa_2) | 0 \rangle \quad (6.8)$$

It is worth noticing that in Eq. (6.8) the intermediate states are physical on-shell states, and that the Pomeron vertex is a physical one, since the factor $|t(z)|^{2-(k^2/2)-2n}$ which breaks the covariance of the cyclic integral is equal to 1 on the mass shell. In order to work out explicitly from (6.8) the Pomeron-Reggeon transition, it is useful to perform again a projective transformation mapping $(0, i, -i)$ into $(1, 0, \infty)$; in such a way we may rewrite Eq. (6.8) in a different form:

$$R_{\ell}^{(2)} = \sum_{\lambda} \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, m, \kappa \rangle \oint_{z=0} \frac{d^2 z}{|z|^2} \langle \kappa_1 | \mathcal{V}(z, \kappa_2) \mathcal{V}_{(\lambda, m)}(1, \kappa) \mathcal{V}(\frac{1}{z}, \kappa_2) | \kappa_1 \rangle \quad (6.9)$$

where $\mathcal{V}_{(\lambda, m)}(1, \kappa)$ is the vertex corresponding to the physical Reggeon state $|\lambda, m, \kappa\rangle$. By inserting a complete set of states at both sides of $\mathcal{V}_{(\lambda, m)}(1, \kappa)$ and introducing a double set of oscillators, we get:

$$R_{\ell}^{(2)} = \sum_{\lambda} \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, m, \kappa \rangle \sum_{\alpha, \beta} \langle \alpha, n, \frac{\kappa}{2} | \mathcal{V}_{(\lambda, m)}(1, \kappa) | \beta, n, \frac{\kappa}{2} \rangle \oint \frac{d^2 z}{|z|^2} \frac{1}{a\bar{a}} \langle \kappa_1 | \mathcal{V}(z, \kappa_2) \bar{\mathcal{V}}(\frac{1}{z}, \kappa_2) | \alpha, \beta, n, \kappa \rangle_{a\bar{a}} \quad (6.10)$$

where $|\alpha, \beta, n, k\rangle_{a\bar{a}}$ stands for $|\alpha, n, k/2\rangle_a |\beta, n, k/2\rangle_{\bar{a}}$ and where α and β are internal quantum numbers labelling the state. Only the states with $R = 1 + \ell$ contribute to the sum because the cyclic integral just projects onto the subspace with $R = n = 1 + \ell$ [see Eq. (6.1)]. If now one takes into account that

$$\begin{aligned} {}_{a\bar{a}}\langle k_1 | \oint \frac{dz}{|z|^2} \mathcal{V}(z, \frac{k_2}{z}) \bar{\mathcal{V}}(\frac{1}{z}, \frac{k_2}{z}) | \alpha, \beta, n, k \rangle_{a\bar{a}} = \\ = {}_{a\bar{a}}\langle k_1 | \mathcal{V}(1, \frac{k_2}{z}) \bar{\mathcal{V}}(1, \frac{k_2}{z}) | \alpha, \beta, n, k \rangle_{a\bar{a}} \end{aligned} \quad (6.11)$$

one finally gets $R_e^{(2)}$ in the completely factorized form of Fig. 12

$$\begin{aligned} R_e^{(2)} = \sum_{\lambda} \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, n, k \rangle \sum_{\alpha, \beta} \langle \alpha, n, \frac{k}{2} | \mathcal{V}_{(\lambda, n)}(1, k) | \beta, n, \frac{k}{2} \rangle \cdot \\ {}_{a\bar{a}}\langle k_1 | \mathcal{V}(1, \frac{k_2}{z}) \bar{\mathcal{V}}(1, \frac{k_2}{z}) | \alpha, \beta, n, k \rangle_{a\bar{a}} \end{aligned} \quad (6.12)$$

where all the lines represent physical on-mass shell states. Instead of the three-Pomeron vertex, the SV amplitude with $M+1$ external legs would obviously appear, as it does in Eq. (5.7), if we would have started from an amplitude with M Pomerons. The previous factorization shows that our unified dual model is actually consistent with both the conventional and the SVM ; it also provides a simple expression for the transition from a closed to an open physical on-mass shell string state. It is identical to a three-Reggeon vertex in the particular case where all three Reggeons are on their mass shell and two of them carry the same momentum. This kinematical relation is reflected, through the on-mass shell condition, into a relation among the eigenvalues of R so that a Pomeron with $R = \bar{R} = 1 + \ell$ may decay only into a Reggeon with $R = 1 + 4\ell$. Of course, even if the kinematical relations are fulfilled, the Pomeron-Reggeon transition may be forbidden from other conservation laws, and the double pole may in that case reduce to a simple one. On the level $k^2 = 0$, for instance, both the "graviton" and the "dilaton" are decoupled from the "photon" because of angular momentum conservation. The double pole is not present even in the case of an amplitude with two external Reggeon lines ; in that case, because of the invariance of $\langle p_1 p_2 | + \langle p_2 p_1 |$ under twist, only the intermediate Reggeon states with an even value of R are coupled to the external Reggeons, whereas only the ones with an odd value of R may be coupled to a Pomeron.

(ii) - When

$$k^2 = 1 - r \quad r \geq 0 \quad r \neq 1 + 4\ell \quad (6.13)$$

the amplitude (6.5) shows a simple Reggeon pole. The residue of the pole, which we shall denote by $R_r^{(1)}$, is given by :

$$R_r^{(1)} = \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, r, k \rangle \sum_{n=0}^{\infty} \oint_{|z|=1} \frac{d^2 z}{|z|^2} \frac{|t(z)|^{2 - \frac{k^2}{2} - 2n}}{2 - \frac{k^2}{2} - 2n} \langle \lambda, r, k | \mathcal{W}(i, -i, k_1) \mathcal{W}(z, \bar{z}, k_2) | 0 \rangle \quad (6.14)$$

where k^2 is given by Eq. (6.13). Let us go back, step by step, up to get the Pomeron-Reggeon amplitude at the right-hand side of Eq. (6.14) in the form of Eq. (5.3). The result is

$$R_r^{(1)} = \sum_{\lambda} \sum_{i=1}^N \langle -p_i, \dots, -p_{i-1} | \lambda, r, k \rangle \int_{|z| < 1} \frac{d^2 z}{|z|^2} \langle \frac{k_1}{2} | \mathcal{V}(z, \frac{k_2}{2}) \mathcal{V}(1, k) \mathcal{V}(1/\bar{z}, \frac{k_2}{2}) | \frac{k_1}{2} \rangle_{(\lambda, r)} \quad (6.15)$$

and the corresponding graphical picture is given in Fig. 13. This result, however, is purely formal, since the usual integral representation of the mixed amplitude diverges, even if the amplitude itself is definite, when a single external Reggeon line is present. The reason of this divergence already shown in Section 4 is clear from Eq. (6.5) ; the analytic continuation of the amplitude in the mass squared k^2 of the Reggeon line has a set of poles due to the transition from the off-shell Reggeon into an on-shell Pomeron. The lowest of these poles is at $k^2 = 4$, and that prevents the integral in Eq. (6.15) from converging, whatever the on-shell Reggeon is. Of course, there is no problem of convergence in Eq. (6.14), where all the poles are explicitly exhibited and where we can safely go on the mass shell of the Reggeon. Equation (6.14) also provides, for any value of n , the coupling between the on-shell Reggeon λ and the projection on the subspace with $R = \bar{R} = n$ of the two-Pomeron coherent state.

7. - THE AMPLITUDE WITH EXCITED POMERONS

In Section 5 we pointed out the problems which are present when there are external excited Pomeron states. In fact in that case, the amplitude is divergent from the very beginning when the Pomeron lines are on-shell independently if a transition from closed to open string state is allowed. However, as it was noticed in Section 3, it is possible to define an off-shell covariant Pomeron vertex :

$$\mathcal{W}^{\alpha}(z, \bar{z}, k) = \mathcal{V}^{\alpha_1}(z, k_1) \mathcal{V}^{\alpha_2}(\bar{z}, k_2) \quad (7.1)$$

where each constituent vertex, considered as a Reggeon one, is on its mass shell but where $\mathcal{W}^{(\alpha)}(z, \bar{z}, k)$ is not on the mass shell of a Pomeron state. The amplitudes built up with these off-shell vertices are well defined, and they have poles in the mass squared of the external Pomerons, as shown in Fig. 14 (the off-shell Pomerons are represented by a wavy line with two moustaches at the end). In particular, we have a pole when the Pomeron is on-shell, namely for $k_1 = k_2$, if some Pomeron-Reggeon transition is allowed. The residue of that pole factorizes into a Pomeron-Reggeon vertex times an amplitude where the Pomeron line is replaced by the Reggeon one inserted in all the different positions (see Section 6). Even if the external Pomerons are off the mass shell, in the sense given by Eq. (7.1), the procedure outlined in Section 6 to factorize in the multi-Pomeron channel may be applied as well, since the covariance of the vertices under the gauge operators is preserved. We get the same Eq. (5.13), but with the Pomeron lines off the mass shell, and with the total momentum q not equally distributed between the two sets of oscillators. The propagator has poles at $L_{0+1} = \bar{L}_{0+1} = 0$, namely if $1 - q_1^2 = n_1$ and $1 - q_2^2 = n_2$. q_1, q_2, n_1 and n_2 are respectively the eigenvalues of $p_0, \bar{p}_0, R$ and $\bar{R}$ relative to the intermediate state $|\lambda, \mu, q\rangle$ ($q = q_1 + q_2$).

The poles in the masses of the external Pomeron lines do not appear in the residue, as these channels are dual with the channel of total momentum q , so that the analytic continuation of the residue on the mass shell may be safely done and the SV amplitude with excited states is recovered in this way.

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APPENDIX A

This appendix is organized in three points. First (A) we find the transformation properties of the vertices under finite reparametrizations of the string, then (B) we prove the equivalence among the various expressions of our amplitudes used in the text and finally (C) we study the gauge properties of the coherent states of closed and open interacting strings.

A. - The transformation properties of the Reggeon vertices under a finite reparametrization of the string are given by

$$e^{\alpha L_f} V(x) f(x) e^{-\alpha L_f} = V(x') f(x') \quad (\text{A.1})$$

where the relation between x and x' is given implicitly by

$$\int_x^{x'} \frac{d\xi}{\xi f(\xi)} = \alpha \quad (\text{A.2})$$

Equations (A.1) and (A.2) follow rather directly from the usual commutation rules of the vertices with the L_f 's [Eq. (2.12)]. In fact, if we introduce the variable

$$y(x) = \int_A^x \frac{d\xi}{\xi f(\xi)} \quad (\text{A.3})$$

the commutation rules (2.12) can be written in the form

$$[L_f, f(x) V(x)] = \frac{d}{dy} [f(x) V(x)] \quad (\text{A.4})$$

The effect of a finite reparametrization is therefore given by Eq. (A.1), where the relation between x and x' is a translation in the variable $y(x)$:

$$y(x') = \alpha + y(x) \quad (\text{A.5})$$

which just coincides with Eq. (A.2).

From Eq. (A.2) the relation between the differentials dx and dx' can be derived :

$$\frac{dx'}{x' f(x')} = \frac{dx}{x f(x)} \quad (\text{A.6})$$

and that allows one to write Eq. (A.1) in the form

$$\frac{dx}{x} e^{\alpha L_f} V(x) e^{-\alpha L_f} = \frac{dx'}{x'} V(x') \quad (\text{A.7})$$

which exhibits the covariance properties of the vertices explicitly.

The transformation properties of the vertices describing the emission of a closed string can be derived in the same way. Let α be a real parameter and $f(\xi)$ a real function, namely satisfying the relation $f(\bar{\xi}) = \overline{f(\xi)}$, then the transformation given by Eq. (A.2) maps the real axis into itself and any couple of complex conjugate points into a couple of complex conjugate ones. Then the transformation properties of $\mathcal{W}(z, \bar{z})$ are given by

$$e^{\alpha L_f} \mathcal{W}(z, \bar{z}) |f(z)|^2 e^{-\alpha L_f} = \mathcal{W}(z', \bar{z}') |f(z')|^2 \quad (\text{A.8})$$

where the relation between z and z' is still given by Eq. (A.2).

Using the transformation properties of the differentials, Eq. (A.8) may be written in a way similar to Eq. (A.7) :

$$\frac{d^2 z}{|z|^2} e^{\alpha L_f} \mathcal{W}(z, \bar{z}) e^{-\alpha L_f} = \frac{d^2 z'}{|z'|^2} \mathcal{W}(z', \bar{z}') \quad (\text{A.9})$$

Let us remark for future convenience that the zeros of $f(x)$ are the points which are mapped into themselves by the transformation (A.2) ; in fact, if $f(x_0) = 0$ the integrand of (A.2) is infinite at $\xi = x_0$ and the limit

$$\lim_{\substack{x \rightarrow x_0 \\ x' \rightarrow x'_0}} \int_x^{x'} \frac{d\xi}{\xi f(\xi)}$$

may be finite and equal to α only if $x'_0 = x_0$.

B. - The projective invariance of the integrand of the dual amplitudes follows from Eqs. (A.7) and (A.9) and from the transformation properties of the vacuum. In fact, since the two equations $e^{-\alpha Lf}|0\rangle = |0\rangle$ and $\langle 0|e^{\alpha Lf} = \langle 0|$ hold simultaneously only if f is of the form :

$$f(\xi) = a_+ \xi + a_0 + a_- \xi^{-1} \quad (A.10)$$

[which correspond, through Eq. (A.2), to projective transformations on the variables], then the vacuum expectation value of a product of vertices is only invariant under the three-parameter subgroup of projective transformations. As a consequence of that the integrand of the amplitude has to be divided by the infinite volume d^3V of the group, in order to make the integral finite. In the case of our unified model, we have :

$$A[NR, MP] = \int \dots \int \frac{\prod_1^N \frac{dx_i}{x_i} \prod_1^M \frac{d^2 z_j}{|z_j|^2}}{d^3V} \langle 0 | T^* \left(\prod_1^N V^{\lambda_i}(x_i, p_i) \prod_1^M W^{\mu_j}(z_j, \bar{z}_j, k_j) \right) | 0 \rangle \quad (A.11)$$

where d^3V can take the two following forms

$$d^3V = \frac{dx_a dx_b dx_c}{(x_a - x_b)(x_b - x_c)(x_c - x_a)} \quad (a, b, c) \in \{1, 2, \dots, N\} \quad (A.12)$$

$$d^3V = \frac{dx_r d^2 z_s}{(x_r - z_s)(z_s - \bar{z}_s)(\bar{z}_s - x_r)} \quad \begin{matrix} r \in \{1, 2, \dots, N\} \\ s \in \{1, 2, \dots, M\} \end{matrix} \quad (A.13)$$

Equation (A.12) coincides with the usual choice of the conventional model and it corresponds to fix three Reggeon variables in the integral of Eq. (A.11), whereas Eq. (A.13) corresponds to fix a Reggeon variable and a Pomeron one (three real variables altogether). We shall prove that the two choices are actually equivalent. Let us start from the amplitude (A.11), with d^3V given by Eq. (A.12) :

$$A[NR, MP] = \int \dots \int \frac{(x_a - x_b)(x_b - x_c)(x_c - x_a)}{x_a x_b x_c} \prod_{\substack{i=1 \\ i \neq a, b, c}}^N \frac{dx_i}{x_i} \prod_{j=1}^M \frac{d^2 z_j}{|z_j|^2} \cdot \langle 0 | T^* \left(\prod_{i=1}^N V^{\lambda_i}(x_i, p_i) \prod_{j=1}^M W^{\mu_j}(z_j, \bar{z}_j, k_j) \right) | 0 \rangle \quad (A.14)$$

Let us introduce a new set of variables x'_i and z'_j defined through the relations

$$\begin{aligned} x_i &= T(x'_i) \\ z_j &= T(z'_j) \end{aligned} \quad (A.15)$$

where

$$\xi = T(\xi') = \alpha \frac{\xi' - \beta}{\xi' - \gamma} \quad (A.16)$$

The coefficients α , β and γ of the projective transformation are functions of x'_a , x'_b and x'_c , and they are determined by the requirement that for any value of x'_a , x'_b and x'_c , $T(x'_a)$, $T(x'_b)$, $T(x'_c)$ are respectively equal to the fixed quantities x_a , x_b , x_c of Eq. (A.14). Moreover, a Reggeon variable (x'_r) and a Pomeron one (z'_s) of the new set are kept fixed. That does not mean that x_r and z_s are fixed, since they depend upon x'_a , x'_b through the parameters of the transformation (A.16).

Now let us choose the function f and the parameter α of Eq. (A.2) in such a way that the corresponding transformation coincides with (A.16), and let us perform that transformation in (A.14). We remark that :

- (i) from the explicit knowledge of the transformation (A.16) the ratio $f(x')/f(x)$ may be evaluated through Eq. (A.6) ; in particular, we have :

$$\frac{f(x'_e) f(x'_m) f(x'_n)}{f(x_e) f(x_m) f(x_n)} = \frac{x_e x_m x_n (x'_e - x'_m)(x'_m - x'_n)(x'_n - x'_e)}{x'_e x'_m x'_n (x_e - x_n)(x_n - x_m)(x_m - x_e)} \quad (A.17)$$

- (ii) from Eqs. (A.1) and (A.17), we have :

$$\begin{aligned} &\langle 0 | \dots \mathcal{V}^{\lambda_a}(x_a, p_a) \mathcal{V}^{\lambda_b}(x_b, p_b) \mathcal{V}^{\lambda_c}(x_c, p_c) \dots | 0 \rangle \frac{(x_a - x_b)(x_b - x_c)(x_c - x_a)}{x_a x_b x_c} = \\ &= \text{idem with primed variables} \end{aligned} \quad (A.18)$$

(iii) from Eqs. (A.7) and (A.9) we have

$$\begin{aligned} & \langle 0 | \dots \prod_{\substack{i=1 \\ i \neq a, b, c, r}}^N \frac{dx_i}{x_i} V^{\lambda_i}(x_i, p_i) \prod_{\substack{j=1 \\ j \neq s}}^M W^{\mu_j}(z_j, \bar{z}_j, k_j) \frac{d^2 z_j}{|z_j|^2} \dots | 0 \rangle = \\ & = \text{idem with primed variables} \end{aligned} \quad (\text{A.19})$$

(iv) from Eqs. (A.1), (A.8) and (A.17) we have

$$\begin{aligned} & \langle 0 | \dots V^{\lambda_r}(x_r, p_r) W^{\mu_s}(z_s, \bar{z}_s, k_s) \dots | 0 \rangle \frac{dx_r}{x_r} \frac{d^2 z_s}{|z_s|^2} = \\ & = \frac{(x'_r - z'_s)(z'_s - \bar{z}'_s)(\bar{z}'_s - x'_r)}{(x_r - z_s)(z_s - \bar{z}_s)(\bar{z}_s - x_r) x'_r |z'_s|^2} \cdot \frac{\partial(x_r, z_s, \bar{z}_s)}{\partial(x'_a, x'_b, x'_c)} dx'_a dx'_b dx'_c \cdot \\ & \cdot \langle 0 | \dots V^{\lambda_r}(x'_r, p_r) W^{\mu_s}(z'_s, \bar{z}'_s, k_s) \dots | 0 \rangle \end{aligned} \quad (\text{A.20})$$

(v) the Jacobian in Eq. (A.20) is given by

$$\frac{\partial(x_r, z_s, \bar{z}_s)}{\partial(x'_a, x'_b, x'_c)} = \frac{(x_r - z_s)(z_s - \bar{z}_s)(\bar{z}_s - x_r)}{(x'_a - x'_b)(x'_b - x'_c)(x'_c - x'_a)} \quad (\text{A.21})$$

Equation (4.21) can be checked directly in a configuration where the calculations are simpler [for instance when $x_a = 0$, $x_b = 1$, $x_c = \infty$, $x'_r = 0$, $z'_s = i$; because in this case the parameters of the projective transformation (A.16) are relatively simple] and then extended to all the possible cases by means of projective transformations on both primed and unprimed variables. By introducing Eqs. (A.18)-(A.21) into Eq. (A.14), we get :

$$\begin{aligned} A[NR, MP] &= \int \dots \int \frac{|z'_s - x'_r| |x'_r - \bar{z}'_s| |\bar{z}'_s - z'_s|}{x_r |z'_s|^2} \prod_{\substack{i=1 \\ i \neq r}}^N \frac{dx'_i}{x'_i} \cdot \\ & \cdot \prod_{\substack{j=1 \\ j \neq s}}^M \frac{d^2 z'_j}{|z'_j|^2} \langle 0 | T^* \left[\prod_{i=1}^N V^{\lambda_i}(x'_i, p_i) \prod_{j=1}^M W^{\mu_j}(z'_j, \bar{z}'_j, k_j) \right] | 0 \rangle \end{aligned} \quad (\text{A.22})$$

which completes the proof of the equivalence between the two volume elements [Eqs. (A.12), (A.13)].

C. - We shall study here the gauge properties of the coherent states of closed and open interacting strings. The coherent states we shall consider here are of the following form :

$$\begin{aligned} & \langle p_1, p_2 \dots p_N; k_1, \dots, k_M | = \\ & = \int_{\mathcal{D}} \dots \int \prod_{i=2}^{N-1} \frac{dx_i}{x_i} \theta(x_{i+1} - x_i) \prod_{j=1}^M \frac{d^2 z_j}{|z_j|^2} \langle p_1 | T^* \left[\prod_{i=2}^N V^{\lambda_i}(\alpha_i, p_i) \prod_{j=1}^M W^{\mu_j}(z_j, \bar{z}_j, k_j) \right] \end{aligned} \quad (A.23)$$

where the integration domain is the whole upper half plane for Pomeron variables z_j 's, and the real axis between $x_1 = 0$ and $x_N = 1$ with the ordering given by the θ functions for the Reggeon variables x_i 's. It is not difficult to check that the coherent states defined in this way have the same gauge operators as the coherent states of open strings in the conventional model. In fact, by using Eqs. (A.7) - (A.9), and Eq. (A.1) for the transformation of $V^{\lambda_N}_{(x_N=1, p_N)}$, we have

$$\begin{aligned} & \langle p_1 \dots p_N; k_1 \dots k_M | e^{-\alpha L_f} = \int_{\mathcal{D}'} \dots \int \prod_{i=2}^{N-1} \frac{dx'_i}{x'_i} \theta(x'_{i+1} - x'_i) \prod_{j=1}^M \frac{d^2 z'_j}{|z'_j|^2} \cdot \\ & \cdot \langle p_1 | e^{-\alpha L_f} T^* \left[\prod_{i=2}^N V^{\lambda_i}(\alpha'_i, p_i) \prod_{j=1}^M W^{\mu_j}(z'_j, \bar{z}'_j, k_j) \right] \lim_{x_N \rightarrow 1} \frac{f(x'_N) x_N}{f(x_N) x'_N} \end{aligned} \quad (A.24)$$

In order to have $\mathcal{D}' \equiv \mathcal{D}$, the transformation must map the real axis into itself, and this requires that α is a real parameter and f a real function. Moreover, the points 0 and 1, namely the values of the Reggeon variables which are kept fixed, must be mapped into themselves and that implies :

$$\lim_{\xi \rightarrow 0} \xi f(\xi) = 0 \quad \Rightarrow \quad f(\xi) \text{ regular in } \xi = 0 \quad (A.25)$$

$$f(1) = 0 \quad (A.26)$$

If the conditions (A.25) and (A.26) are satisfied, we have

$$\langle p_1 | e^{-\alpha L_f} = \langle p_1 | e^{-\alpha f(0) L_0} = e^{\alpha f(0)} \langle p_1 | \quad (\text{A.27})$$

because $\langle p_1 |$ is a physical on-shell state (it is annihilated by all the L_n with $n < 0$) and :

$$\lim_{x_N \rightarrow 1} \frac{f(x'_N)}{f(x_N)} \frac{x_N}{x'_N} = \lim_{x_N \rightarrow 1} \frac{dx'_N}{dx_N} = e^{\alpha f'(1)} \quad (\text{A.28})$$

where the last step can be checked by expanding in Eq. (A.2) the integrand in powers of $\int -1$ and taking into account the condition (A.26). By inserting Eqs. (A.27) and (A.28) into Eq. (A.24), we get :

$$\langle p_1 \dots p_N ; k_1, \dots k_M | e^{-\alpha [L_f + f(0) + f'(1)]} = \langle p_1 \dots p_N ; k_1, \dots k_M | \quad (\text{A.29})$$

The gauge operators are therefore, as in the conventional model, $L_f + f(0) + f'(1)$ where the function f satisfies Eqs. (A.27) and (A.28).

We end with some remarks about the coherent states as defined in Eq. (A.23). We have fixed there the variables relative to the first and the last Reggeon respectively to 0 and 1, whereas we have left the variables relative to the closed string to vary through the whole upper half plane. This means in the string picture that the last Reggeon interacts at a time z conventionally fixed to 0 (remember that $|x_1| = e^{\tau i}$), while the closed strings may interact at any τ and σ , namely on the whole world sheet surface of the string. So if we take the scalar product $\langle p_1 p_2 \dots p_N ; k_1 \dots k_N | \Psi \rangle$, where $|\Psi\rangle$ is a physical on-shell state, we get the whole dual amplitude for interacting closed and open strings.

Moreover, if we consider the quantity

$$A = \langle p_1 p_2 \dots p_N ; k_1 \dots k_M | \frac{1}{L_0 + 1} | p_{N+1} \dots p_R ; k_{M+1} \dots k_S \rangle \quad (\text{A.30})$$

where the ket is a coherent state defined in the same way as (A.23), except of course that the Reggeon variables vary between $x_{N+1} = 1$ and $x_R = \infty$, we should get again the dual mixed amplitude for closed and open strings. However, if we use the standard techniques in order to go from factorized expression (A.30) to a non-factorized one [like (A.11)] we get a formally divergent expression since the closed string vertices are not T ordered in some regions of the integration domain. This happens because, starting from (A.30), the closed string vertices belonging to the bra come before the ones belonging to the ket in the ordering, whereas their variables vary independently over the whole upper half plane so that the T ordering is violated. This does not happen for the Reggeons just because we have fixed the variables relative to p_N and p_{N+1} , and that is essential in the Reggeon case since the ordering has in that case an intrinsic topological meaning and it is not merely a requirement to have convergent series. Anyway the right-hand side of Eq. (A.30) can be obtained by a procedure of analytic continuation starting from a well-defined quantity, and it turns out to coincide with the dual amplitude. In fact, let us consider the quantity :

$$A(\epsilon) = \underset{\epsilon}{\langle} p_1 \dots p_N ; k_1 \dots k_M | \frac{1}{L_0 + 1} | p_{N+1} \dots p_R ; k_{M+1} \dots k_S \rangle_{\epsilon} \quad (A.31)$$

where the states $\underset{\epsilon}{\langle} p_1 \dots p_N ; k_1 \dots k_M |$ and $| p_{N+1} \dots p_R ; k_{M+1} \dots k_S \rangle_{\epsilon}$ are defined as in Eq. (A.23) except that the integration domains are modified by adding respectively the conditions $|z_j| < \epsilon$ ($j = 1, \dots, N$) and $|z_j| > 1/\epsilon$ ($j = N+1, \dots, S$). If $\epsilon < 1$ these conditions assure that the T^* ordering is satisfied (all the strings of the bra interact at a time τ less than 0, all the ones of the ket at a time greater than 0) and in that case, by using the standard techniques, the right-hand side of Eq. (A.31) can be written as

$$A(\epsilon) = \int \prod_{\substack{i=2 \\ 0 \leq i \leq N \\ x_N=1}}^{R-1} \frac{dx_i}{x_i} \delta(x_{i+1} - x_i) \int \prod_{\substack{j=1 \\ \text{Im } z_j > 0 \\ |z_j| < \epsilon, j \in \{1, \dots, M\} \\ |z_j| > \frac{x_{N+1}}{\epsilon}, j \in \{M+1, \dots, S\}}} \frac{d^2 z_j}{|z_j|^2} \quad (A.32)$$

$$\cdot \langle p_1 | T^* \left[\prod_{i=2}^{R-1} V(x_i, p_i) \right] \prod_{j=1}^S W(z_j, \bar{z}_j, k_j) | p_R \rangle$$

The integrand of (A.32) coincides with the one of the dual mixed amplitude, and since the latter is represented by the same analytic function of the x_i 's, z_j 's and $\bar{z}_j$'s in the whole integration domain, the analytic continuation of (A.32) up to $\epsilon = \infty$ reproduces the whole dual amplitude of R open and S closed interacting strings. Since the same analytic continuation in (A.31) gives Eq. (A.30), our definition of the coherent states is justified.

APPENDIX B

In this Appendix, we derive Eq. (6.1) and its generalization to the case of an arbitrary number of Pomerons.

Let us first introduce a projection operator $\mathcal{P}_n$, which projects any state of the Fock space of closed string, characterized by the two sets of oscillators a_m and $\bar{a}_m$, into the subspace with $R = \bar{R} = n$. It is

$$\mathcal{P}_n = \frac{1}{4i\pi^2} \oint \frac{d^2z}{|z|^2} z^{R-n} \bar{z}^{\bar{R}-n} = \frac{1}{4i\pi^2} \oint \frac{d^2z}{|z|^2} z^{1-\frac{p_0^2}{2}-n} \bar{z}^{1-\frac{\bar{p}_0^2}{2}-n} z^{-L_0-1} \bar{z}^{-\bar{L}_0-1} \quad (\text{B.1})$$

where the relations $L_0 = -(p_0^2/2) - R$ and $\bar{L}_0 = -(\bar{p}_0^2/2) - \bar{R}$ have been used. In (B.1) we have used the definition :

$$\oint \frac{d^2z}{|z|^2} = \oint_{\mathcal{P}} \frac{d\varphi}{\mathcal{P}} \int_0^{2\pi} d(\arg z) \quad (\text{B.2})$$

where the modulus $\mathcal{P}$ of z is considered as a complex variable and the integration is done on a path surrounding the origin in the complex plane of $\mathcal{P}$. By using the definition (B.2), we get

$$\mathcal{P}_n = \frac{1}{4i\pi^2} \oint_{\mathcal{P}} \frac{d\varphi}{\mathcal{P}} |z|^{R+\bar{R}-2n} \int_0^{2\pi} d[\arg z] e^{i(R-\bar{R})\arg z} \quad (\text{B.3})$$

from which the properties of $\mathcal{P}_n$ follow directly.

Let us consider now the coherent state of M ground states of momenta k_i ($i=1, \dots, M$) in the SVM. As shown in Ref. 13), it can be written as

$$\langle R | \mathcal{D} = \langle k_1 | \mathcal{W}(1, k_2) \mathcal{D} \mathcal{W}(1, k_3) \mathcal{D} \dots \mathcal{D} + \text{all permutations of } k_2 \dots k_M \quad (\text{B.4})$$

where $\mathcal{D}$ is the SV propagator

$$\mathcal{D} = \frac{1}{L_0 + \bar{L}_0 + 2} \frac{\sin \pi (L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} \quad (\text{B.5})$$

and $\mathcal{W}_{(1,k)}$ is the SV vertex evaluated at $z = \bar{z} = 1$:

$$\mathcal{W}(1, k_i) = \mathcal{V}^{(a)}(1, \frac{k_i}{2}) \mathcal{V}^{(\bar{a})}(1, \frac{k_i}{2}) \quad (\text{B.6})$$

Since in Eq. (B.5) the eigenvalues of p_0 and $\bar{p}_0$ always coincide, $L_0 - \bar{L}_0$ is equal to $R - \bar{R}$ and the factor $(\sin \pi(L_0 - \bar{L}_0)) / (\pi(L_0 - \bar{L}_0))$ is a projector on $R = \bar{R}$. So we can make the substitution

$$\mathbb{D} = \sum_{n=0}^{\infty} \frac{P_n \pi}{L_0 + \bar{L}_0 + 2} = \sum_{n=0}^{\infty} \frac{P_n \pi}{2 - \frac{q^2}{2} - 2n} \quad (\text{B.7})$$

where q is the total momentum of the state to which $\mathbb{D}$ is applied, and the last relation comes from the explicit evaluation of $L_0 + \bar{L}_0 + 2$ in the subspace with $R = \bar{R} = n$.

With the substitution (B.7) and by using the explicit form of P_n , we get

$$\begin{aligned} \langle k | = \sum_{n_1, \dots, n_M} \oint_{|z_2|=0} \frac{d^2 z_2}{|z_2|^2} \frac{|z_2|^{2 - \frac{q_2^2}{2} - 2n_2}}{2 - \frac{q_2^2}{2} - 2n_2} \dots \oint_{|z_M|=0} \frac{d^2 z_M}{|z_M|^2} \frac{|z_M|^{2 - \frac{q_M^2}{2} - 2n_M}}{2 - \frac{q_M^2}{2} - 2n_M} \cdot \\ \cdot \langle k_1 | \mathcal{W}(1, k_2) z_2^{-L_0-1} \bar{z}_2^{-\bar{L}_0-1} \mathcal{W}(1, k_3) \dots z_M^{-L_0-1} \bar{z}_M^{-\bar{L}_0-1} + \text{permutations} \end{aligned} \quad (\text{B.8})$$

where

$$q_i = k_1 + k_2 + \dots + k_i \quad i = 2, \dots, M \quad (\text{B.9})$$

The transformation properties of the vertices imply :

$$\begin{aligned} \langle k_1 | \mathcal{W}(k_2, 1) z_2^{-L_0-1} \bar{z}_2^{-\bar{L}_0-1} \dots \mathcal{W}(1, k_M) z_M^{-L_0-1} \bar{z}_M^{-\bar{L}_0-1} = \\ = \langle k_1 | \mathcal{W}(k_2, \xi_2) \mathcal{W}(k_3, \xi_3) \dots \mathcal{W}(k_M, \xi_M) \end{aligned} \quad (\text{B.10})$$

where

$$\xi_i = z_i z_{i+1} \dots z_M$$

By taking the ξ_i 's as a new set of independent variables, we finally get

$$\begin{aligned} \langle k | = & \sum_{n_2 \dots n_M} \frac{1}{\prod_{j=2}^M (2 - \frac{q_j^2}{2} - 2n_j)} \prod_{j=2}^M \oint \frac{d^2 \xi_j}{|\xi_j|^2} |\xi_j|^{-\frac{q_{j-1}^2 - q_j^2}{2} + 2(n_{j-1} - n_j)} \\ & \cdot \langle k_1 | \mathcal{W}(\xi_2, \bar{\xi}_2, k_2) \dots \mathcal{W}(\xi_M, \bar{\xi}_M, k_M) + \\ & + \text{permutations} \end{aligned} \quad (B.11)$$

where n_1 and q_1^2 have to be put equal to 0 and 4, respectively. The same method we have outlined above may be applied as well when the coherent state of M Pomeron appears in an open string formalism, as in Eq. (5.3) characterized by a single set of oscillators. In that case, in place of Eq. (5.3), we have

$$\begin{aligned} \langle \frac{k_1}{2} | \dots | \frac{k_2}{2} \rangle = & \sum_{n_2 \dots n_M} \frac{1}{\prod_{j=2}^M (2 - \frac{q_j^2}{2} - 2n_j)} \oint \frac{d^2 \xi_j}{|\xi_j|^2} |\xi_j|^{-\frac{q_j^2}{2} + \frac{q_{j-1}^2}{2} - n_j + n_{j-1}} \\ & \quad \quad \quad |\xi_j|=0 \\ \langle \frac{k_1}{2} | \mathcal{V}(\xi_2, \frac{k_2}{2}) \dots \mathcal{V}(\xi_M, \frac{k_M}{2}) \dots \end{aligned} \quad (B.12)$$

$$\dots \mathcal{V}(\frac{1}{\xi_M}, \frac{k_M}{2}) \dots \mathcal{V}(\frac{1}{\xi_2}, \frac{k_2}{2}) | \frac{k_1}{2} \rangle + \text{permutations}$$

By using this result we may write the mixed amplitude of N Reggeons and M Pomerons in a form convenient for the factorization on a double pole :

$$\begin{aligned} A[NR, MP] = & \sum_{n_2 \dots n_M} \frac{1}{\prod_{j=2}^M (2 - \frac{q_j^2}{2} - 2n_j)} \prod_{j=2}^M \oint \frac{d^2 \xi_j}{|\xi_j|^2} |\xi_j|^{\frac{q_{j-1}^2}{2} + n_{j-1} - \frac{q_j^2}{2} - n_j} \\ & \cdot \langle \frac{k_1}{2} | \mathcal{V}(\xi_2, \frac{k_2}{2}) \dots \mathcal{V}(\xi_M, \frac{k_M}{2}) \int_0^{2\pi} \prod_{j=1}^N d\theta_j \prod_{j=1}^{N-1} \theta(\theta_{j+1} - \theta_j) \mathcal{V}(1, p_1) \dots \mathcal{V}(e^{i\theta_N}, p_N) \cdot \\ & \cdot \mathcal{V}(\frac{1}{\xi_M}, \frac{k_M}{2}) \dots \mathcal{V}(\frac{1}{\xi_2}, \frac{k_2}{2}) | \frac{k_1}{2} \rangle \end{aligned} \quad (B.13)$$

In the case of $M = 2$ this equation reduces to Eq. (6.1) of the text. It is worth noticing that the state

$$\begin{aligned} \langle k_1, \dots, k_M; n_2, \dots, n_M | = \lim_{\xi_1 \rightarrow 0} \langle 0 | \frac{\mathcal{W}(\xi_1, \bar{\xi}_1, k_1)}{|\xi_1|^2} \prod_{j=2}^M \oint_{|\xi_j|=0} \frac{d^2 \xi_j}{|\xi_j|^2} \cdot \\ \cdot |\xi_j|^{\frac{q_{j-1}^2 - q_j^2}{2} + n_{j-1} - n_j} \mathcal{W}(\xi_2, \bar{\xi}_2, k_2) \dots \mathcal{W}(\xi_M, \bar{\xi}_M, k_M) \end{aligned} \quad (\text{B.14})$$

corresponds to the graph of Fig. 15 and is an off-mass shell generalization of the states defined in Ref. 2) for the conventional model and for the SV model. The covariance under the L_n 's is broken in each addendum of Eq. (B.13) by the factors

$$|\xi_j|^{\frac{q_{j-1}^2 - q_j^2}{2} + n_{j-1} - n_j}$$

which do not transform at all ; when all the intermediate states are on-shell all these non-covariant factors disappear.

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FIG.1

FIG.2

FIG.3

FIG.4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG. 9a

FIG. 9b

FIG. 9c

FIG. 10

FIG.11

FIG.12

FIG.13

FIG.14

FIG.15