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SCALE DIMENSION OF CHIRAL SYMMETRY BREAKING
AND RIGOROUS CONSTRAINTS ON $K\ell_3$ DECAY

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A B S T R A C T

Using new techniques for deriving bounds satisfying the requirements of unitarity and analyticity, optimal constraints on the scalar $K\ell_3$ form factor are obtained, given the s wave, $I = \frac{1}{2}$, $K\pi$ elastic phase shift. It is assumed that the propagators of current divergences satisfy unsubtracted dispersion relations, that axial divergences are dominated by pion and kaon poles and that the chiral symmetry breaking Hamiltonian transforms as $(3, \bar{3}) + (\bar{3}, 3)$. Recent SLAC-Santa Cruz data satisfy these constraints, as does the Callan-Treiman relation. The constraint on the slope of the form factor, given either the Callan-Treiman relation or the data, is badly violated by the current algebra prediction of Dashen and Weinstein. This violation suggests the necessity for subtraction, corresponding to a chiral symmetry breaking Hamiltonian density with scale dimension greater than or equal to two. The results depend only weakly on the elastic phase shift.

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1. - INTRODUCTION

Bounds involving both form factors and propagators and exploiting analyticity have been derived and investigated numerically by various authors ¹⁾. In the case of $K\ell_3$ decay the current algebra (CA) and partially conserved axial current (PCAC) constraints on the propagators of current divergences permit one to bound the scalar form factor and its derivatives ²⁾⁻⁴⁾ provided the propagator of the divergence of the strangeness changing vector current at zero momentum transfer, $\Delta(0)$, is given by an unsubtracted dispersion relation. These bounds can be made more stringent if the s wave, $I=\frac{1}{2}$, $K\pi$ scattering amplitude is known over some portion of the physical region ⁵⁾. A detailed account of bounds for two- and three-point functions, when relevant partial wave amplitudes are known, will be given elsewhere ¹⁾. Here we are concerned with the specific case of $K\ell_3$ decay, when the elastic $K\pi$ phase shift is given from threshold up to some maximum energy (which may be somewhat above inelastic threshold if four-body states couple weakly to $K\pi$). Our treatment is an essential improvement over that of Bourrely ⁵⁾, who used Schwarz inequalities. Our bounds are optimal ; that is they can in principle be saturated consistent with both analyticity and unitarity. In addition, we are readily able systematically to incorporate an arbitrary amount of discrete information about the form factor and its derivatives on the real line and on the elastic cut.

Our result takes the form of a lower bound for $\Delta(0)$ in terms of a quadratic form of the values of the form factor or any of its derivatives at given points on the real line or elastic cut. The coefficients of the quadratic form are given by the resolvent of an integral equation whose kernel depends upon integrals of the elastic phase shift. The lower bound is valid if $\Delta(0)$ is given by an unsubtracted dispersion relation, for which it is necessary that d , the scale dimension of the chiral symmetry breaking Hamiltonian density, is less than two ⁶⁾. In the $(3,\bar{3})+(\bar{3},3)$ model of chiral symmetry breaking there is an upper bound for $\Delta(0)$ in terms of the axial divergence propagators $\Delta_\pi(0)$ and $\Delta_K(0)$ ⁷⁾. These we approximate by pion and kaon pole terms, to obtain a bound which permits numerical investigation with a variety of inputs. Our principal results are as follows.

- (i) To satisfy the bound the scalar form factor must have positive curvature throughout the $K\mu_3$ Dalitz plot and may not fall below its value at zero momentum transfer by more than 1%. This is consistent with the recent SLAC-Santa Cruz results ⁸⁾ but not with some previous analyses ⁹⁾.

- (ii) The bound is approximately 50% saturated using either the Callan-Treiman relation ¹⁰⁾ or the SLAC-Santa Cruz slope. This is inconsistent with certain predictions for $\Delta(0)$ ¹¹⁾ and is on the borderline of what is allowed by a value of the Gell-Mann, Oakes and Renner ¹²⁾ chiral symmetry breaking parameter $c = -1.25$.
- (iii) Using either the Callan-Treiman relation or the SLAC-Santa Cruz slope, the bound is badly violated unless the slope at the unphysical point $t = m_K^2 + m_\pi^2$ exceeds the current algebra prediction of Dashen, Li, Pagels and Weinstein ¹³⁾ by 40%.

These results are not sensitive to small deviations from the $(3, \bar{3}) + (\bar{3}, 3)$ pattern of chiral symmetry breaking or from π and K pole dominance of axial divergence propagators. They depend only weakly on the form of the $K\pi$ phase shift. From them we conclude that either spontaneously broken $SU(3) \times SU(3)$ is not an acceptable symmetry of the strong interactions or that d , the scale dimension of the chiral symmetry breaking Hamiltonian density, is greater than or equal to 2. This latter possibility is consistent with the quark model, where in the absence of interactions $d = 3$.

The paper is organized as follows. In Section 2 we formulate the problem and by the introduction of an appropriate conformal mapping and of incomplete Omnès functions we reduce it to a general extremal problem in which the phase shift does not appear explicitly. This is solved in Section 3. Further details (together with a treatment of the more general case when partial wave amplitudes in the inelastic region are known) will be given elsewhere ¹⁾. In Section 4, we report and discuss the numerical results obtained with a variety of theoretical and experimental inputs. Our conclusions are summarized in Section 5. Discussions of the further constraints on chiral symmetry breaking imposed by Weinberg sum rules and of the subtraction constants required in the quark model will be given elsewhere ¹⁴⁾.

2. - EXTREMAL PROBLEM

We first define the two-point functions

$$\Delta_{ij}^{(5)}(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T^* (\partial_\mu J_i^{\mu(5)}(x) \partial_\nu J_j^{\nu(5)}(0)) | 0 \rangle \quad (2.1)$$

where i and j are $SU(3)$ labels and $J_i^{\mu(5)}$ the vector (axial) currents. We assume throughout that the time-ordered product is less singular than x^{-4} at short distances. In the theory of broken scale invariance the leading c number singularity in the operator product expansion behaves as x^{-2d} , so we require $d < 2$ (6), 15). In this case the T product may be made covariant such that the propagators satisfy unsubtracted dispersion relations 16).

We are specifically concerned with

$$\Delta(t) \equiv \Delta_{44}(t)$$

to whose absorptive part s wave, $I = \frac{1}{2}$, $K\pi$ states contribute. In addition, we require

$$\begin{aligned} \Delta_\pi(t) &\equiv \Delta_{11}^{(5)}(t), \\ \Delta_K(t) &\equiv \Delta_{44}^{(5)}(t). \end{aligned}$$

With a chiral symmetry breaking Hamiltonian density belonging to a $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$ 12),

$$\mathcal{H}_{S8}(x) = U_0(x) + c U_8(x), \quad (2.2)$$

$\Delta(0)$ and $\Delta_{\pi,K}(0)$ are given by the vacuum expectation values of $U_{0,8}$ 17). Eliminating these we obtain

$$\Delta(0) = \Delta_K(0) + \Delta_\pi(0) - \left[\chi \Delta_K(0) + \frac{1}{\chi} \Delta_\pi(0) \right], \quad (2.3)$$

where

$$\chi = \left(\frac{\sqrt{2} + c}{\sqrt{2} - c/2} \right). \quad (2.4)$$

In the $SU(3)$ limit $\chi = 1$ and in the $SU(2) \times SU(2)$ limit $\chi = 0$. From Eq. (2.3) it follows that either

$$\Delta(0) \leq \left([\Delta_K(0)]^{\frac{1}{2}} - [\Delta_\pi(0)]^{\frac{1}{2}} \right)^2, \quad (2.5a)$$

for $\chi \geq 0$ (being an equality when $\chi = [\Delta_\pi(0)/\Delta_K(0)]^{\frac{1}{2}}$), or

$$\Delta(0) \geq \left([\Delta_K(0)]^{\frac{1}{2}} + [\Delta_\pi(0)]^{\frac{1}{2}} \right)^2, \quad (2.5b)$$

for $\chi \leq 0$ (being an equality when $\chi = -[\Delta_\pi(0)/\Delta_K(0)]^{\frac{1}{2}}$). [Note that the positivity of $\Delta_{\pi,K}(0)$ is ensured by the assumption of unsubtracted dispersion relations for current divergence propagators.]

We now appeal to continuity with the $SU(3)$ limit ($\chi = 1$) to reject (2.5b). This is consistent with the Gell-Mann, Oakes and Renner estimate $c \approx -1.25$, corresponding to $\chi \approx (m_\pi/m_K)^2 > 0$. Approximating $\Delta_{\pi,K}(0)$ by pion and kaon pole terms we obtain from inequality (2.5a)

$$\Delta(0) \leq (m_K f_K - m_\pi f_\pi)^2. \quad (2.6)$$

(The normalization is such that pion and kaon decays give $f_\pi = 0.66 m_\pi$, $f_K/f_\pi \approx 1.27$.)

The upper bound (2.6) is the starting point for the analysis of Bourrely⁵⁾. We have spelt out the details of the derivation in order to expose the assumptions. This is pertinent, since we shall show that inequality (2.6) is inconsistent with the $K\pi$ phase shift, combined with the Dashen-Weinstein prediction¹³⁾ and either the Callan-Treiman relation¹⁰⁾ or the experimental slope of the scalar $K_{\mu 3}$ form factor⁸⁾.

It has been assumed that :

- (i) $\mathcal{H}_{SB}$ has scale dimension $d < 2$;
- (ii) $\mathcal{H}_{SB}$ belongs to a $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$;
- (iii) $\chi \geq 0$ [see Eq. (2.4)] ;
- (iv) $\Delta_{\pi, K}(0)$ are approximately given by the pion and kaon pole terms.
[In fact, we shall only need kaon pole dominance to within a factor of two to violate inequality (2.6).]

We now proceed to investigate the lower bound for $\Delta(0)$ given by the $K\pi$ contributions to the dispersion relation ²⁾ :

$$\Delta(0) \geq \frac{3t_+t_-}{64\pi} [f_+(0)]^2 \mathcal{F}, \quad (2.7)$$

where

$$\mathcal{F} = \frac{1}{\pi} \int_{t_+}^{\infty} \frac{dt}{t^2} [(t-t_+)(t-t_-)]^{\frac{1}{2}} |K(t)|^2, \quad (2.8)$$

$$t_{\pm} = (m_K \pm m_{\pi})^2, \quad (2.9)$$

$$K(t) = [f_+(t) + \left(\frac{t}{m_K^2 - m_{\pi}^2}\right) f_-(t)] / f_+(0), \quad (2.10)$$

and $f_{\pm}(t)$ are the standard $K_{1/2}$ form factors, normalized such that $f_+(0) \approx 1$ by the Ademollo-Gatto theorem ¹⁸⁾.

The extremal problem we must solve is that of finding the optimal lower bound for $\mathcal{F}$ given :

- (i) the analyticity of $K(t)$ in the t plane cut along $[t_+, \infty]$;
- (ii) the phase of $K(t)$ on the elastic cut $[t_+, t_m]$;
- (iii) the values of $K(t)$ or any of its derivatives at points on the real line $(-\infty, t_+)$ or the elastic cut $[t_+, t_m]$.

We use the conformal mapping

$$\left(\frac{1-z}{1+z} \right) = (1-t/t_m)^{\frac{1}{2}} \quad (2.11)$$

so that

$$t(z) = \frac{4t_m z}{(1+z)^2} \quad (2.12)$$

and from Eq. (2.8)

$$J = \frac{1}{\pi} \int_{z_+}^1 dz |G(z)|^2 + \frac{1}{\pi} \int_0^\pi d\theta |G(e^{i\theta})|^2, \quad (2.13)$$

where

$$G(z) = h(z) K(t(z)), \quad (2.14)$$

$$h(z) = \frac{1}{z} \left[\frac{(z-z_+)(z-z_-)}{(1-zz_+)(1-zz_-)} \right]^{\frac{1}{4}} \left[\frac{(1-z)(1-zz_+)(1-zz_-)}{(1+z)(1+z_+)(1+z_-)} \right]^{\frac{1}{2}}, \quad (2.15)$$

$$t(z_{\pm}) = t_{\pm} = (m_K \pm m_\pi)^2. \quad (2.16)$$

Next we construct the function $F(z)$ satisfying :

(i) $F(z) \in \mathcal{H}_2$ [i.e., $F(z)$ is analytic for $|z| < 1$ and quadratically integrable on $|z| = 1$];

(ii) $|F(e^{i\theta})| = |G(e^{i\theta})|;$ (2.17)

(iii) $F(z)/K(t(z)) \neq 0$ for $|z| < 1$.

This is achieved with the help of the functions ^{19),20)}

$$N(z) = \exp \left[\left(\frac{1-z^2}{\pi} \right) \int_{z_+}^1 \frac{dz' \delta(z')}{(z'-z)(1-zz')} \right], \quad (2.18)$$

$$\begin{aligned}
 D(z) &= \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \log |R(e^{i\theta})| \right] \\
 &= \left[\frac{(1-z)(1-\bar{z}z_+)(1-\bar{z}z_-)}{(1+z)(1+\bar{z}z_+)(1+\bar{z}z_-)} \right]^{\frac{1}{2}},
 \end{aligned} \tag{2.19}$$

where

$$\delta(z) = \arg K(t(z+i\epsilon)), \quad z \in [z_+, 1]. \tag{2.20}$$

[Note that $\delta(z)$ must be defined to be continuous on the elastic cut and satisfy $\delta(z_+) = 0$ at elastic threshold.]

With these definitions we have, almost everywhere on the unit circle,

$$\begin{aligned}
 |D(e^{i\theta})| &= |R(e^{i\theta})|, \\
 |\mathcal{N}(e^{i\theta})| &= 1
 \end{aligned}$$

and on the elastic cut

$$\operatorname{Im} [K(t(z))/\mathcal{N}(z)] = 0, \quad z \in [z_+, 1].$$

Hence

$$F(z) = K(t(z))D(z)/\mathcal{N}(z) \in \mathcal{H}_2 \tag{2.21}$$

is the required function and from Eqs. (2.13), (2.14), (2.17) and (2.21), we obtain

$$\mathcal{I} = \int_{z_+}^1 dz [v(z)F(z)]^2 + \frac{1}{\pi} \int_0^{\pi} d\theta |F(e^{i\theta})|^2, \tag{2.22}$$

where

$$V(z) = \frac{1}{\pi^{\frac{1}{2}}} \frac{R(z)}{D(z)} |\mathcal{N}(z)|$$

$$= \frac{1}{\pi^{\frac{1}{2}} z} \left[\frac{(z-z_+)(z-z_-)}{(1-zz_+)(1-zz_-)} \right]^{\frac{1}{4}} \exp \left[\left(\frac{1-z^2}{\pi} \right) \mathcal{P} \int_{z_+}^1 \frac{dz' \delta(z')}{(z'-z)(1-zz')} \right], \quad (2.23)$$

from Eqs. (2.15), (2.18) and (2.19).

The problem has now been reduced to that of finding the optimal lower bound for $\mathcal{J}$, as given by Eq. (2.22), given the values of the real analytic function $F(z)$ (or combinations of its derivatives) on the real line $z \in (-1, 1)$. The phase shift appears only implicitly through the incomplete Omnès function $\Omega(z)$ in Eqs. (2.21) and (2.23).

This general problem is solved in the next section, without reference to the specific details of $K\ell_3$ decay.

3. - SOLUTION OF EXTREMAL PROBLEM

For $|z| < 1$ we may expand the real analytic function $F(z)$ in terms of its Fourier coefficients

$$F(z) = \sum_{\mu=0}^{\infty} F_{\mu} z^{\mu} \quad (3.1)$$

where

$$F_{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta F(e^{i\theta}) e^{-i\mu\theta} = F_{\mu}^* \quad (3.2)$$

and according to the completeness condition for $\mathcal{H}_2$ functions

$$\sum_{\mu=0}^{\infty} (F_{\mu})^2 = \frac{1}{\pi} \int_0^{\pi} d\theta |F(e^{i\theta})|^2 < \infty. \quad (3.3)$$

Equation (2.22) may be written as

$$\mathcal{F} = \sum_{\mu, \nu} F_{\mu} T^{\mu\nu} F_{\nu}, \quad (3.4)$$

where

$$T^{\mu\nu} = V^{\mu\nu} + \delta^{\mu\nu}, \quad (3.5)$$

$$V^{\mu\nu} = \int_{z_+}^1 dz [V(z)]^2 z^{(\mu+\nu)} \quad (3.6)$$

[The interchange of the order of integration and summation is legitimate for the functional $\mathcal{F}_{\epsilon} < \mathcal{F}$ for which the upper limit of integration in Eqs. (2.22) and (3.6) is taken as $(1-\epsilon)$ instead of 1. If the limit $\epsilon \rightarrow 0^+$ is taken the optimal lower bound for $\mathcal{F}_{\epsilon}$ coincides with the optimal lower bound for $\mathcal{F}$ deduced in this section.]

Suppose we are given

$$X_i = \vec{D}_i F(z_i) \quad i=1, n \quad z_i \in (-1, 1) \quad (3.7)$$

where $\vec{D}_i$ is some differential operator polynomial in (d/dz_i) . Then

$$\begin{aligned} \mathcal{F} = \mathcal{L}(F_{\mu}) &= \sum_{\mu, \nu} F_{\mu} T^{\mu\nu} F_{\nu} \\ &+ 2 \sum_{i=1}^n \lambda_i \left[X_i - \sum_{\mu} F_{\mu} \vec{D}_i z_i^{\mu} \right]. \end{aligned} \quad (3.8)$$

To obtain the extremum of $\mathcal{F}$ we vary the Lagrangian $\mathcal{L}$ with respect to $\{F_\mu | \mu = 0, \omega\}$. This gives

$$\sum_\nu T^{\mu\nu} F_\nu = \sum_i \lambda_i \vec{D}_i z_i^\mu, \quad (3.9)$$

$$\mathcal{F} \geq \mathcal{F}_{\min} = \sum_i \lambda_i X_i, \quad (3.10)$$

where the Lagrange multipliers $\{\lambda_i | i=1, n\}$ are determined by Eqs. (3.1), (3.7) and (3.9). The result of Eq. (3.10) may therefore be written as

$$\mathcal{F}_{\min} = \sum_{i,j} X_i (M^{-1})_{ij} X_j \quad (3.11)$$

where M^{-1} is the inverse of the $n \times n$ matrix M given by

$$M_{ij} = \vec{D}_i M(z_i, z_j) \overleftarrow{D}_j, \quad (3.12)$$

$$M(z_i, z_j) = \sum_{\mu, \nu} z_i^\mu (T^{-1})_{\mu\nu} z_j^\nu, \quad (3.13)$$

and T^{-1} is the inverse of the infinite matrix given by Eq. (3.5).

Let

$$(T^{-1})_{\mu\nu} = \delta_{\mu\nu} - V_{\mu\nu} + T_{\mu\nu} \quad (3.14)$$

then from Eqs. (3.5) and (3.6) it may be verified that

$$T_{\mu\nu} = \iint_{z_1}^1 dz_1 dz_2 z_1^\mu V(z_1) R(z_1, z_2) V(z_2) z_2^\nu, \quad (3.15)$$

where $R(z_1, z_2)$ is the resolvent of the kernel

$$K(z_1, z_2) = \frac{v(z_1)v(z_2)}{(1-z_1z_2)}, \quad (3.16)$$

such that

$$KR = RK = K - R. \quad (3.17)$$

Finally, from Eqs. (3.6) and (3.14)-(3.16) we obtain

$$M(z_i, z_j) = \frac{1}{1-z_i z_j} - \iint_{z_+} dz_1 dz_2 \frac{v(z_1)[\delta(z_1-z_2)-R(z_1, z_2)]v(z_2)}{(1-z_i z_1)(1-z_2 z_j)}. \quad (3.18)$$

Equations (3.11), (3.12) and (3.16)-(3.18) give the solution of the extremal problem formulated in the previous section.

We note the following points.

- (i) In evaluating the diagonal elements of M the differentiation (if any) in Eq. (3.12) is to be performed before setting $z_i = z_j$.
- (ii) For $z_i, z_j \in (z_+, 1)$

$$M(z_i, z_j) = \frac{R(z_i, z_j)}{v(z_i)v(z_j)}$$

so that it is particularly simple to calculate a bound if the modulus of $K(t(z))$ [and hence the value of $F(z)$] is given on the elastic cut.

- (iii) The kernel K of Eq. (3.16) is not quadratically integrable. It belongs, however, to the class of completely continuous operators, which ensures the existence of R in the space of quadratically

integrable functions [see Ref. 1) for details]. If we choose a sequence of finite rank operators K_N , of which K is the uniform limit, then R is the limit of the sequence of operators R_N , obtained from K_N by finite matrix inversion [using Eq. (3.17)]. Thus, to obtain an approximate result for the integral of Eq. (3.18), we choose the points

$$z_i = z_+ + \frac{(2i-1)}{2N} (1-z_+) \quad i = 1, N,$$

invert the $N \times N$ matrix

$$\delta_{ij} \left(\frac{N}{1-z_+} \right) + K(z_i, z_j) \quad (3.19)$$

to obtain an approximation to the operator $1 - R$, and evaluate the double integral of Eq. (3.18) by finite summation. For the results presented in the next section we used $N = 81$. We have verified that little changes for $81 \geq N \geq 41$. [For example, none of the numbers given in column 1 of Table I change by more than 1% if N is reduced from 81 to 41.]

We conclude this section by comparing our optimal bound with previous less restrictive ones. In a calculation of a bound for the nucleon's renormalization constant ²⁰⁾ we ignored the elastic contribution to $\mathcal{G}$ in obtaining a lower bound. The result is given by replacing the matrix M , defined by Eqs. (3.12) and (3.18), by M_0 , where

$$(M_0)_{ij} = \vec{D}_i \left(\frac{1}{1-z_i z_j} \right) \overleftarrow{D}_j.$$

A more efficient (but still not optimal) bound was given by Bourrely ⁵⁾, who used a Schwarz inequality for the elastic contribution to Eq. (2.22) :

$$\int_{z_+}^1 dz [v(z)F(z)]^2 \geq \frac{\left[\int_{z_+}^1 dz w(z)v(z)F(z) \right]^2}{\int_{z_+}^1 dz [w(z)]^2},$$

where $w(z)$ is any quadratically integrable function. Repeating our general Lagrange multiplier method (for an arbitrary input) we find a result given by replacing M by M_1 where

$$(M_1)_{ij} = (M_0)_{ij} - \frac{1}{C} \vec{D}_i (n(z_i) n(z_j)) \vec{D}_j^{\leftarrow},$$

where

$$n(z) = \int_{z_+}^1 \frac{dz' w(z') v(z')}{1 - z z'},$$

and

$$C = \iint_{z_+}^1 dz_1 dz_2 w(z_1) [\delta(z_1 - z_2) + K(z_1, z_2)] w(z_2).$$

To evaluate this bound for a given $w(z)$ it is not necessary to solve an integral equation. The result is, however, less restrictive than our optimal bound unless we choose the weight function

$$w(z) \propto v(z) F^{\text{ext}}(z)$$

where $F^{\text{ext}}(z)$ is the solution to our general extremal problem. In this case the Schwarz inequality is saturated and one should obviously reproduce the optimal bound. Indeed it is easy to show that for this weight function the following matrix equation holds

$$M_1 = M_0 - \frac{(M_0 - M) \lambda \tilde{\lambda} (M_0 - M)}{\tilde{\lambda} (M_0 - M) \lambda},$$

where

$$X = M \lambda$$

from which we find that

$$X = M_1 \lambda$$

and

$$\tilde{\lambda} (M_1)^{-1} X = \tilde{\lambda} M^{-1} X$$

and the two bounds coincide.

In fact Bourrelly restricted his investigation to the class of weight functions

$$w(z) = \frac{1}{v(z)} \left[a_1 + \frac{1}{z} \right],$$

where various values for a_1 were tried. Such a choice lessens the burden of computation. It will, however, considerably weaken the bound if there is a narrow resonance in the elastic region. In this case $F^{\text{ext}}(z)$ is fairly smooth near resonance whilst $v(z)$ contains the rapidly varying resonant behaviour. This will mean that Bourrelly's weight function will differ from the optimal one by a factor $|\Omega(z)|^2$ near resonance and much of the elastic contribution to the optimal bound will be lost by using the Schwarz inequality. But because of the resonance the elastic contribution will be important. Thus the bound will be far from optimal.

After completing the calculations of Section 4, we became aware of a paper by Okubo²¹⁾ in which a solution is given to the extremal problem considered here [with the slight restriction that the input of Eq. (3.7) does not involve derivatives]. His solution agrees with ours and he observes that the resolvent R of Eq. (3.17) is related to a "self-reproducing kernel" known to mathematicians. Okubo discusses an iterative solution to Eq. (3.17). In fact, the numerical computation of bounds is more straightforward using the method outlined above. As long as K is completely continuous [see Ref. 1] the resolvent R exists in $\mathcal{L}_2$ space and may be approximated by $1 - (1 + K)^{-1}$ using finite matrix inversion.

4. - RESULTS

We write the bound as

$$V \equiv f_{\text{MIN}} / f_{\text{MAX}} \leq 1, \quad (4.1)$$

where $f_{\min}$ is given by Eq. (3.11) and

$$f_{\max} = \frac{64\pi}{3} \left[\frac{m_K f_K - m_\pi f_\pi}{f_+(0)(m_K^2 - m_\pi^2)} \right]^2, \quad (4.2)$$

using Eqs. (2.6) and (2.7). Experimentally ²²⁾ $f_\pi = 0.66 m_{\pi^+}$ and ⁹⁾ $f_K/f_\pi f_+(0) = 1.27 \pm 0.03$. The Ademollo-Gatto theorem ¹⁸⁾ tells us $f_+(0) \approx 1$. Note that $f_{\max}$ is fairly insensitive to small variations of $f_+(0)$ from unity. We evaluate it with $f_K/f_\pi = 1.27$, $f_+(0) = 1$.

For the purpose of orientation we give the old result of Okubo ¹¹⁾, using only the normalization $K(0) = 1$ and setting $t_m = (m_K + m_\pi)^2$, so that the phase shift is not required. In this case

$$f_{\min} = [F(0)]^2 = [D(0)]^2 = \frac{[m_K^{\frac{1}{2}} + m_\pi^{\frac{1}{2}}]^2}{4(m_K + m_\pi)}, \quad (4.3)$$

yielding

$$V = 0.17. \quad (4.4)$$

In other words the bound is 17% saturated, without using phase shifts, experimental $K_{\ell 3}$ results or CA predictions for the form factor. This is the base line from which we start.

Next we consider what improvement results from using the phase shift. Older analyses of the $K\pi$ system ²³⁾⁻²⁵⁾ were unable to distinguish between a non-resonant S wave phase shift below c.m. energy of 1 GeV ("down" solution) and one with a narrow resonance (of width less than, say, 30 MeV) "hiding" underneath the P wave $K^*(890)$ ("up" solution). Accordingly, we have calculated with both the "down" and "up" solutions of Bingham et al. ²⁵⁾. However, the recent LBL analysis of Barbaro-Galtieri et al. ²⁶⁾ was able to exclude an "up" solution, unless there is a resonance whose width is less than 7 MeV. We take the LBL "down" solution as our preferred phase shift.

The "down" solutions are parametrized by a scattering length and effective range :

$$\cot \delta^{\text{down}}(t) = \frac{1}{a R(t)} + \frac{1}{2} r R(t), \quad (4.5)$$

where

$$R(t) = \left[\frac{(t - t_+)(t - t_-)}{4t} \right]^{\frac{1}{2}}, \quad (4.6)$$

is the c.m. momentum. For the "up" solution a resonant phase is added :

$$\delta^{\text{up}}(t) = \delta^{\text{down}}(t) + \delta^{\text{res}}(t), \quad (4.7)$$

$$\tan \delta^{\text{res}}(t) = \frac{m_r \Gamma_r}{m_r^2 - t} \left[\frac{R(t)}{R(m_r^2)} \right], \quad (4.8)$$

where m_r and Γ_r are the resonance mass and width, respectively.

The three phases we use are listed with their parameters in Table II and are plotted in Fig. 1.

It remains to consider over what energy region these phases may be used. Both analyses were performed for $0.79 \text{ GeV} \leq \sqrt{t} \leq 1.0 \text{ GeV}$. We will use them down to threshold, $\sqrt{t} = m_K + m_\pi \approx 0.63 \text{ GeV}$. Our conclusions do not depend critically on the precise form of the phase shift near threshold. It is only necessary that there be no dramatic structure between threshold and 0.8 GeV . A more difficult choice is that of the maximum c.m. energy. Strictly, the phase of $K(t)$ is only given by $K\pi$ scattering below inelastic threshold, $\sqrt{t} \leq (m_K + 3m_\pi) \approx 0.91 \text{ GeV}$. However, neither the $K\pi\pi\pi$ nor the $K\eta$ channel is known to be important below 1.4 GeV ²⁴⁾. According to Bingham et al. ²⁵⁾ the S wave $K\pi$ amplitude may be described by an elastic phase up to 1.1 GeV . We have chosen to evaluate bounds for each of the three phase shifts (A), (B) and (C), using

$$\begin{aligned}
 (1) \quad \sqrt{t_m} &= m_K + 3m_\pi \approx 0.91 \text{ GeV}, \\
 (2) \quad \sqrt{t_m} &= 1.0 \text{ GeV}, \\
 (3) \quad \sqrt{t_m} &= 1.2 \text{ GeV},
 \end{aligned}$$

and we consider that our results for the phase shift (A2) are both reliable and reasonably exploit the information available. The spread in our results will indicate the sensitivity to the precise form of the phase shift and the energy range over which it is known.

In column 1 of Table I, we give the values of V obtained with different phase shift information using only the normalization $K(0)=1$. It can be seen that V increases from 0.17 to between 0.31 and 0.38 using the phase shift.

In column 2 of Table I, we give the values of V which result from using, in addition, the Callan-Treiman relation ¹⁰⁾

$$K(m_K^2) = f_K/f_\pi f_+(0) = 1.27 \pm .03, \quad (4.9)$$

in each case giving the smallest value of V consistent with the error bars. The bound is now between 42% and 56% saturated. This is already inconsistent with certain predictions of $\Delta(0)$ (remember our bound only uses the CA upper bound). For example, Mathur and Okubo ^{11),17)} estimated

$$\begin{aligned}
 \Delta(0) &\approx 2.0 m_\pi^2 f_\pi^2, \\
 f_K/f_\pi &\approx 1.08,
 \end{aligned}$$

which would require $V \lesssim 0.25$. From an analysis by Brandt and Preparata ²⁷⁾ Okubo ¹¹⁾ estimated

$$\begin{aligned}
 \Delta(0) &\approx 4.5 m_\pi^2 f_\pi^2, \\
 f_K/f_\pi &\approx 1.20,
 \end{aligned}$$

which would require $V \lesssim 0.42$. Already our bound is surprisingly restrictive.

Before discussing the $K_{\mu 3}$ data we show what restrictions the requirement $V \leq 1$ imposes on the form factor and its slope. In Fig. 2, we give upper and lower bounds for $K(t)$, using the preferred phase shift (A2). Without imposing the Callan-Treiman relation the form factor must lie between curves (i) and (ii), for the bound to be satisfied. Curve (iii) gives that form factor which gives the smaller V ($= 0.31$). Imposing $K(m_K^2) = 1.27 \pm 0.03$ the form factor must lie between curves (iv) and (v). It is incorrect to suppose that any form factor which traces out a curve between (iv) and (v) is consistent with the bound and the Callan-Treiman relation, since there are also constraints on the derivatives of the form factor. In Fig. 3, we give the corresponding curves for the slope of the form factor.

The constraints of Figs. 2 and 3 are clearly inconsistent with older $K_{\mu 3}$ analyses, which claimed that $K(t)$ was significantly below 1 over much of the Dalitz plot⁹⁾. However, we now have the results of the SLAC-Santa Cruz collaboration⁸⁾. They measured 1.6×10^6 $K_L^0 \rightarrow \pi \mu \nu$ decays, were largely able to resolve the quadratic kinematic ambiguity, binned the events in 5×5 MeV sectors of the Dalitz plot, and applied radiative corrections. The values of the scalar form factor they obtained are given in Fig. 4, together with the lower and upper bounds consistent with our bound and the Callan-Treiman relation [curves (iv) and (v) of Fig. 2]. Apart from an over-all normalization discrepancy of about 5% the trend of values is not glaringly inconsistent with our limits. In Ref. 8) it was remarked that a straight line fit to the data gives

$$K(t) = (1.06 \pm .03) + (.017 \pm .005) \frac{t}{m_{\pi}^2},$$

but the failure to satisfy the normalization condition at $t=0$ was not explained. Rather, it was observed that an excellent two-parameter fit to the whole Dalitz plot data could be achieved with

$$f_+(t)/f_+(0) = 1 + (.030 \pm .003) \frac{t}{m_{\pi}^2}, \quad (4.10)$$

$$K(t) = 1 + (.020 \pm .003) \frac{t}{m_{\pi}^2}, \quad (4.11)$$

where the quoted errors are twice those imposed by the statistics. Accordingly we interpret the SLAC-Santa Cruz experiment as giving

$$m_\pi^2 K'(3m_\pi^2) = .020 \pm .003, \quad (4.12)$$

$$m_\pi^4 |K''(3m_\pi^2)| \leq 10^{-3}, \quad (4.13)$$

where we choose $t = 3m_\pi^2$ as being in the middle of the accessible range and require that the slope not vary by more than 30% for $(m_K - m_\pi)^2 \geq t \geq m_\mu^2$. We have checked that (4.13) is consistent with all the bounds we present. The result (4.12) is consistent with our bound and the Callan-Treiman relation, which require (see Fig. 3)

$$.020 \geq m_\pi^2 K'(3m_\pi^2) \geq .015$$

for our preferred phase shift.

In column 3 of Table I, we give the values of V which result from using the experimental slope (4.12) (again taking the smallest values consistent with the errors). The bound is now between 50% and 66% saturated. No further improvement results from using both the Callan-Treiman relation and the experimental slope, since the form factors giving these values of V are all consistent with (4.9), within the error limits.

Thus the SLAC experiment can accommodate our bound and the Callan-Treiman relation. But the bound is suspiciously close to saturation, bearing in mind that we have neglected the inelastic contributions to $\Delta(0)$ and allowed it to have its largest possible value consistent with CA and PCAC. In fact it is instructive to use the Gell-Mann, Oakes and Renner value ¹²⁾

$$\chi \approx (m_\pi/m_K)^2$$

in Eq. (2.3). With PCAC, this gives

$$\Delta(0) \approx (m_K^2 - m_\pi^2)(f_K^2 - f_\pi^2)$$

and, assuming $f_+(0)=1$, requires $V \lesssim 0.58$, which is just on the borderline of what is allowed by the experimental slope. One should keep in mind that such estimates of $\Delta(0)$ are not very reliable since $\Delta(0)$ is a quantity of second order in chiral symmetry breaking but has been estimated in terms of the first order quantities $\Delta_{\pi,K}(0)$.

Finally we add the Dashen-Weinstein relation ¹³⁾. This is the last straw, which breaks the back of our bound and demonstrates that it is not possible to satisfy all the constraints of current algebra [with $(3, \bar{3}) + (\bar{3}, 3)$ chiral symmetry breaking] and an unsubtracted relation for $\Delta(0)$.

Dashen, Li, Pagels and Weinstein ¹³⁾ derived the result :

$$2 f_+(0)(m_K^2 - m_\pi^2) K'(m_K^2 + m_\pi^2) \approx \frac{f_K}{f_\pi} - \frac{f_\pi}{f_K} . \quad (4.14)$$

The right-hand side of Eq. (4.14) is only the leading term in the $SU(3) \times SU(3)$ limit and is of order $\epsilon \log \epsilon$ where ϵ measures the breaking of $SU(3) \times SU(3)$ and $SU(3)$. There are in addition terms of order ϵ , which are calculable given $(3, \bar{3}) + (\bar{3}, 3)$ chiral symmetry breaking, and high order terms of order ϵ^2 . We write their result as

$$m_\pi^2 K'(m_K^2 + m_\pi^2) = (.021 \pm .002)(1 + 3.38)(1 + \eta), \quad (4.15)$$

where the errors come from taking $f_K/f_\pi f_+(0) = 1.27 \pm 0.03$, the second factor gives the sensitivity to

$$\delta \equiv f_+(0) - 1 , \quad (4.16)$$

assuming $|\delta| \ll 1$, and the third factor includes the order $(1/\log \epsilon)$ corrections, η . In Ref. 13), the higher order contributions due to $K\pi$ states were calculated giving

$$\eta \approx \frac{3(m_K^2 f_K^2 + m_\pi^2 f_\pi^2)(m_K^4 - m_\pi^4 - 4m_K^2 m_\pi^2 \log(m_K/m_\pi))}{128 \pi^2 f_\pi f_K (f_K^2 - f_\pi^2)(m_K^2 - m_\pi^2)^2} \approx 0.10, \quad (4.17)$$

using $f_K/f_\pi \approx 1.27$. It remains to consider the sensitivity to δ (which is of order ϵ^2). Quinn and Bjorken²⁸⁾ have shown that octet dominance of the sum rule²⁹⁾ for $[f_+(t)]^2$ requires $f_+(t) < 1$ for $t \leq 0$ and hence that δ is negative. Various authors³⁰⁾ have estimated $\delta \approx -0.04$. We shall use the Dashen-Weinstein result in the form of a conservative CA upper bound, for which we assume

$$\delta \leq 0. \quad (4.18)$$

From Eqs. (4.15), (4.17) and (4.18), we take the result

$$m_\pi^2 K'(m_K^2 + m_\pi^2) \leq .026 \quad (4.19)$$

as very conservative statement of CA and PCAC.

In Ref. 13), Dashen et al. state that if the scalar form factor in $K_{\mu 3}$ decay is linear over the Dalitz plot and its slope fails to agree with their prediction at the unphysical point $t = m_K^2 + m_\pi^2$, then one must critically reexamine the hypotheses that chiral $SU(3) \times SU(3)$ is an approximate symmetry of the strong interaction and that weak currents can be related to almost conserved $SU(3) \times SU(3)$ currents. We are in a position directly to test their prediction, even in its attenuated form of inequality (4.19), without appeal to $K_{\mu 3}$ experiments.

In column 4 of Table I, we give the values of V which result from the Dashen-Weinstein and Callan-Treiman relations, represented by (4.19) and (4.9). For every phase shift the bound is badly violated. It will be seen that there is a large scatter in the values of V obtained with different phase information. This is because the quadratic form of Eq. (3.11) is a very rapidly varying function of $K'(m_K^2 + m_\pi^2)$ [once both $K(0)$ and $K(m_K^2)$ are specified] and the value which minimizes V is slightly different according to what phase shift we use. A better expression of our result is the lower bound on $m_\pi^2 K'(m_K^2 + m_\pi^2)$ imposed by the Callan-Treiman relation and our bound, $V \leq 1$. This lower bound is given in column 1 of Table III and varies by less than 5% from one set of phase information to the next. We have also given the lower bound for $V \leq 2$ (allowing kaon PCAC to be wrong by as much as a factor of 2) in column 2 of Table III. In every case the weakened Dashen-Weinstein result (4.19) is violated.

The same situation applies if we substitute the experimental slope of Eq. (4.12) for the Callan-Treiman relation. The bounds are given in column 5 of Table I and the lower limits for the Dashen-Weinstein slope are given in columns 3 and 4 of Table III (corresponding to $V \leq 1$ and $V \leq 2$). The violation of our bound is again dramatic. (As before we find that the combination of the Callan-Treiman relation and the experimental slope, within their errors, does not give any result more restrictive than the results of Tables I and III, which are obtained using these two inputs singly.)

Finally in Fig. 5 we plot V as a function of the slope at the Dashen-Weinstein point with the Callan-Treiman relation or the experimental slope as input, taking into account their errors.

Our bound finally violated, we conclude that the conventional structure of CA and PCAC can only be maintained, in the face of $K\pi$ data and the requirements of unitarity and analyticity, at the expense of having a subtraction in the dispersion relation for $\Delta(0)$.

5. - CONCLUSION

We have systematically and efficiently investigated the constraints on the scalar $K_{\mu 3}$ form factor imposed by unitarity and analyticity and following from the requirements of :

- (i) unsubtracted dispersion relations for the propagators of current divergences ;
- (ii) approximate $SU(3) \times SU(3)$, broken by a piece of the Hamiltonian transforming as $(3, \bar{3}) + (\bar{3}, 3)$;
- (iii) strong PCAC, in the form of approximate pion and kaon pole dominance of $\Delta_{\pi, K}(0)$ and the soft pion and kaon results of Callan and Treiman and of Dashen and Weinstein (it should be emphasized that these PCAC results should be reliable if spontaneous symmetry breaking is realized in the manner proposed by Goldstone and Nambu³¹⁾).

We have shown that it is not possible to satisfy all three of these requirements. Specifically we have found that a very conservative statement of the Dashen-Weinstein result, when combined with the Callan-Treiman relation (or, equivalently, the SLAC-Santa Cruz experimental slope) badly violates our bound (by a factor of 2 at the very least). This result is not critically dependent upon the $K\pi$ phase shift. We do not consider that the violation may be resolved by small modifications in the pattern of chiral symmetry breaking or by a small failure of kaon pole dominance. Instead we believe that the approximate low energy theorems of PCAC, combined with unitarity and analyticity, indicate the necessity of a subtraction in the dispersion relation for $\Delta(0)$. Such a situation is not new to physics. For the last twenty years it has been clear that the low energy theorem for Compton scattering³²⁾, combined with unitarity (in the form of the optical theorem) and the analytic structure of the spin-averaged forward Compton amplitude (derived by Gell-Mann, Goldberger and Thirring³³⁾), is inconsistent with an unsubtracted dispersion relation³⁴⁾. The violation of the inequality

$$-\frac{\alpha}{M} = \int d\nu \sigma_T(\nu) > 0$$

is glaringly obvious. In our case the analysis is more complicated, depending as it does upon the approximate low energy theorems of "partially" conserved currents and upon more subtle (but no less rigorous) inequalities required by analyticity and unitarity. Our tentative conclusion is, however, relevant to the asymptotic behaviour of a two-point function, $\Delta(t)$, (rather than a four-point function in the case of Compton scattering) and hence reflects the short distance behaviour of an operator product. If we have a scale dimension of current divergences $d \geq 2$ then a subtraction in the dispersion relation for $\Delta(t)$ is mandatory, due to its asymptotic behaviour t^{d-2} , determined by Wilson's approach to broken scale invariance⁶⁾. Of course, it is conceivable to have $d < 2$, but $\Delta(0)$ (as given by equal time commutators) not equal to the finite dispersive integral

$$\frac{1}{\pi} \int_{t_+}^{\infty} \frac{dt}{t} \text{Im } \Delta(t).$$

Such a situation would be very unpalatable, indicating that nature requires subtraction constants when none is obligatory. Rather, we suggest that our bound is badly violated because $d \geq 2$. This conclusion is of some importance, since d is also the scale dimension of the chiral symmetry breaking Hamiltonian density (assuming only that the latter is a Lorentz scalar³⁵⁾). In the quark-gluon model, from which leading light cone singularities are often abstracted³⁶⁾,

$$\mathcal{H}_{SB}(x) = \bar{\Psi}(x) m \Psi(x)$$

where m is the quark mass matrix. The naïve scale dimension of $\mathcal{H}_{SB}$ is 3, and in light cone analyses of the chiral symmetry breaking structure functions of deep inelastic neutrino scattering³⁷⁾ $d=3$ is normally assumed. Unfortunately it is not in general possible to say how the interaction modifies d from its free field value. (We note, however, that the anomalous dimension of $\bar{\Psi}\Psi$ is negative to lowest order in QED³⁸⁾). But at least our conclusion is not inconsistent with the much used quark model.

In low energy applications of current algebra it is normal practice to posit unsubtracted dispersion relations, unless these are at variance with the asymptotic behaviour of Regge theory. Our work indicates that one should also beware of any asymptotic behaviour imposed by having $d \geq 2$. Brandt and Preparata³⁹⁾, for example, have questioned applications of current algebra and pion and kaon pole dominance which do not take into account the question of subtractions required with $d=3$. It is sobering to remark that some derivations of the Gell-Mann, Oakes and Renner result that c is close to $-\sqrt{2}$ [i.e., that $SU(2) \times SU(2)$ is an appreciably better symmetry than $SU(3) \times SU(3)$] rest critically upon the assumption of unsubtracted dispersion relations¹⁷⁾ for Δ_π and Δ_K . It is usually argued that low energy applications of current algebra are insensitive to high energy behaviour, since even if the dispersion relations employed need subtraction, the subtraction constants will be "small"⁴⁰⁾ compared with the low energy contributions. This is a defensible position to adopt for the dispersion relations for $\Delta_{\pi,K}(0)$, since the subtraction constants will be of higher order in chiral symmetry breaking than the pole terms generated by Goldstone bosons. In the case of $\Delta(0)$, however, both the subtraction constant and the low energy $K\pi$ states will be of second order in chiral symmetry breaking [assuming that $SU(3)$ is not spontaneously broken by a scalar Goldstone boson : there is little experimental evidence for a kappa meson]. It is reasonable therefore that the $K\ell_3$ complex investigated in this paper is indeed sensitive to high energy behaviour and that the violation of our bound may be resolved by concluding that $d \geq 2$, without otherwise doing violence to the conventional structure of spontaneously broken $SU(3) \times SU(3)$.

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Input Phase	1 -	2 CT	3 SS	4 CT + DW	5 SS + DW
A1	0.31	0.42	0.50	4.4	4.4
A2	0.31	0.44	0.53	7.7	7.3
A3	0.33	0.45	0.54	18.3	16.9
B1	0.32	0.50	0.59	3.4	3.5
B2	0.32	0.54	0.64	5.5	5.5
B3	0.35	0.56	0.66	11.4	11.1
C1	0.33	0.46	0.54	2.6	2.8
C2	0.33	0.51	0.61	2.7	2.9
C3	0.38	0.55	0.65	3.5	3.6

TABLE I - Values of V for various phases and inputs
(CT = Callan-Treiman relation ; SS = SLAC-Santa Cruz
slope ; DW = Dashen-Weinstein relation).

Phase	Ref.	Type	$a(\text{Fm})$	$r(\text{Fm})$	$m_r(\text{MeV})$	$\Gamma_r(\text{MeV})$
A	26	down	0.31	-1.4	--	--
B	25	down	0.24	-1.9	--	--
C	25	up	0.16	-3.3	890	30

TABLE II - Phase shift parameters.

Input Phase	1	2	3	4
	CT $V \leq 1$	CT $V \leq 2$	SS $V \leq 1$	SS $V \leq 2$
A1	0.029	0.028	0.031	0.029
A2	0.029	0.029	0.031	0.030
A3	0.030	0.029	0.032	0.031
B1	0.028	0.027	0.030	0.028
B2	0.029	0.028	0.030	0.029
B3	0.029	0.028	0.031	0.030
C1	0.028	0.027	0.030	0.027
C2	0.028	0.027	0.030	0.027
C3	0.029	0.027	0.031	0.028

TABLE III - Lower bounds for $m_{\pi}^2 K'(m_K^2 + m_{\pi}^2)$, corresponding to saturation of bound or violation by a factor of 2, for various phases and inputs (CT = Callan-Treiman relation ; SS = SLAC-Santa Cruz slope).

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FIGURE CAPTIONS

Figure 1 Phase shifts A, B and C (modulo π) as functions of $\sqrt{t}$. The parameters are given in Table II.

Figure 2 The scalar form factor, $K(t)$, must lie between curves (i) and (ii) to satisfy the bound. Curve (iii) gives the smallest value of $V=0.31$. If the Callan-Treiman relation is imposed, $K(m_K^2)=1.27\pm0.3$, then $K(t)$ must lie between curves (iv) and (v) to satisfy the bound.

Figure 3 As for Fig. 2, giving limits on $m_\pi^2 K'(t)$.

Figure 4 The SLAC-Santa Cruz data [Ref. 8] with the envelope allowed by the Callan-Treiman relation and our bound.

Figure 5 V as a function of the slope at the Dashen-Weinstein point. The dashed curve uses the Callan-Treiman relation. The solid curve uses the SLAC-Santa Cruz slope. The discontinuities result from calculating the smallest V consistent with the uncertainties in the input.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5