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ANALYTIC REPRESENTATION OF THE SIX-POINT FUNCTION
AND INCLUSIVE PROCESSES

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A B S T R A C T

A new representation is proposed for the N point retarded function defined as the Fourier transform of the matrix element between incoming states of the R product of two field operators. Locality, crossing and normal cut structure in the invariant variables containing the Fourier vector are correctly incorporated in this representation and an explicit expression for the spectral function can be given. The case $N = 6$ is discussed in detail and structure functions for deep inelastic as well as purely hadronic inclusive processes are obtained from the representation.

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1. - INTRODUCTION

Deep inelastic and inclusive processes have received very much attention over the past few years. The deep inelastic processes of second generation

- (i) single inclusive scattering

$$e + N \rightarrow e' + \text{hadron } (h) + \text{anything}$$

- (ii) double inclusive annihilation

$$e^+ + e^- \rightarrow h + h' + \text{anything}$$

- (iii) massive lepton pair production

$$p + p \rightarrow \text{lepton pair} + \text{anything}$$

require the knowledge of the discontinuities of the six-point function off shell in the mass of the electromagnetic currents. Analogously, inclusive hadronic processes of the kind

$$(iv) \quad h + h' \rightarrow h'' + \text{anything}$$

are determined by the on shell six-point function.

It is thus of importance to study the analytic structure of the six-point function. Obviously a dispersion-like representation would be best suited for this purpose because discontinuities can then be taken in a very transparent manner.

In fact, such a representation has already been used to investigate the continuation problem from deep inelastic scattering to annihilation ¹⁾.

If one looks at the expression for the structure function, e.g., for process (iii) :

$$\sum_X \langle in, p_1 p_2 | j(0) | X, out \rangle_c \langle out, X | j(0) | p_1 p_2 in \rangle_c \delta^{(4)}(p_1 + p_2 + q - p_X),$$

one sees that it is related to a certain discontinuity of the following matrix element :

$$T_6^{\text{ret}}(q, p_1, p_2, p'_1, p'_2) = i \int d^4x e^{iqx} \langle \text{in } p'_1 p'_2 / \theta(x_0) [j(x), j(0)] / p_1 p_2 \text{ in} \rangle_c \quad (1)$$

(Here, c means connected part).

Our aim will therefore be to obtain a representation for the six-point function T_6^{ret} . Note that T_6^{ret} is not a scattering amplitude because (a) in-states appear on both sides, (b) the R product reduction formalism is only equivalent to the T product formalism for 2-to-N reactions ²⁾. Property (a) is necessary because only then can we arrive at absolute squares by inserting intermediate states. As is well known from dispersion theory, the choice of the retarded product - property (b) - is required for the discussion of analytic properties.

A complete knowledge of the analytic structure of T_6^{ret} is, of course, out of question. We want to confine ourselves to a much narrower scope and consider only singularities of T_{6*}^{ret} with fixed thresholds. These so-called "basic cuts" are also the subject ³⁾ of recent investigations by Stapp and co-workers ³⁾ in the context of "Muellerism" ⁴⁾. It is very difficult to say something about other types of singularities, e.g., anomalous thresholds, whose position depends on the external masses of the amplitude.

The restriction to fixed branch points is very essential because it admits a multiple Cauchy representation. We even restrict our investigation further to those variables which contain the vector q, which tests locality via the Fourier transform in Eq. (1). Within this very restricted framework we propose a representation for T_6^{ret} which satisfies the spectrum property of physical states, locality and crossing. For its construction, the Steinmann ⁵⁾ relations will be a crucial ingredient. It is a dispersion-type

*) The cuts dealt with in this work are not quite the same as the "basic cuts" of Ref. 3) since in T_6^{ret} two legs are off shell.

representation ^{*)} and has the advantage that the multiple discontinuity can be given an explicit expression. This is a marked difference to the rigorous Jost-Lehmann-Dyson or (less rigorous) Deser-Gilbert-Sudarshan representations whose spectral functions are too general and do not seem to have much direct physical significance ⁹⁾.

Within the limited framework outlined above, our representation, being causal, also constitutes an answer to the problem raised by Feynman ⁹⁾ concerning the general form of a causal scattering amplitude. In addition it allows a unified treatment of the deep inelastic processes (i), (ii) and (iii).

The organization of the paper is as follows. In Section 2, we exhibit the representation and describe its building principle. Section 3 is devoted to a discussion of the scope of our representation. Further motivation for it is cited and several important properties are commented on. In Section 4 discontinuities and structure functions are obtained. Some remarks on crossing relations between different processes are made in Section 5. We end in Section 6 with a few model-independent scaling considerations following from our ansatz.

^{*)} A multiple spectral representation for the three- and four-point function was proposed by Nambu, Ref. 6). Integral representations in perturbation theory were derived by Nakanishi, Ref. 7). Cauchy-type representations for the form factor have recently been written down by B. Andersson (Lund preprints LU-TP 73-2, 3 and 74-1). A general representation for the N particle function using the Bergmann-Weil integral formula is treated in Ref. 8). We are indebted to D. Olive for calling Ref. 8) to our attention.

2. - THE REPRESENTATION

We take scalar particles and currents. Using the R product contraction to bring particles p_1' and p_2' to the other side, we can choose all particle momenta as ingoing and define : $p_1' = -p_3$, $p_2' = -p_4$ with $p_i^2 = m^2$.

The two currents $J(x)$ ($J(0)$) carry momenta q and $Q = q - p_1 - p_2 - p_3 - p_4$, respectively. We subdivide all invariant (Mandelstam) variables into two sets. The first one comprises the variables formed with the vector q and is given by

$$q^2, Q^2, s_i = v_i^2, s_{ij} = v_{ij}^2, s_{ijk} = v_{ijk}^2, \quad (2)$$

where

$$\begin{aligned} v_i &= q - p_i \\ v_{ij} &= q - p_i - p_j & i \neq j \\ v_{ijk} &= q - p_i - p_j - p_k & i \neq j \neq k \\ i, j, k &= 1, \dots, 4 \end{aligned}$$

The second set does not contain q . We call them "external variables" :

$$t_{ij} = (p_i + p_j)^2, \quad i \neq j; \quad i, j = 1, \dots, 4 \quad (3)$$

Then the representation is :

$$\begin{aligned} T_6^{\text{ret}}(q, p_1 \dots p_4) &= \pi^{-5} \sum_{\text{Perm}} \int_{q_{th}^2}^{\infty} \frac{dq'^2}{q'^2 - q_+^2} \int_{q_{th}^2}^{\infty} \frac{dQ'^2}{Q'^2 - Q_+^2} \\ &\cdot \int_{\sigma_{th}}^{\infty} \frac{d\sigma_1}{\sigma_1 - s_{i+}} \int_{\sigma_{th}}^{\infty} \frac{d\sigma_2}{\sigma_2 - s_{ij+}} \int_{\sigma_{th}}^{\infty} \frac{d\sigma_3}{\sigma_3 - s_{ijk+}} \cdot \\ &\cdot g_6(q'^2, Q'^2, \sigma_1, \sigma_2, \sigma_3 | t_{ij}, t_{ke}, t_{ik}, t_{je}, t_{ie}, t_{jk}) \end{aligned} \quad (4)$$

Here the + sign means that q_0 has to be given a small positive imaginary part : $q = (q_0 + i\epsilon, \vec{q})$.

The sum over permutations runs over all groups of three variables s_i, s_{ij}, s_{ijk} which are non-dual to each other. This is most easily visualized by drawing a hexagon familiar from dual theories (Fig. 1).

Figure 1 - Dual diagram for the choice of variables.

Non-dual, q dependent channel variables correspond to non-intersecting diagonals drawn from the common meeting point of q and Q . Each term in the representation corresponds to a particular permutation of the four external momenta $p_1 \dots p_4$ in the hexagon keeping q and Q fixed. Hence there are 24 terms altogether.

Furthermore, the spectral weight g_6 is given by an explicit expression :

$$\begin{aligned}
 & \theta(q_0) \theta(Q_0) \theta(v_{i,0}) \theta(v_{ij,0}) \theta(v_{ijk,0}) \theta(q^2) \theta(Q^2) \theta(s_i) \theta(s_{ij}) \theta(s_{ijk}) \\
 & \cdot g_6(q^2, Q^2, s_i, s_{ij}, s_{ijk} / t_{ij}, t_{ke}; t_{ik}, t_{je}; t_{ie}, t_{jk}) \\
 & = 2^{-5} \int d^4x d^4y_1 \dots d^4y_4 e^{iqx} e^{-i(p_i y_1 + p_j y_2 + p_k y_3 + p_l y_4)} \\
 & \langle 0 | j(x) j(y_1) j(y_2) j(y_3) j(y_4) j(0) | 0 \rangle
 \end{aligned} \tag{5}$$

where the $j(y_i)$ are the sources of the external particles.

The lower limits of integration q_{th}^2 and σ_{th} in Eq. (4) are the fixed, normal thresholds determined by the lowest intermediate states in the channels characterized by the variables of Eq. (2). Due to the $i\epsilon$ prescription the Fourier transform of $T_6^{ret}(q)$ is the convolution of retarded functions and therefore itself retarded (i.e., has support only in the forward light cone). Thus, the two requirements of normal spectrum and retard- edness are built into our representation.

From Eq. (5) it would seem as if each term in the representation had its own spectral weight g_6 depending on the assignment of momenta p_i to the sources $j(y_1) \dots j(y_4)$. Each assignment in turn defines a triplet s_i, s_{ij}, s_{ijk} also appearing in the denominator of the same term. However, in the scalar model the spectrum in the three channels s_i, s_{ij}, s_{ijk} is always the same. Hence g_6 is one and the same function of the integration variables for every term [as anticipated by the unified notation $\sigma_1, \sigma_2, \sigma_3, q_{th}^2$ and σ_{th} in Eq. (4)], only taken at different points in the space of the external variables. The latter play the rôle of parameters in g_6 and are uniquely determined by the triplet s_i, s_{ij}, s_{ijk} in each term. In Eq. (5), the convention for these external variables is such that in forward direction ($\sum_{i=1}^4 p_i = 0$) they become equal in pairs ($t_{ij} = t_{kl}$, etc). Furthermore, from PT invariance g_6 is real (for real four vectors). Using this, complex conjugation of the right-hand side of Eq. (5) yields the symmetry relation for g_6 :

$$g_6(q'^2, Q'^2, \sigma_1, \sigma_2, \sigma_3 / t_{ij}, t_{kl}; t_{ik}, t_{jl}; t_{ie}, t_{jk}) \\ = g_6(Q'^2, q'^2, \sigma_3, \sigma_2, \sigma_1 / t_{kl}, t_{ij}; t_{jl}, t_{ik}; t_{ie}, t_{jk}) \quad (6)$$

In the same manner we obtain the "crossing" relation for T_6^{ret} :

$$T_6^{ret}(q, p_1, p_2, p'_1, p'_2) = T_6^{ret}(-q, p'_1, p'_2, p_1, p_2)^* \quad (7)$$

which is satisfied by virtue of the symmetric ansatz (4) and the reality of g_6 .

3. - ON THE SCOPE AND THE PROPERTIES OF THE REPRESENTATION

The six-point function T_6^{ret} is a complex function of the invariant variables introduced in the preceding chapter. From perturbation theory we know that T_6^{ret} has essentially two types of singularities : the normal cuts with fixed thresholds in certain variables and moving singularities (e.g., anomalous thresholds). We restrict our discussion to that part of T_6^{ret} which is completely determined by the normal cuts in the variables of the set Eq. (2). All other variables [i.e., those given in Eq. (3)] are treated as parameters. A complex function with the normal cut structure allows a multiple Cauchy representation ¹⁰⁾. In treating these cuts as independent, we ignore the restriction coming from the Gram determinant, to which we return below. The Steinmann relations ⁵⁾ tell us that double discontinuities with respect to variables, which are dual to each other, are zero for physical momentum four vectors. In the case of the six-point function with two external legs off-shell this implies that except for subtraction terms T_6^{ret} consists of a sum of five-fold dispersion integrals. The dispersed variables in each term are the two mass variables q^2 and Q^2 of the off-shell legs and three other non-dual variables of the set (2). To obtain the most general form one has to sum over all possibilities. This reflects itself in the general structure of Eq. (4).

In order to obtain an explicit expression for the five-fold spectral function we contract all particles in T_6^{ret} until we reach the vacuum expectation value of a retarded product of six operators :

$$T_6^{\text{ret}} = i \int d^4x d^4y_1 \dots d^4y_4 e^{iqx} e^{-i \sum_{j=1}^4 p_j y_j} \langle 0 | R(j(x); j(0)j(y_1) \dots j(y_4)) | 0 \rangle \quad (8)$$

By Fourier transforming the step functions contained in the R product, one arrives at a multiple Low representation for T_6^{ret} . This representation consists of a sum of 120 terms with five-fold dispersion integrals involving the zero components of momentum vectors. In twenty-four of these terms, q_0 appears in each of the five denominators. In fact, the zero components in the denominators of these terms are exactly those of the vectors defined in Eq. (2). The twenty-four terms are given by :

$$\begin{aligned}
& \frac{1}{(2\pi)^5} \sum_{\text{Perm}} \int \frac{d^4 q'_0}{q'_0 - q_0 - i\epsilon} \int \frac{d^4 Q'_0}{Q'_0 - Q_0 - i\epsilon} \int \frac{d^4 v'_{i0}}{v'_{i0} - v_{i0} - i\epsilon} \int \frac{d^4 v'_{ij0}}{v'_{ij0} - v_{ij0} - i\epsilon} \\
& \cdot \int \frac{d^4 v'_{ijk0}}{v'_{ijk0} - v_{ijk0} - i\epsilon} \int d^4 x d^4 y_1 \dots d^4 y_4 e^{iq'x} e^{-i(p'_i y_1 + p'_j y_2 + p'_k y_3 + p'_e y_4)} \\
& \cdot \langle 0 | [[[[[j(x), j(y_1)] \dots] j(y_4)] j(0)]] | 0 \rangle \\
& q' = (q'_0, \vec{q}) , \quad p'_i = (p'_{i0}, \vec{p}_i) \\
& v'_{i0} = q'_0 - p'_{i0} \quad \text{etc.} \quad i = 1, \dots, 4
\end{aligned} \tag{9}$$

The sum over permutations is the same as in formula (4).

Taking the five-fold discontinuity with respect to $q_0, Q_0, v_{i0}, v_{ij0}, v_{ijk0}$ we are left with the Fourier transform of the multiple commutator. This discontinuity differs from the one taken with respect to the invariants $q^2, Q^2, s_i, s_{ij}, s_{ijk}$ by factors $\epsilon(q_0), \epsilon(Q_0) \dots \epsilon(v_{ijk0})$ ($\epsilon(x) = \text{sign function}$). Since $\langle 0 | j(0) | p_i, p_i^2 = m^2 \rangle = 0$, only those two members of the commutator survive in which j is not next to the vacuum state. In this way one finally obtains Eq. (5) for g_6 , where the inclusion of the step functions for the zero components selects one of the two members.

As regards the remaining 96 terms they contain at least one q_0 independent denominator. Therefore they do not contribute to the five-fold discontinuity discussed above. They are to be considered as subtractions.

Further arguments for the validity of our representation come from the work of the Heidelberg group a few years ago, which attempted a local saturation scheme of current algebra ¹¹⁾. Their procedure to saturate commutator matrix elements was an iterative one, each order being defined by the number of particles kept in the intermediate states, but with locality preserved in every order of approximation. The resulting expressions for form factors and scattering functions are of a multiple pole type. They can be recovered by inserting the intermediate states of a given order into the spectral function g of the appropriate retarded amplitude T_3^{ret} or T_4^{ret} using representations analogous to Eq. (4) [see Ref. 12]. The most concise

way to see how the pole structure is built up is to use the formulation of the local approximation scheme in terms of the Haag expansion ¹³⁾ for a field operator as described in Ref. 14). With this technique it has been checked explicitly for T_6^{ret} that the systematics of the multiple denominators is exactly as stated in Eq. (4).

Although the arguments sketched in the last paragraph are not entirely independent of our initial reasoning they shed some further light on our representation. In particular the appearance of only non-dual variables in one single term (Steinmann relations !) is seen to be a consequence of field theory.

It should be clear from the preceding that it is straightforward to generalize the representation to the case of the N point retarded function. All one has to do is to draw the appropriate polygon to sort out the non-dual variables and to insert $N-2$ sources for the external particles between $J(x)$ and $J(0)$ in the expression for the spectral weight.

The great advantage of this representation - as compared to the Jost-Lehmann-Dyson or Deser-Gilbert-Sudarshan representations - is the explicit form of the spectral weight which allows one to make models. Also all terms share the same spectral function. This enables one to correlate the structure functions for different processes obtained by taking different discontinuities ¹⁾.

The representation analogous to Eq. (4) for the Compton amplitude (T_4^{ret} in our nomenclature) restricted to forward direction is very similar to the form guessed by Feynman ⁹⁾. However, one important lesson to be drawn from this work is the fact that one should not represent the square form factor by one single dispersion integral in q^2 , but one has to treat the form factors separately. Inasmuch as anomalous thresholds are neglected, our causal expression Eq. (4) is an answer to the problem posed by Feynman (loc.cit.).

For a higher-point function there is the problem of independent variables. The amplitude T_6^{ret} is a function of the invariants formed with the five kinematically unrestricted vectors $q, p_1 \dots p_4$, of which q is off-shell. Hence there are eleven independent invariants, namely the six external variables t_{ij} [Eq. (3)] and five q dependent ones. The sixteen variables of Eq. (2) are therefore connected by Mandelstam type relations :

$$s_i + s_j = q^2 + s_{ij} + 2m^2 - t_{ij} \quad (10a)$$

$$s_{ij} + s_{kl} = q^2 + Q^2 + 8m^2 + t_{ij} + t_{kl} - \sum_{m < n} t_{mn} \quad (10b)$$

$$s_i + s_{jkl} = q^2 + Q^2 + 6m^2 + t_{jk} + t_{jl} + t_{kl} - \sum_{m < n} t_{mn} \quad (10c)$$

$$s_{ij} + s_k = s_{ijk} + q^2 + 4m^2 - t_{ik} - t_{jk} \quad (10d)$$

$$s_i + s_j + s_k = s_{ijk} + 2q^2 + 6m^2 - t_{ij} - t_{ik} - t_{jk} \quad (10e)$$

(All indices have to be different.)

Equations (10) altogether constitute 29 relations, but only eleven of them are independent as it should be. They are such that the five q dependent channel variables of one and the same term in the representation (4) are never connected to each other. Alternatively speaking, the Mandelstam relations (10) can be characterized by observing that they always interrelate "crossed" or "overlapping" channel energies of the set Eq. (2). By construction, such variables never appear in one term of the representation as a consequence of the Steinmann relations.

We now turn to the important question of how far the known analytic properties of T_6^{ret} in the variables t_{ij} are reproduced by our representation.

We observe that for the six-point function there cannot be more than three non-dual variables (in addition to q^2 and Q^2). If we denote by τ any variable of the set $\{t_{ij}, t_{kl}, \dots, t_{jk}\}$ appearing as argument in a particular g_6 , then τ is necessarily "crossed" (i.e., dual) to at least one member of the non-dual triplet s_i, s_{ij}, s_{ijk} . Therefore the Steinmann relations imply that

$$\text{disc}_{\tau} g_6(q^2, Q^2, s_i, s_{ij}, s_{ijk} / t_{ij}, \dots, t_{jk}) = 0 \quad (11)$$

holds in the physical domain (for real vectors), where the discontinuity is taken across the normal cut in τ . In view of Eq. (11) one might ask how the normal cut of T_6^{ret} in τ is generated in our representation as we move from τ values below threshold to $\tau = 4m^2$. Since the spectral weight does not have the cut, it must be created by a singularity in one of the dispersion integrals of the representation. In fact, such a mechanism works in the familiar case of a fixed t dispersion relation for the ordinary four-point function. We discuss this problem in Appendix A.

Apart from cuts in t_{ij} , T_6^{ret} also has single particle poles in variables like $(p_1 + p_2 + p_3)^2$, which are linear combinations of t_{ij} 's. This is evident ^{*)} from the cluster decomposition of the matrix element occurring in T_6^{ret} , which will be taken up in the next section. The spectral function g_6 as given by Eq. (5) does not possess such poles. Therefore we interpret our representation as being valid for the difference $T_6^{\text{ret}} - (T_6^{\text{ret}})_{\text{poles}}$. It should be stressed at this point that the multiple dispersion integral together with the explicit form of g_6 given above constitute only a part of the full T_6^{ret} since we threw away the subtraction terms which do contain the above-mentioned poles. Our ansatz could be considered as a model for the analyticity of that part of T_6^{ret} which (hopefully) is the relevant one for structure functions.

Finally, there is a geometrical restriction coming from the dimensionality of space-time : the five vectors $q, p_1 \dots p_4$ are linearly dependent, thus the Gram determinant vanishes :

$$G(q, p_1, p_2, p_3, p_4) = \begin{vmatrix} q \cdot q & q \cdot p_1 & q \cdot p_2 & q \cdot p_3 & q \cdot p_4 \\ p_1 \cdot q & p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot p_4 \\ p_2 \cdot q & p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 & p_2 \cdot p_4 \\ p_3 \cdot q & p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 & p_3 \cdot p_4 \\ p_4 \cdot q & p_4 \cdot p_1 & p_4 \cdot p_2 & p_4 \cdot p_3 & p_4 \cdot p_4 \end{vmatrix} = 0 \quad (12)$$

^{*)} See Appendix B.

Only the first row and the first column contain q dependent variables, therefore Eq. (12) is of second order in s_i, s_{ij}, s_{ijk} and q^2 .

This restriction arises for the first time at the level of the six-point function. It implies that at fixed t_{ij} the cuts in the q dependent variables are no longer independent. [With the help of relations (10) one can always reformulate Eq. (12) in such a manner that it contains only the variables of a fixed term in the representation (4).] We have to neglect this dependence in order that the five-fold discontinuity of T_6^{ret} still be given by g_6 as in Eq. (5). One can, however, incorporate the Gram condition a posteriori by restricting each representation term to the manifold given by Eq. (12) [suitably rewritten with Eqs. (10)]. Obviously this restriction does not destroy the locality property of the representation, since the Fourier transform of Eq. (4) into space vectors runs over the momentum vectors $q, p_1 \dots p_4$. Therefore, it extends only over that surface in the space of invariant variables which is defined by the vanishing Gram determinant.

One might add that T_6^{ret} is only an auxiliary quantity. Physically measurable quantities are derived from it in forward direction where the Gram restriction becomes empty.

4. - DISCONTINUITIES AND STRUCTURE FUNCTIONS

The total q_0 discontinuity of T_6^{ret} is equal to the Fourier transform of the commutator :

$$\text{disc}_{q_0} T_6^{\text{ret}}(q) = i \int d^4x e^{iqx} \langle \text{in } p'_1 p'_2 | [j(x), j(0)] | p_1 p_2 \text{ in} \rangle_c \quad (13)$$

The amplitude T_6^{ret} depends on q_0 through the invariant variables formed with the vector q . Hence, by successively adding and subtracting terms in which the imaginary part of q_0 is reversed - as compared to the preceding term - in only one of these invariants, one can convert $\text{disc}_{q_0} T_6^{\text{ret}}$ into a

sum of discontinuities with respect to the variables of Eq. (2), multiplied by sign functions for the zero components of the vectors v of Eq. (2). These discontinuities correspond to the various semi-disconnected terms obtained by inserting a complete set of intermediate states into the right-hand side of Eq. (13) and using the cluster decomposition for each factor. According to how many external particles act as spectators there are sixteen different types of intermediate states, whose invariant energies are the channel energies (2). However, the individual terms look quite differently depending on whether the saturation is done with "in" or with "out" states, due to different cluster decompositions.

For "out" states we have ^{*)} :

$$\begin{aligned} \langle in p'_1 p'_2 | J | N out \rangle &= \langle in p'_1 p'_2 | J | N out \rangle_c + \langle p'_1 | J | \hat{N} out \rangle_c \langle p'_2 | \hat{p} \rangle \\ &+ (p'_1 \leftrightarrow p'_2) + \sum_{a+b=N} \langle 0 | J | a, out \rangle \langle in p'_1 p'_2 | b, out \rangle \end{aligned} \quad (14)$$

since $\langle in p'_1 p'_2 | J | N out \rangle$ is basically an S matrix element. We list the various contributions to (13) arising from (14) in a diagrammatic notation, where c in a blob symbolizes the connected matrix element of the current (wavy line) and $a, b, \dots$ are intermediate cut states (the sum is always understood). We also indicate the mass of the intermediate state connecting the two current matrix elements :

no spectator

^{*)} Compare Ref. 15) ; the cluster decomposition given there is incomplete.

one spectator

(15b)

two spectators

(15c)

or

(15d)

where $S^{-1} = \langle \text{in } p_1' p_2' | a \text{ out} \rangle$ is the full S matrix element ;

three spectators

(15e)

or

(15f)

here only the connected part (T^+) of S^{-1} contributes since the matrix element in (13) is connected as a whole ;

four spectators

(15g)

or

(15h)

or

or

here $q = p_a + p_d$, $Q = p_a + p_c$.

The other possible diagrams are obtained by permutations. There is an analogous set of diagrams originating from the opposite order of the currents in Eq. (13). The two sets together reproduce the sign functions mentioned before ¹⁶⁾. Thus each graph can be attributed to a discontinuity according to the mass of the intermediate state as indicated [the funny diagram (15j) contributes if both q^2 and Q^2 are timelike]. The above diagrams are analogous to those found in S matrix theory ¹⁷⁾.

Things are different for "in" saturation. In the last term of Eq. (14) only $|b \text{ in} \rangle = |\text{two-particle state} \rangle$ survives yielding a product of two δ functions for the three momenta. This can be seen by expanding $J(x)$ into a complete set of in-field operators [Ref. 13], working out the matrix element between in-states with Wick's theorem and remembering that

the coefficients in the Haag expansion ¹³⁾ are retarded functions. Hence their Fourier transforms are boundary values of analytic functions, i.e., free of δ functions and therefore connected ^{*)}.

Consequently the diagrams for the "in" saturation of Eq. (13) are obtained from the set (15) by crossing out all S matrix elements, replacing the label "out" by "in" for all intermediate states and dropping those graphs which contain a T blob. Thus the discontinuities of T_6^{ret} are represented in two different forms. We have found no direct way to demonstrate their equivalence.

As can be seen from (15a), the structure function for lepton pair production (with $q_0 < 0$)

$$p_1 + p_2 \rightarrow \text{"current"} + \text{anything}$$

$$W_e = \sum_N \langle \text{in } p_1 p_2 | \mathcal{J} | N \text{out} \rangle_c \langle N \text{out} | \mathcal{J} | p_1 p_2 \text{in} \rangle_c \delta^{(4)}(p_1 + p_2 + q - p_N) \quad (16)$$

is equal to the s_{34} discontinuity of T_6^{ret} in forward direction. Similarly from (15c), the structure function for single inclusive scattering

$$\text{"current"} + p_1 \rightarrow p_2 + \text{anything}$$

$$W_S = \sum_N \langle p_1 | \mathcal{J} | N, p_2 \text{out} \rangle_c \langle N, p_2 \text{out} | \mathcal{J} | p_1 \rangle_c \cdot \delta^{(4)}(q + p_1 - p_2 - p_N) \quad (17)$$

is given by the s_{23} discontinuity.

The structure function for double inclusive annihilation

$$\text{"current"} \rightarrow p_1 + p_2 + \text{anything}$$

$$W_a = \sum_N \langle 0 | \mathcal{J} | N, p_1 p_2 \text{out} \rangle \langle N, p_1 p_2 \text{out} | \mathcal{J} | 0 \rangle \delta^{(4)}(q - p_1 - p_2 - p_N) \quad (18)$$

^{*)} We are indebted to V. Glaser for making this clear to us.

can be identified with the s_{12} discontinuity using CPT invariance and looking at the diagram corresponding to (15g) for in saturation (remember that then the S matrix elements are absent !).

We can now use ^{*)} our representation, Eq. (4), to work out these discontinuities. In order to convert them into the physical structure functions one has to keep in mind the different $i\epsilon$ prescriptions holding for the latter. For a general structure function depicted in Fig. 2, these prescriptions are :

Figure 2 - The general structure function

$$\begin{aligned} & Q^2 + i\epsilon, (Q + p_a)^2 + i\epsilon, (Q + p_e)^2 + i\epsilon \\ & q^2 - i\epsilon, (q - p_i)^2 - i\epsilon, (q - p_j)^2 - i\epsilon \end{aligned}$$

as indicated in Fig. 2 - or all signs of the ϵ 's reversed which is irrelevant in forward direction. But it is necessary to start from the non-forward case to keep track of the opposite $i\epsilon$'s ! (Whether or not the $i\epsilon$ prescription is really necessary for a particular variable depends on the kinematics.) These rules are of course nothing but the recipes of "Muellerism" ⁴⁾ which were put on a more solid basis by Stapp et al. ³⁾. Explicitly, we get from our representation :

$$\begin{aligned} W_S(q, p_1, p_2) = & \theta(v_{23,0}) \theta(s_{23} - s_{th}) \pi^{-5} \int_{q_{th}^2}^{\infty} \frac{dq'^2}{q'^2 - q^2} \int_{q_{th}^2}^{\infty} \frac{dQ'^2}{Q'^2 - q^2} \\ & \int_{\sigma_{th}}^{\infty} d\sigma_1 d\sigma_3 \left\{ \frac{g^{(1)}(q'^2, Q'^2, \sigma_1, s_{23}, \sigma_3; t)}{(\sigma_1 - s_{234} + i\epsilon)(\sigma_3 - s_{234} - i\epsilon)} + \frac{g^{(1)}(q'^2, Q'^2, \dots; t)}{(\sigma_1 - s_2)(\sigma_3 - s_2)} \right. \\ & \left. + \frac{g^{(2)}(q'^2, Q'^2, \sigma_1, s_{23}, \sigma_3; t)}{(\sigma_1 - s_{234} + i\epsilon)(\sigma_3 - s_2)} + \frac{g^{(2)}(q'^2, Q'^2, \dots; t)}{(\sigma_1 - s_2)(\sigma_3 - s_{234} + i\epsilon)} \right\} \end{aligned} \quad (19)$$

^{*)} The pole terms do not contribute to the discontinuities. See Appendix B.

where, in forward direction, we put :

$$(p_1 - p_2)^2 = t$$

$$g^{(1)}(q'^2, Q'^2, \epsilon_1, s_{23}, \epsilon_3; t) \equiv g_6(q'^2, Q'^2, \epsilon_1, s_{23}, \epsilon_3 | t, t; 4m^2 - t, 4m^2 - t; 0, 0)$$

$$g^{(2)}(q'^2, Q'^2, \epsilon_1, s_{23}, \epsilon_3; t) \equiv g_6(q'^2, Q'^2, \epsilon_1, s_{23}, \epsilon_3 | t, t; 0, 0; 4m^2 - t, 4m^2 - t)$$

From kinematics $s_{234} = (q + p_1)^2 \geq m^2$ is the only variable which requires an $i\epsilon$ prescription. With the help of the crossing relation Eq. (6) for g_6 one easily verifies that W_s is real.

The other two structure functions are obtained by making the following replacements in W_s :

$$\begin{array}{ll} W_\ell & \\ \hline \nu_{23,0} & \rightarrow \nu_{34,0} \\ s_{23} & \rightarrow s_{34} \\ s_2 & \rightarrow s_{134} \\ t & \rightarrow s = (p_1 + p_2)^2 \end{array}$$

Physical region : q^2 above threshold, requires $i\epsilon$ prescription ; s_{134} and s_{234} below threshold, no $i\epsilon$ prescription necessary.

$$\begin{array}{ll} W_a & \\ \hline \nu_{23,0} & \rightarrow \nu_{12,0} \\ s_{23} & \rightarrow s_{12} \\ s_{234} & \rightarrow s_1 \\ t & \rightarrow s = (p_1 + p_2)^2 \end{array}$$

Physical region : q^2, s_1 and s_2 all above threshold, need $i\epsilon$ prescriptions.

Again W_ℓ and W_a are real.

In the case of inclusive hadron-hadron scattering, e.g.,

$$p + p \rightarrow \pi + \text{anything}$$

the appropriate discontinuity of the on-shell six-point function is needed :

$$t_{on\ shell}^{ret} = i \int d^4x e^{iqx} \langle in p'_1 p'_2 | \theta(x_0) [j_\pi(x), j_\pi(0)] | p_1 p_2 in \rangle_c \quad (20)$$

where j_π is the source of the pion field ϕ_π , $(\square + m_\pi^2)\phi_\pi(x) = j_\pi(x)$.

The above on-shell amplitude is the residue of the off-shell amplitude t_6^{ret} at the poles $q^2 = m_\pi^2$, $Q^2 = m_\pi^2$:

$$t_{on\ shell}^{ret} = \lim_{\substack{q^2 \rightarrow m_\pi^2 \\ Q^2 \rightarrow m_\pi^2}} (q^2 - m_\pi^2)(Q^2 - m_\pi^2) t_6^{ret} \quad (21)$$

where t_6^{ret} has the same form as Eq. (20) with the operators j_π replaced by the fields ϕ_π .

Obviously t_6^{ret} has the same type of representation as Eq. (4), its spectral function is obtained from Eq. (5) by replacing $J(x)$ by $\phi_\pi(x)$. Because of Eq. (21) only the pion pole terms in the variables q^2 and Q^2 contribute to the on-shell amplitude $t_{on\ shell}^{ret}$. This then leads to the following representation :

$$t_{on\ shell}^{ret} = \pi^{-3} \sum_{perm} \int_{\sigma_{th}}^{\infty} \frac{d\sigma_1}{\sigma_1 - s_i +} \int_{\sigma_{th}}^{\infty} \frac{d\sigma_2}{\sigma_2 - s_{ij} +} \int_{\sigma_{th}}^{\infty} \frac{d\sigma_3}{\sigma_3 - s_{ijk} +} \quad (22)$$

$$\cdot g_{on\ shell}(\sigma_1, \sigma_2, \sigma_3 | t_{ij}, t_{ke}; t_{ik}, t_{je}; t_{ie}, t_{jk})$$

with

$$\theta(v_{i,0}) \theta(v_{ij,0}) \theta(v_{ijk,0}) g_{on\ shell}(s_i, s_{ij}, s_{ijk} | t_{ij}, t_{ke}; t_{ik}, t_{je}; t_{ie}, t_{jk})$$

$$= (4\pi)^{-3} \int d^4y_1 d^4y_2 d^4y_3 e^{-ip_1 y_1} e^{-ip_j y_2} e^{-ip_k y_3} \quad (23)$$

$$\cdot \langle \pi(q) | j(y_1) j(y_2) j(y_3) j(0) | \pi(Q) \rangle$$

The representation (22) is the direct generalization of the fixed t dispersion relation for the four-point function. It fulfils all relevant Steinmann relations. The structure function of inclusive hadron-hadron scattering is given by the s_{34} discontinuity of $t_{\text{on-shell}}^{\text{ret}}$ in forward direction.

For the on-shell amplitude $t_{\text{on-shell}}^{\text{ret}}$ one can dispose ^{*)} of the Gram restriction from the beginning by considering the amplitude only on a manifold $\sum_{i=1}^4 \alpha_i p_i = 0$, which is away from the forward direction for general α_i . One can convince oneself that no new dependences among the three s variables of one term in the representation (22) are introduced thereby. However, the explicit form of g [Eq. (23)] cannot be established that way, since for it the five-fold discontinuity would be needed.

5. - CROSSING RELATIONS BETWEEN STRUCTURE FUNCTIONS

Given the formulae for the structure functions described before, one may now investigate the problem of crossing from one process to the other. In a previous work, concerned with deep inelastic scattering and annihilation, a generalized crossing relation was derived by two of the authors ¹⁾. The situation is now more complicated since more variables are involved.

From the discussion of the $i\epsilon$ prescriptions, we see that the structure functions are made real by choosing opposite $i\epsilon$'s (wherever necessary) for variables which become numerically equal in forward direction. Hence, one takes the variable at opposite sides of the cut. This is the reason why an analytic continuation in this variable to values below the branch point leads to points outside the physical sheet. Thus the structure functions for the various processes cannot be connected by analytic continuation. One can, however, derive relations of a similar type as in Ref. 1) by just comparing the expressions for the various structure functions [Eq. (19)].

*) This was suggested to us by N.S. Craigie.

The significance of these relations is, however, rather limited because of the presence of one additional parameter (s or t) in the spectral function g_6 in forward direction. This parameter has to be changed in going, say, from W_s to W_a , from negative to positive values. This means that the values of the spectral weight g_6 which determine W_s are different from those determining W_a . Nevertheless a model will permit a construction of g_6 in both regions via the explicit expression Eq. (5). In view of this additional parameter in which g_6 has to be taken from negative to positive values we shall not write down the explicit crossing relations.

6. - GENERAL SCALING CONSIDERATIONS

Even without making a detailed model for g_6 one can discuss some general features of a possible scaling behaviour of the structure functions W .

Let us denote by M the missing mass, $M^2 = s_{12}, s_{23}$ or s_{34} . As before, let τ stand for t or s . Scaling behaviour for the deep inelastic structure functions in the limit $M^2 \rightarrow \infty$ with fixed ratios of q dependent variables can be achieved if one assumes that the spectral weight itself scales, as was proposed in the case of the Compton amplitude ¹⁾. In principle the two functions $g^{(1)}$ and $g^{(2)}$ could scale differently thus allowing for two scaling components for W_s, W_ℓ and W_a (but only simultaneously for all three-structure functions!). For simplicity let us disregard this alternative. Otherwise the subsequent remarks will apply to each component separately.

There are two possible scaling limits depending on whether τ is kept fixed or grows with M^2 :

$$(I) \quad g_6(q'^2, Q'^2, \epsilon_1, M^2, \epsilon_3; \tau) \xrightarrow[M^2 \rightarrow \infty]{\tau \text{ fixed}} (M^2)^n \tilde{g}\left(\frac{q'^2}{M^2}, \frac{Q'^2}{M^2}, \frac{\epsilon_1}{M^2}, \frac{\epsilon_3}{M^2}; \tau\right) \quad (24a)$$

$$(II) \quad g_6(q'^2, Q'^2, \epsilon_1, M^2, \epsilon_3; \tau) \xrightarrow[M^2 \rightarrow \infty]{M^2/\tau \text{ fixed}} (M^2)^m \hat{g}\left(\frac{q'^2}{M^2}, \frac{Q'^2}{M^2}, \frac{\epsilon_1}{M^2}, \frac{\epsilon_3}{M^2}; \frac{\tau}{M^2}\right) \quad (24b)$$

In (24a), the exponent n could still depend on τ . Only the limit (II) is physical for W_ℓ , since $\tau = s$ is bound to grow with $M^2 = s_{34}$ ($s > s_{34}$ from kinematics).

With the assumed limiting forms (I) and (II) for g_6 , the structure functions will scale in M^2 with the same power as g_6 while only fixed ratios appear in the denominators. The two limits (I) and (II) characterize different regions in phase space. Limit (I) controls the target fragmentation region for W_s and the region of finite subenergy of the produced two-hadron system for W_a . Limit (II) controls the entire phase space of W_ℓ , the current fragmentation region for W_s and the region of a very heavy produced two-hadron system for W_a .

For W_s , in the scaling regime (i.e., neglecting masses) t is equal to the (transverse momentum)² of the produced secondary times a function of the other ratios. Unless a special mechanism operates (as, e.g., in the parton model), which makes this function small of $O(1/q^2)$, one will expect a $p_\perp^2/q^2$ scaling¹⁸⁾ in the current fragmentation region because only ratios appear. Similarly, for W_ℓ one will in general expect a $q_\perp^2/s$ scaling.

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APPENDIX A - GENERATION OF CUTS IN EXTERNAL VARIABLES

Let us consider a fixed t ($t < 0$) dispersion relation ^{*}) for the four-particle scattering amplitude $A(s, t)$:

$$A(s, t) = \int_{4m^2}^{\infty} \frac{A_s(s', t)}{s' - s} ds' \quad (A.1)$$

It is known from S matrix theory ¹⁹⁾ that

$$\text{disc}_t A_s(s', t) \Big|_{t=4m^2} = 0 \quad (s' \geq 4m^2) \quad (A.2)$$

where disc_t is the discontinuity across the normal t cut. Equation (A.2) is basically again a Steinmann relation except that it is extended to an unphysical point. As we continue in t from negative values to $t = 4m^2$, we should encounter the normal t cut of $A(s, t)$. Since the integrand does not possess a discontinuity at $t = 4m^2$, it must come from the integration. In fact, $A_s(s', t)$ has singularities coming from higher Landau curves, namely along the well-known Mandelstam boundary which asymptotically touches the line $t = 4m^2$ (see Fig. 3).

Figure 3 - Singularity curve for $A_s(s, t)$

^{*}) This example was pointed out to us by D. Olive.

Thus, at $t = 4m^2$ an end-point singularity of the s' integration arises which is responsible for the branch point.

A similar mechanism can work for T_6^{ret} . If we write a Low representation :

$$T_6^{\text{ret}} = \frac{1}{2\pi} \int d^4 q'_0 \frac{\int d^4 x e^{i q'_0 x} \langle \text{in } p'_1 p'_2 / [\gamma(x), \gamma(0)] / p_1 p_2 \text{ in} \rangle_c}{q'_0 - q_0 - i\epsilon} \quad (\text{A.3})$$

then the dependence of T_6^{ret} on the t_{ij} is contained in the numerator which after insertion of intermediate (out) states has the form of unitarity integrals. The singularities of such integrals are determined by the Landau-Cutkosky rules ¹⁷⁾. These singularities can then also show up in g_6 . (Singularities in g_6 are no contradiction to g_6 being real, if g_6 is not globally analytic. It is to be expected that g_6 is only a piecewise analytic function.) Thus, as in the previous example, a higher Landau curve (whose details now depend on the external masses) can give rise to a singularity (end-point or otherwise) in one of the integrations in the representation (4), thereby creating a t_{ij} cut.

APPENDIX B - POLES IN EXTERNAL VARIABLES

Single particle poles in external variables occur in the cluster decomposition of Eq. (13) and hence also in T_6^{ret} . As an example, consider graph (15d) for out saturation, which (for $S=I$) contains the diagram

In (B.1), the connected matrix element $\langle \text{out } N, p_1' p_2' | J | p_1 p_2 \text{ in} \rangle_c$ is still one particle reducible ²⁰⁾, i.e., it contains a pole, e.g., in the variable $(p_1' + p_2' - p_1)^2$ at the value m^2 . For in saturation the same diagram (B.1) occurs with the matrix element $\langle \text{in } N, p_1' p_2' | J | p_1 p_2 \text{ in} \rangle_c$. The latter matrix element is only a different boundary value of the former, therefore it also contains the same pole ^{*}). In order to compensate for the difference stemming from this other boundary value prescription, there is diagram (15f) for out saturation, which for $a = \text{one-particle state of mass } m$ supplies the necessary δ function $\delta_+[(p_1' + p_2' - p_1)^2 - m^2]$ [note that (15f) contributes to the q^2 discontinuity as does (B.1) !]. As mentioned in the text, the representation (4) should be considered valid for the difference $T_6^{\text{ret}} - (T_6^{\text{ret}})_{\text{poles}}$. For obtaining structure functions this does not matter since the discontinuity in s_{ij} of the pole term $(T_6^{\text{ret}})_{\text{poles}}$ vanishes by virtue of the Steinmann relations.

Proof

If $\text{disc}_{s_{ij}} T_6^{\text{ret}}$ had a pole in a crossed variable (call it ξ), then from the Steinmann relations we would have

$$\text{disc}_{s_{ij}} T_6^{\text{ret}} = \frac{R}{\xi - m^2 + i\epsilon} + Z(\xi + i\epsilon) = \frac{R}{\xi - m^2 - i\epsilon} + Z(\xi - i\epsilon) \quad (\text{B.2})$$

where Z is regular at $\xi = m^2$. But then :

^{*}) This was pointed out to us by V. Glaser.

$$Z(f+i\epsilon) - Z(f-i\epsilon) = 2\pi i R \delta(f-m^2) , \quad (B.3)$$

which contradicts the regularity of Z . Hence the pole is in fact absent in (B.2). Note that the same observation also applies to standard "Muellerism" involving the on-shell 3-to-3 S matrix element !

REFERENCES

- 1) H.D. Dahmen and F. Steiner - Phys.Letters 43B, 217 and 521 (1973).
- 2) E.g. :
R. Hagedorn - "Field Theory and Dispersion Relations", Fortschr.Phys.
5. Sonderband (1963).
- 3) H.P. Stapp - Phys.Rev. D3, 3177 (1971) ;
K.E. Cahill and H.P. Stapp - Phys.Rev. D6, 1007 (1972) ; ibid D8,
2714 (1973).
- 4) A.H. Mueller - Phys.Rev. D2, 2963 (1970).
- 5) O. Steinmann - Helv.Phys.Acta 33, 275 and 347 (1960) ;
H. Araki and N. Burgoyne - Nuovo Cimento 18, 342 (1960) ;
H. Araki - J.Math.Phys. 2, 163 (1960) ;
See also Ref. 3).
- 6) Y. Nambu - Phys.Rev. 100, 394 (1955).
- 7) N. Nakanishi - "Graph Theory and Feynman Integrals", Gordon and Breach,
New York (1971).
- 8) H.P. Stapp and J.L. Wright - Nuovo Cimento 57A, 475 (1968).
- 9) R.P. Feynman - "Photon-Hadron Interactions", W.A. Benjamin, Inc.
(1972), p. 182 ff.
- 10) E.G. :
V.S. Vladimirov - "Methods of the Theory of Functions of Many Complex
Variables", M.I.T. Press (1966).
- 11) H.D. Dahmen and K.D. Rothe - Nuclear Phys. B15, 77 (1970) ;
B. Stech - Z.Phys. 239, 387 (1970) ;
H.D. Dahmen, K.D. Rothe and B. Stech - Phys.Letters 34B, 83 (1971) ;
H.D. Dahmen, D. Gromes, K.D. Rothe and B. Stech - Phys.Letters 35B,
335 (1971).
- 12) D. Gromes - "The Local Saturation Scheme", Habilitationsschrift,
Heidelberg (1973).
- 13) R. Haag - Dan.Math.Fys.Medd. 29, 13 (1955) ;
V. Glaser, H. Lehmann and W. Zimmermann - Nuovo Cimento 6, 1122 (1957).
- 14) H.D. Dahmen - Springer Tracts in Modern Physics, Vol. 62, p. 29 (1972).
- 15) V.A. Matveev, R.M. Muradyan and A.N. Tavkhelidze - Dubna Preprint
P2-4543 (1969).
- 16) See :
V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti - "Currents in
Hadron Physics", North Holland Publishing Company (1973),
p. 431 ff.

- 17) R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne - "The Analytic S Matrix", Cambridge (1966).
- 18) M. Greco - Nuclear Phys. B63, 398 (1973) ;
M.A. Gonzalez and J.H. Weis - CERN Preprint TH. 1778 (1973).
- 19) D.I. Olive - Nuovo Cimento 37, 1422 (1965).
- 20) E.g. :
K. Hepp - Brandeis University Summer Institute (1965), Gordon and Breach, Inc., New York (1966), Vol. I, p. 137.