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EFFICIENT ALGORITHMS FOR VOLTERRA
INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT

Spline functions of degree m , deficiency $J-1$, i.e. in C^{m-J} , are used in conjunction with (Gaussian) quadrature rules to construct algorithms for the numerical solution of a general Volterra integral equation of the second kind. For a given m , the method is of order $(m+1)$ and, in general, requires $O(N)$ evaluations of the kernel. This is in sharp contrast to the $O(N^2)$ evaluations required by hitherto known methods.

It is shown that the method for spline functions with full continuity ($J = 1$) is numerically unstable for all $m > 2$. However, stability is established for $J = m, m-1$, for all m . Furthermore, for $m = 3, J = 1$, it is demonstrated that by appropriately modifying the original method, a whole family of stable methods is obtained.

Geneva - 10 June 1974

WIRKSAME ALGORITHMEN FÜR VOLTERRASCHE
INTEGRALGLEICHUNGEN ZWEITER ART

ZUSAMMENFASSUNG

Zur Konstruktion von Algorithmen zur numerischen Lösung einer allgemeinen Volterraschen Integralgleichung zweiter Art werden Splinefunktionen vom Grade m und der Defizienz $J - 1$, d.h. in C^{m-J} , zusammen mit Gausssschen Quadraturformeln benutzt. Die Methode ist, für gegebenes m , von der Ordnung $(m+1)$, und sie erfordert im allgemeinen $O(N)$ Auswertungen des Kerns. Die bisher bekannten Methoden erfordern $O(N^2)$ Auswertungen.

Es wird gezeigt, dass die Methode für Splinefunktionen mit voller Stetigkeit ($J=1$) numerisch instabil ist für alle $m > 2$. Dagegen wird die Stabilität bewiesen für $J = m$, $m - 1$ und m beliebig. Weiter wird für $m = 3$, $J = 1$ gezeigt, dass bei geeigneter Modifikation der ursprünglichen Methode eine ganze Familie stabiler Methoden gewonnen werden kann.

1. INTRODUCTION

In this paper we present some numerical methods for the solution of the general non-linear Volterra integral equation of the second kind

$$\phi(x) = f(x) + \int_0^x w(x,y)K(x,y,\phi(y)) dy, \quad 0 \leq x \leq X, \quad (1.1)$$

where

$$\int_0^x |w(x,y)| dy \leq C < \infty, \quad 0 \leq x \leq X. \quad (1.2)$$

We assume that (1.1) has a unique solution $\phi(x) \in C[0,X]$.

When both $w(x,y)$ and $K(x,y,z)$ are sufficiently smooth (in which case we talk of non-singular equations) there exist in the literature several satisfactory numerical methods from which the interested reader may choose. There are, on the one hand, discrete variable methods, a comprehensive description of which may be found in Linz [1], and there are, on the other hand, continuous approximation methods presented in Hung [2], Brunner [3] and El Tom [4], [5]. All of these methods require $O(N^2)$ evaluations of the kernel; N being the number of sub-intervals into which $[0,X]$ is divided. Thus it is important to consider the construction of more economical algorithms.

The situation, however, is less satisfactory for singular equations, i.e. when $w(x,y)$ or some of its derivatives are undefined, a case which is very important in applications (see, for instance, [6]). For, although it is demonstrated in [1], p. 44, that, in general, discrete variable methods for non-singular equations may be extended to singular ones by employing the technique of product integration, the resulting methods require, in addition to the $O(N^2)$ kernel evaluations, the computation of $O(N^2)$ coefficients. More recently, El Tom [7] gave arbitrarily high-order spline function methods which, however, require a closed expression for the integral in (1.1) with $\phi(x)$ replaced by an approximating spline function. The latter methods together with block-by-block methods require at each step of the computation the solution of, in general, a system of non-linear equations of order one less than that of the method.

The procedures we present below have the following features. The approximate solution is a spline function of degree m , deficiency $(J-1)$, i.e. in the continuity class C^{m-J} ; the method is of order $m+1$; and in contrast to the above-mentioned methods, requires only $O(N)$ evaluations of the kernel and, at each stage of the computation, the solution of J equations which are linear in $[x^*, X]$ for some x^* . However, it is shown in Section 3 that the method is divergent for $J = 1$ for all $m > 2$. On the other hand, it is also established in Section 3

that the method is numerically stable for $J = m$, $m-1$ for all m , and in Section 5 we present a modified version of the method for $m = 3$, $J = 1$ that is fourth-order accurate and numerically stable. Finally, in Section 6 we consider some numerical examples.

We conclude this section by observing that the results for the case $J = m$ are given, without proofs, in El Tom [8].

2. CONSTRUCTION OF THE APPROXIMATING SPLINE FUNCTION

For each x , let us be given a quadrature rule described by

$$\left\{ \begin{array}{l} \int_0^x w(x,y)g(y) dy = G_x[g] + E_x[g] , \\ G_x[g] = \sum_{i=1}^n A_i g(y_i) , \quad 0 \leq y_1 < \dots < y_n \leq x , \\ E_x[y^r] = 0 , \quad r = 0(1)d , \end{array} \right. \quad (2.1)$$

where n and d may depend on x . Choose two positive integers J, m ; introduce the mesh

$$0 = x_0 < x_1 < \dots < x_{JN} = X$$

and write

$$h_i = x_{i+1} - x_i , \quad i = 0(1)JN-1 ,$$

$$h = \max_i h_i ,$$

$$x_{kJ+j} - x_{kJ} = hc_{k,j} , \quad k = 0(1)N-1 ; \quad j = 1(1)J .$$

For $x_{kJ} \leq x \leq x_{kJ+J}$, $k = 0(1)N-1$, define the function $S(x)$ by

$$\left\{ \begin{array}{l} S^{(r)}(x_0) = \phi^{(r)}(x_0) , \quad r = 0(1)m-J , \\ S(x) = \sum_{r=0}^{m-J} (x - x_{kJ})^r S^{(r)}(x_{kJ})/r! + \sum_{j=1}^J (x - x_{kJ})^{m-J+j} a_{k,j} , \\ S(x_{kJ+j}) = f(x_{kJ+j}) + G_{k,j}[K(S)] , \quad j = 1(1)J , \end{array} \right. \quad (2.2)$$

where the coefficients $a_{k,j}$ are determined according to the last of relations (2.2) and where

$$G_{k,j}[K(S)] = G_x[K(x,y,S(y))]\Big|_{x=x_{kJ+j}}.$$

Throughout the rest of the paper the following will be assumed:

$$(2a) \quad K(x,y,z) \in C[D]; \quad D = \{(x,y,z) : 0 \leq y \leq x \leq X, \quad |z| \leq Z < \infty\};$$

$$(2b) \quad |K(x,y,z) - K(x,y,z^*)| \leq L|z - z^*|, \quad (x,y,z) \in D;$$

$$(2c) \quad f(x) \text{ is bounded in } [0,X];$$

$$(2d) \quad E_x^{(n)}[g] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{for all } g \in C[0,X].$$

We now proceed to show that the above construction is well-defined.

THEOREM 1

There exist a constant h_0 and a positive integer n_0 such that the construction (2.2) uniquely defines $S(x)$ as a spline function of degree m , deficiency $(J-1)$, i.e. in the continuity class C^{m-J} , for all $h \leq h_0$, $n \geq n_0$.

PROOF

The proof is by induction and we observe that it suffices to show that there exist a unique set of coefficients $a_{k,j}$ satisfying the last of relations (2.2). Denote by $\underline{a}_k$ and $\underline{S}_k$ the column vectors

$$\begin{cases} \underline{a}_k = h^{m-J} (h a_{k,1} \quad h^2 a_{k,2} \quad \dots \quad h^J a_{k,J})^T, \\ \underline{S}_k = \left(S(x_{kJ}) \frac{h}{1!} S^{(1)}(x_{kJ}) \quad \dots \quad \frac{h^{m-J}}{(m-J)!} S^{(m-J)}(x_{kJ}) \right)^T, \end{cases} \quad (2.3)$$

and, for $j = 1(1)J$, write

$$\begin{cases} \tilde{\underline{c}}_j = (1 \quad c_{k,j} \quad c_{k,j}^2 \quad \dots \quad c_{k,j}^{m-J})^T, \\ \underline{c}_j = (c_{k,j}^{m-J+1} \quad c_{k,j}^{m-J+2} \quad \dots \quad c_{k,j}^m)^T, \end{cases} \quad (2.4)$$

and define $u_j(\underline{a}_k)$ by

$$u_j(\underline{a}_k) = f(x_{kJ+j}) - \tilde{\underline{c}}_j \underline{S}_k + G_{k,j}[K(S)].$$

Then it is not difficult to see that in view of assumptions (2a)-(2d) we may find h^* and n^* so that $u_j(\underline{a}_k)$ defines a contraction mapping for all $h \leq h^*$, $n \geq n^*$.

The proof is now completed by observing that from (2.2) we have, in a self-explanatory notation

$$C\bar{a}_k = \underline{u}^T(\bar{a}_k) ,$$

C being essentially a Vandermonde matrix.

Q.E.D.

Henceforward, we shall refer to the method described by (2.2) simply as an (m,J)-method. We shall study below the convergence and stability properties of these methods for $J = 1, m, m-1$.

We end this section by observing that if $G_x[\cdot]$ has degree $d \geq r-1$, if $g(y)$ is continuously differentiable through order $r-1$ in $[0, x]$ and if it has an absolutely integrable r^{th} derivative therein, then (see Krylov [9], p. 135)

$$\begin{cases} E_x[g] = \int_0^x F_{r,x}(t) g^{(r)}(t) dt , \\ F_{r,x}(t) = \frac{1}{(r-1)!} \left\{ \int_0^x w(x,y) (y-t)_+^{r-1} dy - G_x[(y-t)_+^{r-1}] \right\} , \\ (y-t)_+^i = \begin{cases} (y-t)^i , & y \geq t , \\ 0 , & y < t . \end{cases} \end{cases} \quad (2.5)$$

3. NUMERICAL STABILITY OF THE (m,J)-METHOD

The concept of stability is to be understood in the following sense.

DEFINITION

The (m,J)-method is said to be numerically stable if all solutions $\{S(x_{kJ})\}$ remain bounded as $h \rightarrow 0$, $k \rightarrow \infty$, with x_{kJ} fixed, when it is applied to any integral equation of the form

$$\phi(x) = 1 + \lambda \int_0^x w(x,y) \phi(y) dy , \quad (3.1)$$

where λ is a constant.

In addition to condition (1.2) we shall require $w(x,y)$ to satisfy the following conditions. We assume there is a function $\rho(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, such that for $0 \leq x < X$ we have

$$(3a) \quad \int_x^{x+\epsilon} |w(x+\epsilon, y)| dy = O(\rho(\epsilon)),$$

$$(3b) \quad \int_0^x |w(x+\epsilon, y) - w(x, y)| dy = O(\rho(\epsilon)).$$

We shall also require the rules $G_x[\cdot]$ to satisfy the following conditions:

$$(3c) \quad G_x \text{ has degree } d(x) \geq m-J;$$

$$(3d) \quad n(x) \geq n(x^*) \text{ , } x > x^* ;$$

$$(3e) \quad (G_{x+\varepsilon} - G_x) [(y-t)_+^{m-J}] = O(\rho(\varepsilon)), \quad 0 \leq t \leq x ;$$

and finally, in order to simplify our analysis, we assume

$$(3f) \quad h_i = h, \text{ for all } i, \text{ i.e. we have a uniform mesh.}$$

Now, in view of assumption (3f) and using notation (2.3), we obtain on differentiating (2.2)

$$\underline{S}_{k+1} = P \underline{S}_k + Q \underline{a}_k, \quad (3.2)$$

where P is the upper triangular matrix of order $m-J+1$ with

$$P_{ij} = \begin{cases} \binom{j}{i} J^{j-i}, & j \geq i ; \quad i, j = 0(1)m-J, \\ 0 & , \quad j < i, \end{cases}$$

and where Q is the matrix defined by

$$q_{ij} = \binom{m-J+j}{i} J^{m-J+j-i}, \quad i = 0(1)m-J ; \quad j = 1(1)J.$$

If we now apply the (m, J) -method to (3.1), we find in view of (2.5) and assumptions (3b), (3c), and (3e)

$$\begin{aligned} S(x_{kJ+p}) - S(x_{kJ}) &= \lambda \int_{x_{kJ}}^{x_{kJ+p}} w(x_{kJ+p}, y) S(y) dy + \\ &+ \lambda \int_{x_{kJ}}^{x_{kJ+p}} F_{k,p}(t) S^{(m-J+1)}(t) dt + O(\rho(h)), \end{aligned} \quad (3.3)$$

where $F_{k,p}(t) = F_{x_{kJ+p}}(t)$. Now, let

$$\psi_{k,p}(t) = \frac{1}{(m-J)!} \int_t^{x_{kJ+p}} w(x_{kJ+p}, y) (y-t)^{m-J} dy.$$

Then, for sufficiently small h we find, in view of assumption (3d), that the second integral on the right-hand side of (3.3) is given by

$$\tilde{\psi}_{k,p} = \int_{x_{kJ}}^{x_{kJ+p}} \psi_{k,p}(t) S^{(m-J+1)}(t) dt ,$$

and since $\psi_{k,p}(t)$ is continuous in $x_{kJ} \leq t \leq x_{kJ+p}$, we may use the second-mean value theorem of integral calculus to obtain

$$\tilde{\psi}_{k,p} = h \underline{u}_p^{(k)} \underline{a}_k , \quad (3.4)$$

where $\underline{u}_p^{(k)}$ is the row vector with

$$u_{p,q}^{(k)} = \frac{(m-J+q)!}{(q-1)!} \theta_{k,p}^{q-1} U_{k,p} , \quad q = 1(1)J .$$

Here

$$U_{k,p} = \frac{p^{m-J+1}}{(m-J)!} \iint_{00}^{11} w(x_{kJ+p}, x_{kJ} + ph(t + (1-t)y)) y^{m-J} \times (1-t)^{m-J+1} dy dt ,$$

and

$$0 < \theta_{k,p} < 1 .$$

Moreover, if we write

$$\tilde{w}_{p,q}^{(k)} = p^{q+1} \int_0^1 w(x_{kJ+p}, x_{kJ} + phy) y^q dy$$

and if we denote by $V^{(k)}$ and $W^{(k)}$ the matrices whose $(p,q)^{th}$ elements are, respectively, given by

$$\begin{cases} v_{p,q}^{(k)} = p^{m-J+q} - \lambda h \left(\tilde{w}_{p,m-J+q}^{(k)} - u_{p,q}^{(k)} \right) , & p, q = 1(1)J , \\ w_{p,q}^{(k)} = \begin{cases} \lambda h \tilde{w}_{p,0}^{(k)} , & q = 0 , \quad p = 1(1)J , \\ \lambda h \tilde{w}_{p,q}^{(k)} - p^q , & q = 1(1)m-J , \quad p = 1(1)J , \end{cases} \end{cases} \quad (3.5)$$

we find from (3.3) and (3.4)

$$V^{(k)} \underline{a}_k = W^{(k)} \underline{s}_k + O(\rho(h)) ;$$

and since, for sufficiently small h , $V^{(k)}$ has a bounded inverse, (3.2) gives

$$\underline{s}_{k+1} = B^{(k)} \underline{s}_k + O(\rho(h)) , \quad (3.6)$$

where

$$B^{(k)} = B^{(k)}(h) = P + Q(V^{(k)})^{-1} W^{(k)} . \quad (3.7)$$

Henceforward, we shall drop the superscript k and simply write B , V , etc. We denote by μ_i the eigenvalues of B and observe that $h\tilde{w}_{p,q}$ and $h u_{p,q}$ are both $O(\rho(h))$ by assumption (3a).

With the above preliminaries, we are ready to state and prove the main result of this section. For part of the proof, however, we shall need the following auxiliary result which may be of independent interest since it relates to Vandermonde matrices.

LEMMA 1

Let R be the matrix of order $m-1$ with

$$r_{ij} = i^{j+1}, \quad i, j = 1(1)m-1;$$

let $\underline{p}(x)$ be the row vector

$$\underline{p}(x) = (x^2 \ x^3 \ \dots \ x^m)$$

and let

$$p_j(x) = \underline{p}(x)\tilde{r}_j, \quad j = 1(1)m-1,$$

where $\tilde{r}_j$ denotes the j^{th} column of R^{-1} . Then we have

- i) $\sum_{j=1}^{m-1} j p_j(m-1) = m-1$,
- ii) $\sum_{j=1}^{m-1} j p'_j(m-1) = 1 + (-1)^m$, where the prime indicates differentiation.

PROOF

The first part of the lemma is immediate, since

$$\underline{p}^{(m-1)}\tilde{r}_j = \delta_{m-1,j}.$$

As for the second part, let us observe that $p_j(x)$ is completely determined by the following properties:

- a) it is a polynomial of degree m in x ;
- b) it has a double zero at $x = 0$ and simple zeros at $x = 2, 3, \dots, j-1, j+1, \dots, m-1$;
- c) $p_j(j) = 1$.

Indeed, we have

$$j p'_j(m-1) = \begin{cases} (-1)^{m-j-1} \binom{m-1}{j} \frac{m-1}{m-j-1}, & j = 1(1)m-2, \\ 1 + (m-1) \sum_{i=1}^{m-1} \frac{1}{m-i}, & j = m-1, \end{cases}$$

and the result follows on noting that

$$\sum_{k=1}^t \frac{1}{k} = \sum_{k=1}^t (-1)^{k+1} \frac{1}{k} \binom{t}{k}$$

$$\binom{t}{k} = \binom{t}{t-k} . \quad \text{Q.E.D.}$$

Now, in so far as the stability of the method depends on the eigenvalues of B, we have:

THEOREM 2

Let conditions (3a)-(3c) hold. Then we have

- i) the (m,1)-method is unstable for all $m > 2$;
- ii) the (m,m)-method is stable for all m;
- iii) the (m,m-1)-method is stable for all m.

PROOF

- i) On computing the diagonal elements of B, for $J = 1$, we easily find that its trace is given by

$$\text{tr } (B) = m + 2 - 2^m + O(\rho(h)) ;$$

whence the first part of the theorem since

$$\text{tr } (B) = \sum_{i=1}^m \mu_i$$

and $\mu = 1 + O(\rho(h))$ is an eigenvalue of B for all m.

- ii) Observing that for $h = 0$, $J = m$

$$QV^{-1} = (0 \ 0 \ \dots \ 0 \ 1) ,$$

we find

$$B = 1 + \lambda h \tilde{w}_{m,0} + O((\rho(h))^2)$$

which proves the second part.

- iii) Using the notation introduced in the proof of lemma 1, we find

$$QV^{-1}W = \begin{pmatrix} \lambda h \tilde{w}_{m-1,0} + O(\rho(h)^2) & \lambda h \tilde{w}_{m-1,1} - (m-1) + O(\rho(h)) \\ \underline{p}'(m-1)V^{-1}\underline{w}_1 & \underline{p}^1(m-1)V^{-1}\underline{w}_2 \end{pmatrix} ,$$

where $\underline{w}_i$ denotes the i^{th} column of W and where V and W are defined by (3.5) with $J = m-1$. Hence, by lemma 1

$$B = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{m+1} \end{pmatrix} + O(\rho(h))$$

and the stability of the method follows since, for h sufficiently small, the eigenvalues of B are distinct. Q.E.D.

Remark

For the case $w(x,y) \equiv 1$ and $G_x[\cdot]$ a repeated rule based on a given quadrature formula, the results of El Tom [10] suggest that the (m,m) -methods possess stronger stability properties than the $(m,m-1)$ -methods. Indeed, for $m \leq 3$, the former are A-stable.

4. ORDER OF CONVERGENCE

For non-singular equations, it is known that the order of a finite difference method is determined, provided the starting values are sufficiently accurate, by the order of the quadrature formula (see [1], p. 11). This much is also true of spline function methods (see [4] and [5]). We shall now give a result which shows that our present methods do indeed conform to this general rule when applied to the general equation (1.1). We assume

$$(4a) \quad \phi(x) \in C^{m+1}[0, X];$$

$$(4b) \quad K(x,y,z) \text{ and } G_x[\cdot] \text{ are such that}$$

$$E_x[K(\phi)] = O(h^q), \quad q \geq m+1.$$

We set in (2.2) for $j = 1(1)J$

$$a_{k,j} = \frac{1}{(m-J+j)!} \phi^{(m-J+j)}(x_{kJ}) + h^{J-j+1} b_{k,j}, \quad (4.1)$$

where $b_{k,j} = b_{k,j}(h)$. We write

$$e(x) = S(x) - \phi(x), \quad 0 \leq x \leq X, \quad (4.2)$$

and for $x = x_{kJ}$, $k = 0(1)N-1$ we define

$$S^{(r)}(x) = \sum_{j=1}^{m-r+1} \alpha_{r,j} S^{(r)}(x_{kJ} + ht_j), \quad r > m-J; \quad 0 < t_j < c_{k,J} \quad \text{for all } j, \quad (4.3)$$

where $\underline{\alpha}_r$ denotes the first column of the inverse of the matrix (t_j^{i-1}) , $i, j = 1(1)m-r+1$. We now have the following basic result.

LEMMA 2

Let conditions (4a) and (4b) hold. Then the coefficients $b_{k,j}$ defined by (4.1) are uniformly bounded and we have

$$e(x_{kJ}) = O(h^{m+1}) , \quad k = O(1)N .$$

PROOF

By induction and is a straightforward extension of that of lemmas 1 and 2 of [5]. We therefore omit it. Q.E.D.

As a direct consequence of lemma 2, we obtain from (2.2) and (4.1):

THEOREM 3

Let the conditions of lemma 2 hold. Then the following holds:

- i) $e^{(r)}(x) = O(h^{m+1-r})$, $r = O(1)m - J$; $0 \leq x \leq X$;
- ii) $e^{(m-J+r)}(x) = O(h^{J+1-r})$, $r = 1(1)J$; $0 < x < X$, $x \neq x_{kJ}$;
- iii) for $x = x_{kJ}$, $k = 1(1)N - 1$, (ii) holds if the corresponding derivatives of $S(x)$ are defined by (4.3).

5. A MODIFIED CUBIC SPLINE METHOD

In this section we shall describe a method for cubic splines with full continuity, show that it is numerically stable in the sense of Section 3, and that it is fourth-order accurate. The idea behind the method is to give up the requirement that the spline function satisfies the integral equation exactly at the nodal points.

For each x define the operator

$$T_x[g] = g(x) - f(x) - G_x[K(g)] \quad (5.1)$$

and for $x_{3k} \leq x \leq x_{3k+3}$, define $S(x)$ by the following relations

$$\left\{ \begin{array}{l} S^{(r)}(x_0) = \phi^{(r)}(x_0) , \quad r = 0, 1, 2 , \\ S(x) = \sum_{r=0}^2 (x - x_{3k})^r \frac{S^{(r)}(x_{3k})}{r!} + (x - x_{3k})^3 a_k , \\ T_{k,3}[S] = \alpha T_{k,1}[S] + \beta T_{k,2}[S] , \end{array} \right. \quad (5.2)$$

where α and β are as yet free real parameters and where we have set $x = x_{3k+j}$, $j = 1, 2, 3$, in (5.1). That the above construction is well-defined is the content of the following result.

THEOREM 4

Provided that the mesh and the parameters α, β are such that

$$c_{k,3}^3 - \alpha c_{k,1}^3 - \beta c_{k,2}^3 \neq 0, \quad (5.3)$$

then the construction (5.2) uniquely defines $S(x)$ as a cubic spline for all $h \leq h_0$, $n \geq n_0$ for some h_0 and n_0 .

PROOF

Similar to that of theorem 1.

Q.E.D.

Remark

We shall see later on that for stable, fourth-order methods, condition (5.3) is always satisfied when the mesh is uniform.

In discussing numerical stability, we shall continue to use the notation of Section 3. We introduce, for each k , the quantities

$$\left\{ \begin{array}{l} w_j^* = \tilde{w}_{3,j} - \alpha \tilde{w}_{1,j} - \beta \tilde{w}_{2,j}, \quad j = 0(1)3, \\ \delta = (27 - \alpha - 8\beta - \lambda h w_3^*)/9, \\ \xi = \lambda h w_0^*/\delta, \\ \eta = [\lambda h w_1^* - (3 - \alpha - 2\beta)]/\delta, \\ \zeta = [\lambda h w_2^* - (9 - \alpha - 4\beta)]/\delta, \end{array} \right. \quad (5.4)$$

where $\tilde{w}_{ij}$ is as in (3.5) with $J = 3$. If we now apply our modified method to (3.1) and assuming that conditions (3a)-(3f) hold, we find

$$h^3 a_k = (\xi \eta \zeta) \underline{S}_k + O(\rho(h)), \quad (5.5)$$

and on differentiating the second of relations (5.2) we obtain, in view of (5.5)

$$\underline{S}_{k+1} = B^* \underline{S}_k + O(\rho(h)), \quad (5.6)$$

where

$$B^* = \begin{pmatrix} 1+3\xi & 3+3\eta & 9+3\zeta \\ 3\xi & 1+3\eta & 6+3\zeta \\ \xi & \eta & 1+\zeta \end{pmatrix}. \quad (5.7)$$

Concerning the eigenvalues of B^* we have the following results.

LEMMA 3

$\mu_1 = 1 + \lambda h \tilde{w}_{3,0} + O(\rho^2(h))$ is an eigenvalue of B^* for all choices of α and β .

PROOF

Let $\pi(\mu)$ denote the characteristic polynomial of B^* . Then direct computation yields

$$\pi(1 + \lambda h \tilde{w}_{3,0}) = O(\rho^2(h)). \quad \text{Q.E.D.}$$

Concerning the remaining two eigenvalues of B^* we have

LEMMA 4

Let

$$\alpha + \beta = 1. \quad (5.8)$$

Then in order that μ_2 and μ_3 have modulus less than 1, it is necessary and sufficient to have

$$\frac{20}{13} \leq \beta \leq 2. \quad (5.9)$$

[Note that condition (5.3) is automatically satisfied.]

PROOF

Set $h = 0$ in (5.4) and denote the corresponding quantities by ξ_0 , etc. In view of lemma 3 we have

$$\begin{aligned} \pi_0(\mu) &= (1-\mu)\pi^*(\mu), \\ \pi^*(\mu) &= \mu^2 - (2+3\eta_0+\zeta_0)\mu + 1 - 3\eta_0 + \zeta_0. \end{aligned}$$

It is not difficult to see that (i) there is no real β for which $\pi^*(\mu)$ has double or complex roots with $|\mu| < 1$; and (ii) condition (5.9) is necessary and sufficient for $\pi^*(\mu)$ to have real distinct roots in $(-1,1)$. These observations complete the proof. Q.E.D.

Summing up, we have

THEOREM 5

In order for the modified (3,1)-method defined by (5.2) and (5.8) to be numerically stable, it is necessary and sufficient that β satisfies (5.9).

We end this section by stating the convergence result corresponding to theorem 3. The proof is similar to that of the latter.

THEOREM 6

Let $\phi(x) \in C^4[0, X]$. Let $K(x, y, z)$ and $G_x[\cdot]$ be such that

$$E_x[K(\phi)] = O(h^q), \quad q \geq 4,$$

and let condition (5.8) be satisfied. Then, for $0 \leq x \leq X$, we have

$$e^{(r)}(x) = O(h^{4-r}), \quad r = 0, 1, 2,$$

and for $0 < x < X$, we have

$$e^{(3)}(x) = O(h),$$

where for $x = x_{3k}$, $k = 1(1)N-1$, $S^{(3)}(x)$ is defined by

$$S^{(3)}(x) = \left[S^{(3)}(x + hc_{k,j}/2) + S^{(3)}(x - hc_{k-1,j}/2) \right] / 2.$$

Remark

On writing down Taylor's expansion for the expression

$$[e(x_{3k+3}) - \alpha e(x_{3k+1}) - \beta e(x_{3k+2})]$$

about x_{3k} , assuming $c_{k,i} = i$, $i = 1, 2, 3$, we find that the coefficients of $he^{(1)}(x_{3k})$ and $h^2e^{(2)}(x_{3k})$ are, respectively, $(2 - \beta)$ and $(8 - 3\beta)$, thus suggesting the choice $\beta = 2$. However, it should be noted that this choice corresponds to the border of the stability interval (5.9).

6. CONCLUDING REMARKS

- i) When solving numerically Volterra integral equations, the principal part of the effort is expended in computing values of the kernel $K(x, y, z)$. Hence, the total number Γ of kernel evaluations required by a given method is a reasonable measure of its computational efficiency. For our presented methods, this measure satisfies

$$\Gamma \leq \begin{cases} J \sum_{k=0}^{N-1} n_k, & \text{for the } (m, J)\text{-method,} \\ 3 \sum_{k=0}^{N-1} n_k, & \text{for the modified } (3, 1)\text{-method,} \end{cases}$$

where $n_k = \max_{1 \leq p \leq J} \{n(x_{kJ+p})\}$. In particular, if $n = \max_{0 \leq k \leq N-1} n_k$, then $\Gamma \leq JnN$, with $J = 3$ for the modified (3,1)-method. This is in sharp contrast with methods presently known to the author for which $\Gamma = cN^2$, c being a constant of the method.

- ii) In using an (m,J)-method, we have to solve, at each stage of the computation, a system of J (in general) non-linear equations. It turns out, however, that for certain integral equations this system is linear over a certain part of the interval $[0, X]$. For instance, let $w(x, y) = w(x - y)$; suppose it is homogeneous; and let us be given a Gaussian formula

$$\sum_{i=1}^n A_i g(y_i) \approx \int_0^1 w(y) g(y) dy.$$

On applying the (m,J)-method in conjunction with this formula over (x_{kJ}, x_{kJ+J}) , the resulting system will be linear whenever $k \geq [(1 - y_1)/y_1]$.

- iii) In view of remark (ii) it would be of interest to construct, in the manner of Section 5, modified (m,1)-methods for $m \geq 4$. The major problem to be considered here is that of stability.

7. NUMERICAL EXAMPLES

In order to illustrate certain aspects of our presented methods, we consider below their application to the following test examples.

EXAMPLE 1

$$\phi(x) = (1+x)^{-1/2} + \frac{\pi}{8} - \frac{1}{4} \arcsin \left(\frac{1-x}{1+x} \right) - \frac{1}{4} \int_0^x (x-y)^{-1/2} \phi(y) dy.$$

Solution

$$\phi(x) = (1+x)^{-1/2}.$$

EXAMPLE 2

$$\phi(x) = (1+x)^{-1/4} + \frac{\pi}{8} - \frac{1}{4} \arcsin \left(\frac{1-x}{1+x} \right) - \frac{1}{4} \int_0^x (x-y)^{-1/2} \phi^2(y) dy, \quad 0 \leq x \leq X.$$

Solution

$$\phi(x) = (1+x)^{-1/4}.$$

In applying our methods to both examples, we used a fixed n-point Gaussian rule (see Krylov [9], p. 120):

$$G[g] = \sum_{i=1}^n A_i g(y_i) \approx \int_0^1 x^{-1/2} g(x) dx .$$

Numerical results were computed for Example 1 using the (2,2)-method with $n = 4$ and the results for the error are given in Table 1, where

$$\hat{e}_0 = \max_{0 \leq i \leq mN} |S(x_i) - \phi(x_i)| .$$

Similar results are obtained for the linear equation

$$\phi(x) = (1+x)^{1/2} + \frac{(1-x)x^{1/2}}{4} + \frac{(1+x)^2}{8} \left[\frac{\pi}{2} - \arcsin \left(\frac{1-x}{1+x} \right) \right] - \int_0^x (x-y)^{1/2} \phi(y) dy .$$

Solution

$$\phi(x) = (1+x)^{1/2} .$$

When applying any of the presented methods with a fixed n-point Gaussian rule to Example 2, the defining equations are linear at the k^{th} step of the computation for all $k \geq k^* = \lceil (1-y_1)/y_1 \rceil$. In order to see the effect of this property on the computed solution, we used the (2,2)-, the (2,1)-, and the modified (3,1)-method, with $n = 4$, and we set $S(x) = \phi(x)$, $0 \leq x \leq x^*$ where $x^* = x_{2k^*}$, x_{k^*} , x_{3k^*} , respectively, for the three methods. Table 2 gives values of

$$\hat{e}_r = \max_{t \leq i \leq T} |S^{(r)}(x_i) - \phi^{(r)}(x_i)| , \quad r = 0, 1, 2 ,$$

$$x_t = x^* , \quad x_T = X .$$

The results demonstrate, at least for the case under consideration, that the above-mentioned property does not have any pronounced effect on the computed solution. The rather surprising results for the modified method are easily understood in considering the expression

$$e^{(j)}(x_{3k+3}) + e^{(j)}(x_{3k+1}) - 2e^{(j)}(x_{3k+2}) , \quad j = 0, 1, 2 ,$$

in the manner of the remark made at the end of Section 5.

Table 1

Solution of Example 1 by the (2,2)-method with $n = 4$

x	h = 1.0	h = 0.5
1	-3.0 E - 3	3.2 E - 4
2	1.3 E - 3	3.0 E - 4
3	-5.1 E - 4	-4.7 E - 4
4	1.1 E - 3	-4.5 E - 5
5	4.6 E - 5	1.4 E - 4
6	-1.8 E - 3	6.6 E - 5
7	-1.1 E - 3	3.6 E - 4
8	-7.4 E - 5	7.0 E - 4
9	2.9 E - 4	7.9 E - 4
10	4.6 E - 4	8.2 E - 4
$\hat{e}_0$	3.0 E - 3	8.2 E - 4

Table 2

Numerical results for Example 2
 $n = 4$, $h = 0.1$ and $S(x) = \phi(x)$, $0 \leq x \leq x^*$

$\hat{e}_r$	(2,2)-method $x^* = 5.6$; $X = 8.2$	(2,1)-method $x^* = 2.8$; $X = 4.1$	Modified (3,1)-method $x^* = 8.4$; $X = 12.3$; $\beta = 2$
$\hat{e}_0$	3.0 E - 4	3.6 E - 5	7.4 E - 5
$\hat{e}_1$		5.1 E - 4	3.0 E - 5
$\hat{e}_2$			2.7 E - 5

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