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BETHE-SALPETER EQUATION AND CURRENT CONSERVATION

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A B S T R A C T

Current conservation in the form of the Ward identity between the electromagnetic vertex and the propagator implies that the energy dependence of the BS kernel is restricted and that the propagator cannot be chosen independently from the kernel. It is rather determined from the BS kernel in terms of an integral equation. Convolution type, energy independent kernels are compatible with current conservation. We study the propagator and form factor resulting from smooth kernels.

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INTRODUCTION

The Bethe-Salpeter equation ¹⁾ is used in high energy physics to treat in a relativistic manner two-body scattering, bound state problems and form factors. It is an exact equation if one uses as kernel the sum over all two-particle irreducible diagrams and includes all self-energy effects in the propagator. Its solutions then fulfil the requirements of quantum field theory such as analyticity and unitarity and show current conservation for the electromagnetic form factors.

For certain phenomenological applications, however, one works with free propagators and simple kernels. This may lead to difficulties with unitarity for the scattering amplitude ²⁾ and with current conservation for the form factors. In particular the Ward identities for non-trivial form factors are not fulfilled with free propagators. There have been attempts to remedy these points by including the lowest order self-energy graphs in the propagator ³⁾. It was found that for strong couplings these lead to large changes in the elastic scattering amplitudes.

In this paper we investigate the problem of current conservation in the BS framework, i.e., we discuss the question of the compatibility of BS kernels with Ward identities relating the form factor and the propagator. We find that the dependence of the BS kernel on the total energy is strongly restricted, if these Ward identities are to be valid. Furthermore, it turns out that the dressed propagator is not independent of the BS kernel, since it is determined by the BS kernel in terms of an integral equation.

This offers the possibility to have the following dynamical, field theoretical framework for studying two-body scattering, the bound state problem and form factors : start with a phenomenologically chosen BS kernel. From this calculate the dressed renormalized propagator and insert these two quantities into the BS equations. The resulting scattering and bound state amplitudes are relativistic co-variant and have the correct analytical properties ; furthermore, the form factor is consistent with the requirements of current conservation. Smooth BS kernels of the convolution type provide a suitable choice, since they fulfil the condition Eq. (15). We shall illustrate the above scheme for this class of kernels.

1) THE BETHE-SALPETER EQUATION

We consider the elastic scattering of two spin zero particles. Let $\phi(x)$ be the field operator in the Heisenberg picture, normalized as follows :

$$\langle 0 | \phi(x) | m, p \rangle = (2\pi)^{-3/2} \cdot e^{-ipx}.$$

The scattering Green's function $\tau(x_1, x_2, x'_1, x'_2)$ is defined by :

$$\tau(x_1, x_2, x'_1, x'_2) = \langle 0 | T \phi(x_1) \phi(x_2) \phi^\dagger(x'_1) \phi^\dagger(x'_2) | 0 \rangle.$$

Its Fourier transform satisfies the momentum space BS equation (Fig. 1a)

$$\begin{aligned} [\Delta_F(\frac{p}{2}+q) \Delta_F(\frac{p}{2}-q)]^{-1} \tau(p, q, p) &= \delta^{(4)}_{(p-q)} + \\ &+ i \int d^4 p' K(p, p', p) \tau(p', q, p). \end{aligned} \quad (1)$$

Here $\Delta_F(p)$ denotes the dressed Feynman propagator

$$\Delta_F(p) = \int e^{ipx} \langle 0 | T \phi(x) \phi^\dagger(0) | 0 \rangle d^4 x$$

and $K(p, p', p)$ denotes the Bethe-Salpeter kernel ; it is given as the sum over all two-particle irreducible diagrams.

The bound states are described by the BS amplitudes

$$\chi(x, p) = (2\pi)^{3/2} \langle 0 | T \phi(x) \phi^\dagger(-x) | M, p; \mathcal{P}^c \rangle.$$

They are determined by the homogeneous BS equation (Fig. 1b)

$$[\Delta_F(\frac{p}{2}+p) \Delta_F(\frac{p}{2}-p)]^{-1} \chi(p, p) = i \int d^4 p' K(p, p', p) \chi(p', p) \quad (2)$$

and the field theoretical normalization condition ⁴⁾

$$(2\pi)^4 i \int d^4p \bar{\chi}_r(p, p) \left\{ \frac{\partial}{\partial p_\mu} [\Delta_F(\frac{p}{2} + p) \Delta_F(\frac{p}{2} - p)] - \frac{\partial}{\partial p_\mu} K(p, p', p) \right\} \chi_{r'}(p, p) \quad (2')$$

$$= 2 p_\mu \delta_{rr'}.$$

2) CURRENT GREEN'S FUNCTIONS

Electromagnetic properties (of the particles) are contained in the current Green's functions

$$\mathcal{J}_\mu(x_1 \dots x_n, x'_1 \dots x'_n, y) = \langle 0 | T \phi(x_1) \dots \phi(x_n) j_\mu^{\text{el}}(y) \phi^\dagger(x'_1) \dots \phi^\dagger(x'_n) | 0 \rangle, \quad (3)$$

where $j_\mu^{\text{el}}(x)$ is the electromagnetic current operator. The electric charge $Q = \int d^3x j_0^\mu(x)$ has the property

$$[Q, \phi(x)] = e \phi(x).$$

In a wide class of field theories one can show the validity ⁵⁾ of the local form of this commutation relation

$$[j_0^\mu(x), \phi(y)]_{x_0=y_0} = e \phi(x) \delta^{(4)}(x-y). \quad (4)$$

This has as consequence the classical Ward identities for the current Green's functions

$$k^\mu \mathcal{J}_\mu(p_1 \dots p_n; p'_1 \dots p'_n) = e \tau(p_1 - k, p_2 \dots p_n; p'_1 \dots p'_n) + \dots + e \tau(p_1 \dots p_{n-1}, p_n - k; p'_1 \dots p'_n) \quad (5)$$

$$- e \tau(p_1 \dots p_n, p'_1 + k, p'_2 \dots p'_n) - \dots - e \tau(p_1 \dots p_n, p'_1 \dots p'_{n-1}, p'_n + k),$$

$$k_\mu = p_1 + \dots + p_n - p'_1 - \dots - p'_n.$$

A special case of these relations is the well-known connection ⁶⁾ between the vertex

$$\Gamma_{\mu}(p, p') = \Delta_F^{-1}(p) \Delta_F^{-1}(p') \mathcal{J}_{\mu}(p, p') \quad (6)$$

and the propagator

$$(p-p')^{\mu} \Gamma_{\mu}(p, p') = e [\Delta_F^{-1}(p) - \Delta_F^{-1}(p')], \quad (7)$$

where on both sides renormalized quantities are to be taken. We notice the limiting case

$$\Gamma_{\mu}(p, p) = e \frac{\partial}{\partial p_{\mu}} \Delta_F^{-1}(p). \quad (7')$$

Now we perform the decomposition of $\Gamma_{\mu}(p, p')$ into covariants and invariant functions. For this we make a change of variables : $k = p - p'$ = four-momentum of the virtual photon, $q = \frac{1}{2}(p + p')$ = relative momentum

$$\Gamma_{\mu}(q, k) = k_{\mu} \cdot \frac{qk}{k^2} \cdot F_1(q, k) + (q_{\mu} - k_{\mu} \cdot \frac{qk}{k^2}) F_2(q, k). \quad (8)$$

Multiplication with k^{μ} gives :

$$k^{\mu} \Gamma_{\mu}(q, k) = qk F_1(q, k).$$

Therefore, only the form factor $F_1(q, k)$ is related via the Ward identity to the propagator

$$F_1(q, k) = e \frac{\Delta_F^{-1}(q + \frac{k}{2}) - \Delta_F^{-1}(q - \frac{k}{2})}{qk} = 2e \cdot \frac{\Delta_F^{-1}(p) - \Delta_F^{-1}(p')}{p^2 - p'^2}. \quad (9)$$

The current operator $j_{\mu}^{el}(x)$ can be expressed in terms of the fields with help of the ansatz

$$j_{\mu}^{el}(x) = Ze [\phi^{\dagger}(x), \vec{\partial}_{\mu} \phi(x)],$$

which is a quite general feature of Lagrangian field theory. The renormalization constant Z is defined such that on the mass shell

$$\lim_{p^2 \rightarrow m^2} \Gamma_{\mu}(p, p) = e \cdot 2p_{\mu}$$

and is identical with the wave function renormalization constant

$$[\phi^{\dagger}(x), \phi(y)]_{x=y} = Z^{-1} \delta^{(4)}(x-y).$$

The BS equation for the vertex has the form (Fig. 1c) :

$$\Gamma_{\mu}(q, k) = Ze \cdot 2q_{\mu} + i \int K(q, q', k) \Delta_F(q' + \frac{k}{2}) \Delta_F(q' - \frac{k}{2}) \Gamma_{\mu}(q', k) d^4 q'. \quad (10)$$

It decomposes into the system of two equations for the two form factors :

$$qk \cdot F_1(q, k) = Ze \cdot 2qk + i \int K(q, q', k) \Delta_F(q' + \frac{k}{2}) \Delta_F(q' - \frac{k}{2}) \cdot q'k \cdot F_1(q', k) d^4 q', \quad (11)$$

$$\begin{aligned} (q^2 - \frac{(qk)^2}{k^2}) F_2(q, k) = & Ze \cdot 2 (q^2 - \frac{(qk)^2}{k^2}) \\ & + i \int K(q, q', k) \Delta_F(q' + \frac{k}{2}) \Delta_F(q' - \frac{k}{2}) [qq' - \frac{qk \cdot q'k}{k^2}] F_2(q', k) d^4 q'. \end{aligned} \quad (12)$$

3) THE INTEGRAL EQUATION FOR THE PROPAGATOR

In the previous sections we presented the BS equations for elastic scattering, the bound state amplitudes and the electromagnetic form factor, all being determined by the same propagator and BS kernel. However, the Ward identity relates the form factor to the propagator. Therefore, propagator and kernel are no longer independent.

In order to get this relation between them more explicitly, we insert the Ward identity Eq. (9) into the BS equation for the form factor (11) :

$$\begin{aligned} \Delta_F^{-1}(q+\frac{k}{2}) - \Delta_F^{-1}(q-\frac{k}{2}) &= Z \cdot 2qk + i \int K(q, q', k) \Delta_F(q'+\frac{k}{2}) \Delta_F(q'-\frac{k}{2}) \cdot \\ &\quad [\Delta_F^{-1}(q'+\frac{k}{2}) - \Delta_F^{-1}(q'-\frac{k}{2})] d^4 q' \\ &= Z \cdot 2qk - i \int K(q, q', k) [\Delta_F(q'+\frac{k}{2}) - \Delta_F(q'-\frac{k}{2})] d^4 q'. \end{aligned} \quad (13)$$

From this equation we can derive a condition telling us whether the BS kernel is altogether compatible with the Ward identity and, this being the case, how to obtain the dressed propagator from the kernel. Let us start with the second point. Taking the limit $k_\mu \rightarrow 0$ in Eq. (13), we have

$$\frac{\partial}{\partial q_\mu} \Delta_F^{-1}(q) = Z \cdot 2q_\mu - i \int K(q, q', 0) \frac{\partial}{\partial q'_\mu} \Delta_F(q') d^4 q' \quad (14)$$

and we obtain a non-linear integral equation for the gradient of the propagator. Only the BS kernel at vanishing four-momentum is involved in this equation.

In order to discuss the consistency problem we differentiate Eq. (13) $\partial/\partial q_\mu$ and insert Eq. (14). The result is :

$$\begin{aligned} \int d^4 q' \left\{ \frac{\partial}{\partial q_\mu} K(q, q' - \frac{k}{2}, k) - \frac{\partial}{\partial q_\mu} K(q, q' + \frac{k}{2}, k) \right. \\ \left. - \frac{\partial}{\partial q'_\mu} K(q - \frac{k}{2}, q', 0) + \frac{\partial}{\partial q'_\mu} K(q + \frac{k}{2}, q', 0) \right\} \Delta_F(q') = 0. \end{aligned} \quad (15)$$

This is the necessary condition on the BS kernel for the validity of the Ward identity. It allows to determine the energy dependence of the BS kernel from $K(q, q', 0)$ in form of a power series expansion.

The condition (15) is fulfilled for example in the phenomenologically interesting case of convolution type kernels :

$$K(q, q', k) = K(q - q') , \quad (16)$$

which we now want to consider in some detail. In this case we can integrate Eq. (14) after the application of Gauss' theorem and obtain :

$$\Delta_F^{-1}(q) = 2(q^2 - m_0^2) - i \int K(q - q') \Delta_F(q') d^4 q' . \quad (17)$$

We have called the integration constant $-2m_0^2$ since in the case of not too singular interactions, m_0 has the meaning of the bare mass. The formal solution of Eq. (17) is the continued fraction :

$$\Delta_F(q) = \frac{1}{2(q^2 - m_0^2) - i \int K \frac{1}{2(q'^2 - m_0^2) - i \int K \frac{1}{2(q''^2 - m_0^2) - \dots}} d^4 q'} . \quad (18)$$

This corresponds in a certain sense to the Born series solution of linear integral equations. Finally, we transform the four-dimensional integral equation into a one-dimensional one. As usual we first make a Wick rotation, i.e., we continue q to Euclidean points : $q_0 = iq_4$; $q_L^2 = -q_E^2$.

$$\Delta_F^{-1}(q) = 2(-q_E^2 - m_0^2) - \int K(q - q') \Delta_F(q') d^4 q' . \quad (19)$$

Now we expand the kernel K into hyperspherical functions $y_{lm}^n(\hat{q})$:

$$K_{(q,q')}^{(n)} = \frac{n+1}{2\pi^2} \iint d\Omega_q d\Omega_{q'} y_{lm}^{*n}(\hat{q}) K_{(q,q')} y_{lm}^n(\hat{q}') \quad (20)$$

$$K_{(q,q')} = 2\pi^2 \sum_{n,lm} K_{(q,q')}^{(n)} y_{lm}^{*n}(\hat{q}') y_{lm}^n(\hat{q}) \cdot \frac{1}{n+1} .$$

The integration then yields :

$$\begin{aligned} \int d^4q' K_{(q,q')} \Delta_F(q') &= 2\pi^2 \int d\Omega_{q'} \int_0^\infty q'^3 dq' \sum_{n,lm} K_{(q,q')}^{(n)} y_{lm}^{*n}(\hat{q}') y_{lm}^n(\hat{q}) \frac{\Delta_F(q'^2)}{n+1} \\ &= 2\pi^2 \cdot \int_0^\infty q'^3 dq' K_{(q,q')}^{(0)} \Delta_F(q'^2) \end{aligned}$$

Because of Lorentz invariance the propagator is determined by the $O(4)$ s wave projection of the BS kernel. We are left with the one-dimensional equation :

$$\Delta_F^{-1}(q^2) = Z(-q^2 - m_0^2) - 2\pi^2 \int_0^\infty q'^3 dq' K_{(q,q')}^{(0)} \Delta_F(q'^2) . \quad (21)$$

4) A SIMPLE EXAMPLE : SMOOTH KERNELS

The propagator Eq. (17) is a complicated non-linear integral equation. This type of equations is much more difficult to handle than the mathematically better known linear integral equations. In order to get a feeling for the properties of its solutions, it is desirable to look for soluble examples. Of course, there are exactly soluble examples like factorizable kernels or kernels which act multiplicatively in momentum space, which seem, however, for our intentions an over-simplification. Instead, we present as example smooth kernels, which are more realistic, although approximations have to be made. Such kernels have been applied in a relativistic quark model to provide the binding forces for quarks and antiquarks and were found to give good results for the properties of the mesons ⁷⁾.

For the smooth kernels we make the ansatz :

$$K_r(q-q') \sim (\mu^2)^{r-2} \cdot \frac{1}{[(q-q')^2 + \mu^2]^r} ; \quad r=1,2,3,\dots \quad (22)$$

In configuration space we have ^{*)} :

$$K_r(R) \sim \frac{2\pi^2\mu}{(r-1)!} \left(\frac{\mu R}{2}\right)^{r-2} \cdot K_{r-2}(\mu R) ; \quad R = \sqrt{\vec{x}^2 + t^2}. \quad (23)$$

We see that $K_r(R)$ is finite at the origin for $r \geq 3$. For the purpose of illustration $K_r(R)$ is shown for $r=1,\dots,5$ in Fig. 2. The kernels with $r \geq 4$ are smooth at the origin. They are related to the one-particle exchange kernel by differentiation :

$$K_r(q-q') \sim \frac{(\mu^2)^{r-2}}{(r-1)!} \left(\frac{\partial}{\partial \mu^2}\right)^{r-1} K_1(q-q'). \quad (22')$$

We make use of this fact in order to calculate their s wave projections from the s wave projection of the one-particle exchange :

$$K_r^{(0)}(q,q') \sim \frac{(\mu^2)^{r-2}}{(r-1)!} \left(\frac{\partial}{\partial \mu^2}\right)^{r-1} \frac{q^2 + q'^2 + \mu^2 - \sqrt{(q^2 + q'^2 + \mu^2)^2 - 4q^2 q'^2}}{2q^2 q'^2} ; \quad (24)$$

for $r=3,4$ we obtain explicitly :

$$K_3^{(0)}(q,q') \sim \frac{1}{8q'^3} \cdot \frac{\mu^2}{\left[\left(\frac{q^2 - q'^2 + \mu^2}{2q'}\right)^2 + \mu^2\right]^{3/2}} , \quad (25)$$

$$K_4^{(0)}(q,q') \sim \frac{1}{16q'^5} \cdot \frac{(q^2 + q'^2 + \mu^2) \cdot \mu^4}{\left[\left(\frac{q^2 - q'^2 + \mu^2}{2q'}\right)^2 + \mu^2\right]^{5/2}} . \quad (25')$$

^{*)} The $K_{r-2}(\mu R)$ on the r.h.s. of Eq. (23) denotes the modified Bessel function.

These expressions are strongly peaked around $q'^2 = q^2 + \mu^2$, with a width of the order of magnitude μ^2 . In fact, for $\mu^2 \rightarrow 0$ they approach δ functions. The approximation, which we now want to make, consists in neglecting finite width effects, but leaving the position of the peak at $q'^2 = q^2 + \mu^2$ in the wave projections of these kernels, which amounts to replacing them by δ functions :

$$K_3^{(0)}(q, q') = \frac{G^2}{4\pi^2} \cdot \frac{\delta(q^2 - q'^2 + \mu^2)}{q'^2}, \quad (26)$$

$$K_4^{(0)}(q, q') = \frac{G^2}{4\pi^2} \cdot \frac{\delta(q^2 - q'^2 + \mu^2)}{q'^2}. \quad (26')$$

With this zero width approximation the propagator equation now reads :

$$\Delta_F^{-1}(q^2) = 2(-q^2 - m_0^2) - \frac{G^2}{2} \int_0^\infty q' dq' \delta(q^2 - q'^2 + \mu^2) \Delta_F(q'^2)$$

or (27)

$$\Delta_F^{-1}(x) = 2(x - m_0^2) - \frac{G^2}{4} \Delta_F(x - \mu^2),$$

where we have put $x = q_L^2 = -q_E^2$. Defining x, m_0^2 in units of μ^2 and substituting

$$A(x) = 2 \Delta_F(x)$$

$$G^2 = 2^2 G_0^2 = 4 2^2 \lambda^2, \quad (28)$$

where A is the unrenormalized propagator and G_0 the unrenormalized coupling constant, this equation reads

$$A(x) = \frac{1}{x - m_0^2 - \lambda^2 A(x-1)}. \quad (29)$$

The solution of this equation is the infinite continued fraction

$$A(x) = \frac{1}{x - m_0^2 - \frac{\lambda^2}{x - m_0^2 - 1 - \frac{\lambda^2}{x - m_0^2 - 2 - \dots}}}$$

(30)

$$= \frac{1/(x - m_0^2)}{1 - \frac{\lambda^2/(x - m_0^2)(x - m_0^2 - 1)}{1 - \frac{\lambda^2/(x - m_0^2 - 1)(x - m_0^2 - 2)}{1 - \dots}}}$$

From the theory of continued fractions ⁸⁾ we know the following properties of $A(x, \lambda)$:

- i) $A(z, \lambda)$ considered as a function of z converges to a Stieltjes integral

$$A(z, \lambda) = \int_{-\infty}^{+\infty} \frac{d\psi(x)}{z + x},$$

where $\psi(x)$ is of bounded variation. The convergence is uniform for all z not on the real axes. In other words : $A(z, \lambda)$ possesses an unsubtracted Lehmann-Källén representation with positive spectral density.

- ii) The finite rational approximations $A_n(x, \lambda)$ of $A(x, \lambda)$ have the following properties :

$$A_n(x, \lambda) = \sum_{j=1}^n \frac{M_j}{x + x_j} \quad \text{with} \quad \begin{matrix} x_j \text{ real} \\ \sum_{j=1}^n M_j = 1, \text{ all } M_j > 0. \end{matrix}$$

They consist of a sum over n simple poles with positive residues, where the sum over the residues gives 1.

- iii) As a function of λ it is meromorphic with simple poles with positive residues.

The infinite continued fraction (30) can be evaluated exactly. The result is

$$A(x) = -\frac{1}{\lambda} \cdot \frac{J_{m_0^2-x}(2\lambda)}{J_{m_0^2-x-1}(2\lambda)},$$

where $J_\nu(z)$ is the Bessel function of first kind.

Reinserting the renormalized quantities we finally obtain for the dressed propagator of the smooth kernels in zero width approximation :

$$\Delta_F(q^2) = -\frac{2}{G} \cdot \frac{J_{m_0^2-q^2}(G/2)}{J_{m_0^2-q^2-1}(G/2)}. \quad (31)$$

Δ_F has an infinite number of simple poles in q^2 with positive residues. They are the remainders of the cuts in the exact propagator, which are contracted to poles by the zero width limit.

We want to discuss the propagator (31) for $G_0 \gg 1$, because its behaviour for $G_0 \ll 1$ is similar to that which we know from perturbation theory. Let us first consider the position and residue of the lowest lying pole, which corresponds to the stable particle, and from which we obtain the mass and wave function renormalization.

The mass m_1^2 of the stable particle is given by the first zero of the Bessel function in the denominator of Eq. (31). For m_1^2 we have⁹⁾ :

$$m_1^2 = m_0^2 - 1 - G_0 + 1.856 G_0^{4/3} - 0.113 G_0^{-1/3} + \dots \quad (32)$$

The residue $R(m_1^2)$ of Δ_F at this pole can be calculated as follows :

$$\begin{aligned} R(m_1^2) &= \frac{2}{G} \cdot \frac{J_{m_0^2-q^2}(G_0)}{\frac{\partial}{\partial q^2} J_{m_0^2-q^2-1}(G_0)} \Big|_{m_1^2-q^2} = \frac{2}{G} \cdot \frac{\frac{\partial}{\partial G_0} J_{m_0^2-q^2-1}(G_0)}{\frac{\partial}{\partial q^2} J_{m_0^2-q^2-1}(G_0)} \Big|_{q^2=m_1^2} \\ &= -\frac{2}{G} \cdot \frac{d m_1^2(G_0)}{d G_0}. \end{aligned}$$

From the renormalization condition $R(m_1^2) \equiv 1$, we obtain the renormalization constant Z :

$$Z = - \frac{1}{G_0} \cdot \frac{dm_1^2}{dG_0} = \frac{1}{G_0} \left(1 - \frac{1.856}{3} G_0^{-2/3} - \frac{0.113}{3} G_0^{-4/3} + \dots \right) = G/G_0. \quad (33)$$

This equation shows that $0 \leq Z \leq 1$. It can be used to express G_0 as a function of G , thereby expressing m_1^2 and Z in terms of the renormalized coupling constant G .

The position of this and the other poles of Δ_F as a function of G_0 is shown in Fig. 3, which is taken from Jahnke-Emde-Lösch¹⁰⁾. The residues of these poles are given by the slope of the curves $m_1^2(G_0)$ divided by the slope of $m_1^2(G_0)$. The asymptotic behaviour of $\Delta_F(q^2)$ can be derived from the power series expansion of the Bessel functions :

$$\Delta_F(q^2) = \frac{Z^{-1}}{q^2 - m_0^2}. \quad (34)$$

It depends on the bare mass and the renormalization constant, as was to be expected.

Having determined the zero width approximation of the dressed propagator we can continue to calculate the form factor $F_1(q,k)$ from its integral Eq. (11) in the same approximation with the aim to check the validity of the Ward identity explicitly. Eq. (11) can be treated by performing an $O(3)$ partial wave expansion in the rest frame of the virtual photon $k=(k_0, \underline{0})$. Because of rotational symmetry, only the s wave projections of the kernel and F_1 are needed :

$$K(q-q') = 4\pi \sum_{lm} K_{(q,q',q^2,q'^2)}^{(l)} Y_{lm}^*(\hat{q}') Y_{lm}(\hat{q}) \quad (35)$$

$$K_{(q,q',q^2,q'^2)}^{(l)} = \frac{1}{2} \int_0^\pi P_l(\cos\theta) K(q,q',q^2,q'^2, \cos\theta) d(\cos\theta).$$

For the one-particle exchange we have the familiar expression :

$$K_1^{(0)} \sim qq' \ln \frac{(q_0 - q'_0)^2 + (q + q')^2 + \mu^2}{(q_0 - q'_0)^2 + (q - q')^2 + \mu^2} \quad (36)$$

From this we get by differentiation :

$$K_3^{(0)} \sim 2qq' \mu^2 \cdot \frac{(q_0 - q'_0)^2 + q^2 + q'^2 + \mu^2}{[(q_0 - q'_0)^2 + (q - q')^2 + \mu^2]^2 [(q_0 - q'_0)^2 + (q + q')^2 + \mu^2]^2} \quad (37)$$

The zero width approximation of this smooth kernel, as well as of $K_4^{(0)}$ is :

$$K_3^{(0)} = G^2 \cdot \frac{1}{q'} \delta(q_0 - q'_0) \cdot \delta(q'^2 - q^2 + \mu^2) \quad (38)$$

Insertion into the s wave projection of the integral equation (11) yields :

$$q_0 F_1(q_0, k, q^2) = 2e \cdot 2q_0 + \frac{G^2}{4} \cdot q_0 \cdot F_1(q_0, k_0, q^2 - \mu^2) \Delta_F((q + k_0)^2 - \mu^2) \Delta_F((q - k_0)^2 - \mu^2)$$

or when introducing $p_\mu = q_\mu + \frac{1}{2}k_\mu$, $p'_\mu = q_\mu - \frac{1}{2}k_\mu$ and writing covariantly :

$$F_1(p^2, p'^2, pp') = 22e + \frac{G^2}{4} F_1(p^2 - \mu^2, p'^2 - \mu^2, pp') \Delta_F(p^2 - \mu^2) \Delta_F(p'^2 - \mu^2) \quad (39)$$

This may be considered as the zero width approximation of the BS equation for the form factor F_1 with smooth kernels. We solve it by summing its Born series ($\mu^2=1$) :

$$\begin{aligned} F_1(p^2, p'^2) = & 22e + \frac{G^2}{4} \Delta_F(p^2-1) \Delta_F(p'^2-1) \cdot 22e \\ & + \frac{G^4}{16} \Delta_F(p^2-1) \Delta_F(p^2-2) \Delta_F(p'^2-1) \Delta_F(p'^2-2) \cdot 22e + \dots \end{aligned}$$

$$= 2Ze \cdot \sum_{\alpha=0}^{\infty} \prod_{\beta=0}^{\alpha} \left(\frac{G^2}{4} \right)^{\alpha} \Delta_F(p^2 - \beta) \Delta_F(p'^2 - \beta).$$

Inserting the result for the dressed propagator (31), this becomes :

$$F_1(p^2, p'^2) = 2Ze \sum_{\alpha=0}^{\infty} \frac{J_{m_0^2 - p^2 + \alpha}(G/2) J_{m_0^2 - p'^2 + \alpha}(G/2)}{J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2}(G/2)}.$$

In order to evaluate the sum over the product of the Bessel functions we make use of the formula ⁹⁾:

$$J_{\nu_1}(z) J_{\nu_2}(z) = \frac{2}{\pi} \int_0^{\pi/2} J_{\nu_1 + \nu_2}(2z \cos \vartheta) \cos(\nu_1 - \nu_2)\vartheta d\vartheta$$

and obtain :

$$F_1(p^2, p'^2) = 2Ze \cdot \frac{2/\pi \sum_{\alpha=0}^{\infty} \int_0^{\pi/2} J_{2m_0^2 - p^2 - p'^2 + 2\alpha}(2G/2 \cos \vartheta) \cos(p'^2 - p^2)\vartheta d\vartheta}{J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2}(G/2)}.$$

From the recursion formulae of the Bessel functions one can derive :

$$\sum_{\alpha=0}^{\infty} J_{\nu+1+2\alpha}(z) = \frac{1}{2} \int_0^z J_{\nu}(x) dx.$$

Hence,

$$F_1(p^2, p'^2) = 2Ze \cdot \frac{1/\pi \int_0^{\pi/2} d\vartheta \int_0^{2G/2 \cos \vartheta} dx J_{2m_0^2 - p^2 - p'^2 - 1}(x) \cos(p'^2 - p^2)\vartheta}{J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2}(G/2)}.$$

Partial integration gives :

$$\begin{aligned} F_1(p^2, p'^2) &= -2Ze \cdot \frac{2G}{2} \cdot \frac{1}{\pi} \cdot \frac{1}{p'^2 - p^2} \cdot \frac{\int_0^{\pi/2} J_{2m_0^2 - p^2 - p'^2 - 1}(2G/2 \cos \vartheta) \sin(p'^2 - p^2)\vartheta \cdot \sin \vartheta d\vartheta}{J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2}(G/2)} \\ &= \frac{-eG}{p'^2 - p^2} \cdot \frac{-J_{m_0^2 - p^2 - 1}(G/2) J_{m_0^2 - p'^2}(G/2) + J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2 - 1}(G/2)}{J_{m_0^2 - p^2}(G/2) J_{m_0^2 - p'^2}(G/2)}. \end{aligned}$$

Using now the result Eq. (31) for the renormalized propagator we have

$$F_1(p^2, p'^2) = 2e \cdot \frac{\Delta_F'(p^2) - \Delta_F'(p'^2)}{p^2 - p'^2} \quad (40)$$

But this is the Ward identity expression for $F_1(q, k)$ [Eq. (9)]

CONCLUSION

When one uses the Bethe-Salpeter equation to describe two-body scattering, bound states and form factors, one should be aware of the fact that current conservation, i.e., the Ward identity for the electromagnetic vertex implies that the ingredients of the BS equation, namely the propagator and the kernel cannot be chosen arbitrarily. The possible energy dependence of the BS kernel has to satisfy Eq. (15) and the propagator is determined from the kernel at zero energy by the non-linear integral Eq. (14).

This suggests for phenomenological applications the following framework, which starts from an appropriately chosen BS kernel :

We have illustrated in a simple example how this scheme works in giving a consistent form factor and propagator for smooth interactions. Of course, if one treats fully the field equations starting from a Lagrangian all these interrelations are automatically fulfilled. But for many practical applications the above described framework may be more useful.

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FIGURE CAPTIONS

- Fig. 1, a) : The Bethe-Salpeter equation for two-particle scattering.
b) : The BS equation for the bound states.
c) : The BS equation for the electromagnetic form factor.
- Fig. 2 : The interaction kernels $K_r(R)$ in configuration space as a function of the four-distance R .
- Fig. 3 : The location of the poles m_1^2 of the propagator as a function of the coupling constant G_0 .

FIG.1 a

FIG.1 b

FIG.1 c

FIG.2

FIG. 3