Summary and Conclusions

Increasing the PSB and CPS intensities to $10^{13}$ ppp will require more powerful octupoles in order to overcome the transverse instabilities that are to be expected. Unfortunately, theory fails to give quantitative predictions at low energies, where direct space-charge forces dominate over image forces. To be able to specify the future needs more precisely, an extension of the theory would be desirable, particularly since experimental evidence is such that it is hard to extrapolate to higher intensities.

The theory presented here solves the dispersion relation, including the influence of external and space-charge induced non-linearities in both transverse dimensions as well as a spread in the LNS coefficient $U$. Results are preliminary in the sense that modulation of the space-charge forces due to synchrotron motion is not included. They show that the combined effect of external and space-charge non-linearities in the two transverse planes can considerably enhance Landau damping in low-energy machines (up to 10 GeV, say). The sign of the octupole current is important especially for a flat beam, and should be chosen in such a way that $\nu$ increases with amplitude in the direction where the beam is wide.

Stability diagrams for several typical conditions are presented and applied to the PSB.

1. Introduction

Octupole lenses have been installed both in the CPS and its Booster years ago, and they have successfully cured transverse instabilities. It is planned to use more powerful lenses in the CPS, and possibly in the Booster too, in order to tame future intensities. It is puzzling, however, to observe that the currents needed to avoid instabilities are considerably lower than predicted by theory1,2. In addition, lower thresholds for radial rather than for vertical resistive wall instability were observed frequently in the PSB. These observations suggest that the influence of space-charge non-linearities and the spreads due to motion in the second transverse direction might be important.

An attempt to include these effects3 already showed the importance of the variations due to the betatron amplitudes in the second transverse plane. This approach, however, still predicted too high octupole strengths and failed to give the correct behaviour in the limit of vanishing external octupole force. In this limit incoherent space-charge forces should have no influence on dipole motion, a fact which can be deduced from the single-particle-equation4 and is fully confirmed by computer simulation5.

To make more reliable extrapolations it was felt necessary to extend the theory in order to remove these deficiencies. The main difference of the present work in comparison with Ref. 3 is that we include space-charge non-linearity in both the driving term (the U-term of Ref. 1) and in the single-particle frequency.

A general model would have to include:

i) frequency spreads due to external non-linearities in both transverse (x and y) directions;

ii) spreads due to $x^2$ and $y^2$ non-linearities of space-charge forces;

iii) spreads due to the energy distribution of the beam;

iv) spreads due to longitudinal variation of transverse space-charge forces;

and probably other effects. In the present paper we content ourselves with considering the effects (i) and (ii). This is a valid approximation at least for the coasting beam experiments which were performed in the Booster and which exhibited the "anomalities" discussed above. We shall find that the combination of external and space-charge non-linearities can considerably enhance the stability conditions, and that the spreads from both transverse directions are important.

2. Dispersion Relation

Let us discuss dipole oscillations in one transverse ($a$) direction. Neglecting energy spread, the dispersion relation of Laslett, Neil and Sessler (LNS) can be written as

$$\frac{\omega}{\omega_c} = \frac{n + \nu(a) - \nu_c}{2\nu_c}.$$

Here the tune $\nu(a)$ depends on $a$, the amplitude of the incoherent betatron oscillation $h(a)$ is the corresponding distribution function, $\nu_c$ is the growth rate, $\nu$ being the collective betatron frequency.

To include space-charge non-linearities as well as y-variation of $\nu_c$ we can use the derivation of Eq. (1) given by Hereward6. We start from the single-particle equation

$$\dot{x}_a = i \nu \left[ \frac{v_1}{(x_a,y_a)} - 2\nu_c \nu_1 \left( x_a,y_a \right) \right] x_a = 2\nu \nu_1 \left[ \nu_c - \nu_1 \left( x_a,y_a \right) \right] x_a$$

where $x_a$ is the dipole motion of the beam and $x_a$ the motion of the test particle (coherent and incoherent). The corresponding dispersion relation is

$$\int_{-\pi}^{\pi} \left( U + V + iV \right) (a,b) \left[ -\left( h'(a) \right)^2 \right] g(b) d\alpha = 1.$$

Here $h$ is the incoherent y-amplitude, $g(b)$ the corresponding distribution, $\int g(b) d\alpha = 1$. The variance of the tune $\nu_1$ includes the incoherent tune shift, and so does $U + V + iV$ through Eq. (2). The terms $\nu_1(a,b)$ and $U + V + iV(a,b)$ are obtained by averaging over the incoherent betatron motion.

To evaluate Eq. (1a) we make some further approximations: We assume that only the incoherent tune shift is non-linear, whereas the image term $\Delta \nu_c$ is the same for all particles. This is a valid approximation at low energy, where $\Delta \nu_c < \nu_c$ and/or for thin beams. The case where both $\Delta \nu_c = \Delta \nu_c(a,b)$ and $\Delta \nu_c = \Delta \nu_c(a,b)$ but where external non-linearities are negligible ($\nu_1 = \text{const}$) was discussed in Ref. 4. We shall further expand $\nu_1(a,b)$ and only retain terms up to the octupole moment.
\[ \Delta \nu_c(a,b) = \Delta \nu_c(0) + \frac{3a}{\Delta \nu_c} a^2 + \frac{3b}{\Delta \nu_c} b^2 = \Delta - \Delta a^2 - \Delta b^2, \]

and
\[ \nu_b(a,b) = \nu_b(0) + \frac{3a}{\Delta \nu_c} a^2 + \frac{3b}{\Delta \nu_c} b^2 = \nu_b + \nu_a^2 + \nu_b^2. \]

The symbols \( \Delta \), \( \Delta a \), \( \Delta b \) (all positive) and \( \nu_a \), \( \nu_b \) are hereby defined and will be used throughout the rest of the paper. Quantities such as \( \nu_a^2 \), \( \nu_b^2 \), etc., will be denoted as "external spread" and "space-charge spread", respectively; \( \Delta \) and \( \delta \) are typical amplitudes to be defined below; *a* corresponds to the plane of the instability.

### 3. Results

Here we shall present the solution of the dispersion relation (1a) for two different distributions and for several typical conditions. We assume \( \nu_a = 0 \), \( \nu_b = 0 \) in Section 3.1; \( \delta \gg \delta \) in Section 3.2 and Figs. 1 and 2; \( \delta = \delta \) in Section 3.3 and Figs. 3 and 4; and \( \delta \gg \delta \) in Section 3.3 and Fig. 5.

#### 3.1 No external non-linearities (i.e. \( \nu_a = 0 \), \( \nu_b = 0 \))

In this case incoherent space-charge has no effect, as was found already in Ref. 4. In fact it is easily verified from the single-particle equation (3) that \( \chi = \chi \) is a solution provided that \( \chi = \chi \) and \( \chi \) is the same for all particles. Under this condition \( \Delta \nu_c \) simply drops out from Eq. (3).

#### 3.2 \( \nu \)-spread due to betatron amplitudes in the plane of the instability only (i.e. \( \delta \gg \delta \))

##### 3.2.1 "Semicircular" distribution

It is instructive to start with the distribution which — without space charge forces — leads to a circular range of stability and gives the rule-of-thumb criterion

\[ \frac{\Delta V}{\Delta \nu_c} = \frac{\Delta V}{\nu_c} \leq \frac{\delta \text{FWHH}}{V}. \]

for the stabilizing \( \nu \)-spread. The derivative of this function which enters into Eq. (1a) is a half circle:

\[ \nu^2 + \left( \frac{U + V}{\Delta \nu_c} \right)^2 = \left( \frac{\nu_c}{\Delta \nu_c} \right)^2, \]

or

\[ 0 \leq a^2 \leq 2 \Delta \nu_c. \]

Including now space-charge, the solution of the dispersion relation gives the stability boundary

\[ \nu^2 + \left( \frac{U + V}{\Delta \nu_c} \right)^2 = \left( \frac{\nu_c}{\Delta \nu_c} \right)^2 \Delta \nu_c. \]

Here we have introduced a parameter \( \nu_a = \nu_a/\Delta \nu_c \), the ratio of external spread to space-charge spread. \( U \) involves an average of \( \Delta \nu_c \)

\[ U = \Delta \nu_c - \left( \Delta - \Delta a^2 \right). \]

For \( 1/\Delta \) we recover the rule of thumb (5). With space-charge spread, the stability circle is distorted into an ellipse with real half-axis:

\[ \nu = \left( \frac{U}{\nu_a} \right) \Delta \nu_c \text{FWHH} \left( 1 + \frac{2}{\nu_a} \right). \]

The stability range is thus increased or decreased depending on the sign of \( \nu_a \), i.e. depending on the polarity of the octupoles. Typically the space-charge non-linearity is such that \( 2/\nu_a = \left( U/\Delta \nu_c \right) \) (see Appendix). Hence for \( V \ll |U| \) — the case of interest in the PSB and in the CPS below 10 GeV — the stability condition reduces to

\[ \frac{\Delta V}{\Delta \nu_c} \leq \frac{\delta \text{FWHH}}{V} = \frac{\delta \text{FWHH}}{V} \frac{2}{1 + \frac{2}{\nu_a}}. \]

and the stabilizing octupole \( \nu \)-spread is about 0.67 or twice the value obtained from conventional theory (1/\( \nu_a = 0 \)).

##### 3.2.2 Parabolic distribution

Figure 2 gives similar results for a parabolic distribution:

\[ h(a) = \frac{1}{2 \pi} \left( 1 - \frac{a^2}{2 \beta^2} \right), \quad 0 \leq a^2 \leq 2 \beta^2. \]

Again for negative \( q_a \) the stable area is largely reduced. One particularity of Fig. 2 is that all curves intersect the point \( \nu = 0, \nu = -2 \). It appears that this effect is due to the sharp cut-off of this distribution.

#### 3.3 \( \nu \)-spread due to betatron amplitudes in both transverse directions (parabolic distribution in each plane)

Now we have to introduce another two parameters:

\[ p = \nu_a^2 - \nu_b^2, \quad r = \frac{\Delta \nu_c}{\nu_c} \nu_a \nu_b \nu_b. \]

\( p \) denotes the ratio of the two external spreads and \( r \) the ratio of the space-charge spreads; \( \delta \) is the equivalent of \( \delta \) for the second transverse direction. Note that \( r \) is determined by the emittance ratio and varies from 0 for \( \delta \gg \delta \) to about 2 for \( \delta \ll \delta \); \( r = 1 \) corresponds to \( \delta = \delta \).

Figures 4 and 5 refer to parabolic distributions \( h(a) \) in Eq. (1a) and demonstrate the influence of the second transverse plane. In Figs. 3 and 4, \( r = 1 \), i.e. the same spreads \( \Delta \beta \) and \( \delta \beta \) have been assumed.

The case \( p = 1 \) as taken in Fig. 3 assumes that two sets of octupoles (in F- and D-sections say) are used such that \( \nu_a \) and \( \nu_b \) have the same sign. In this case we recover the reduction of the stable area for negative \( q_a \) as for the one-dimensional case (Figs. 1 and 2).

In Fig. 4, \( \nu_a \) and \( \nu_b \) have been taken of opposite sign, which is typical for simple lens arrangements.
In this case, the reduction of stable areas is less noticeable, because the tendency of cancellation for the one octupole polarity is to some extent smoothed out: cancellation in one plane goes together with addition in the other plane.

Finally Fig. 5 corresponds to a beam which is wide in the direction perpendicular to the plane of instability ($b \gg a$), as is usually the case for vertical instability of a multiturn beam. Hence we take $r = 2$ and assume in addition "simple octupoles" with large negative $p$.

One finds again that one octupole polarity is favourable although the "right" sign is now the opposite to the preferred one in the case of a beam which is wide in the other direction ($p = 0, r = 0$). We conclude that the octupole moment in the direction where the beam is wide should be chosen with care.

Application to the PSB

The coasting beam and bunched beam instabilities observed in the PSB occur sometimes horizontally, sometimes vertically, in an apparently irregular way. In order to explain this feature, we apply the results of Section 3.3 to the PSB. We compute the thresholds for both planes as a function of the emittance ratio $\epsilon_H/\epsilon_V$, assuming the product $\epsilon_H \epsilon_V$ and hence the area of the beam cross-sections to be a constant. We include image forces and averaging over the strongly varying beam dimensions within a machine period. The arrangement of the octupoles is as described in Ref. 9. Thresholds are expressed by the octupole currents required to stabilize the beam.

Figure 6 shows the result for $N = 2.5 \times 10^{12} p/p$ and $\epsilon_H \epsilon_V = 5200 (\pi \text{ mrad mm})^2$ at 50 MeV. One observes that positive octupole current ($3v/3r^2 > 0$) is more favourable and that for this polarity vertical stability requires stronger octupole currents for $\epsilon_H/\epsilon_V \leq 1.7$ whereas horizontal stability is more critical for $\epsilon_H/\epsilon_V > 1.7$. The intersection point $\epsilon_H/\epsilon_V = 1.7$ falls into the region of emittance ratios actually observed in many machine experiments. This might explain why slight differences in beam parameters can favour the one or the other direction.

The predicted octupole currents are still higher than measured values. This is probably owing to the neglect of synchrotron motion in our model for the case.
Fig. 3: Parabolic amplitude distributions (11) in both transverse directions: \( r = 1 \) means \( \delta = 2\pi \); \( p = -1 \) is typical for one set of octupoles and not too flat a beam.

Fig. 4: As for Fig. 3, but \( p = 1 \) requires two sets of octupoles.

Fig. 5: As for Fig. 3, but for a beam wide in the direction perpendicular to that of the instability \((b >> 2)\).

Fig. 6: Octupole currents required to stabilize the PSB beam versus emittance ratio, calculated for parabolic amplitude distributions (11) in both transverse planes. Nominal parameters at 50 MeV: \( E_{H/V} = 130 \times 40 \) (\( \pi \) mm), \( N = 2.5 \times 10^{12} \) p/p.
of bunched beams and because of neutralization in the
coasting beam case. To conclude, let us make a com-
parison with conventional theory (external spread in
one transverse direction, no spread in Δνic) – the latter
requires stabilizing octupole currents of 400 A (hor-
izontal) and 520 A (vertical instability), for the nomi-
nal E=130 GeV, all other parameters as for Fig. 6.
This is to be compared with the values of 180 A and
70 A taken from Fig. 6.

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Appendix

Calculation of incoherent v-shift and
v-spreads

In order to assure self-consistency with the para-
bolic distribution that we mainly use, we should solve
the potential problem for the corresponding charge dis-
tribution. This seems to be difficult for the kind of
factorized amplitude distribution h(a)b assumed above,
which represents a beam of rectangular cross-section.
For the semicircular distribution (6), a numerical
estimate gives u = 3.8 (horizontal) and u = 4.8 (vertical).
Note that r is approximately given by r = 2(2a + b)
and can only take values of 0 < r < 2 (Eqs.(12), (A5)).
This would imply values of u between -4 and -8, which is
valid only for a machine of small wiggle in the β-func-
tions.

Computed values for the PSB, assuming nominal emit-
tances (ΔH = 130 m, ΔE = 40 mrad at 50 MeV) and in-
cluding image contributions, give values of
u = -3.8 (horizontal) and u = 4.8 (vertical).

This value of u is used to derive the stability
criterion (9). From (6) and (10) we obtain for V<<|u|

\[ \Delta \nu_{ic} = \left( 1 + \frac{\Delta \varphi}{\varphi} \right) \left( 1 + \frac{\Delta \varphi}{\varphi} \right) . \]  

Hence
\[ |u| < \frac{\Delta \varphi}{\varphi} = \frac{1}{2} \varphi \theta^2 b^2 |u + 1| + \frac{1}{2} \theta^2 \]  
and
\[ \Delta \nu_{ic} = \left( 1 + \frac{\Delta \varphi}{\varphi} \right) \left( 1 + \frac{\Delta \varphi}{\varphi} \right) . \]

Performing an averaging process over incoherent
betatron motion (e.g. by the method of harmonic balance),
x - R = a cos(φ), y = b cos(φ), we take
\[ \langle x - R \rangle = \frac{3}{4} a^2 \cos \phi \Delta t = \frac{3}{4} a^2 \Delta \nu_{ic} \]  
\[ \langle y^2 \rangle = \frac{1}{2} b^2 \Delta \nu_{ic} \]  
and hence
\[ \Delta \nu_{ic} = \frac{1}{2} b^2 \Delta \nu_{ic} . \]

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