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ELECTROPRODUCTION AND ANNIHILATION SCALING LAWS
IN A DUAL MODEL

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A B S T R A C T

Scaling laws for large virtual photon mass (q^2) in electroproduction and annihilation are studied in the framework of a simple planar dual model. We find, as has recently been conjectured, that the scaling behaviour depends on the number of space-time dimensions spanned by large momenta. In particular, for a certain range of parameters in the model, we find that the annihilation cross-section is dominated by the one-dimensional configuration and increases with q^2 relative to its canonical behaviour while the electroproduction total hadronic cross-section is dominated by the canonical two-dimensional configuration with small transverse momenta. In general the scaling laws and therefore the structure of events in the model are distinctively different from the conventional parton model. The problem of consistency of planar dual tree diagrams with unitarity sum rules is discussed.

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1. - INTRODUCTION

In this paper we present a study of high energy electron-positron annihilation and deep inelastic lepton scattering in the framework of a dual model. Perhaps the most interesting feature of the model is that it has different scaling behaviours which depend upon the kinematical configuration of the hadrons involved ^{*}). It has recently been proposed ¹⁾ that the exponent which characterizes the scaling behaviour of an arbitrary process should depend on the number of space-time dimensions spanned by the large momenta in the corresponding amplitude. The dual model studied here is, under certain circumstances, a realization of this principle at all levels, i.e., when one, two, or three dimensions are asymptotically excited. In particular, it allows e^+e^- annihilation to have power behaviour in q^2 with an exponent larger than the "canonical" one while deep inelastic scattering still has the "canonical" exponent and scaling behaviour.

Let us recall briefly what is expected on the basis of the dimensional scaling behaviour hypothesis. In high energy e^+e^- annihilation through a single photon, the virtual photon has a large energy but no momentum in the e^+e^- c.m. system. We expect the scaling behaviour with the highest exponent of $q \equiv \sqrt{|q^2|}$ to arise when there is no particle with a large momentum, in the final state, since in this situation only an energy component is anywhere large (that of the virtual photon). For example, we would write the single particle inclusive cross-section as (E fixed as $q \rightarrow \infty$)

$$E \frac{d^3\sigma}{d^3p} \underset{q \rightarrow \infty}{\sim} q^{-d_1} f_1(E). \quad (1.1)$$

We call d_1 the single-dimensional scaling power.

^{*}) By scaling behaviour we mean that as some of the kinematic variables become large the cross-section behaves as a simple power of one of the large variables (e.g., q^2 , the virtual photon mass) times a function of ratios of the large variables and any variables which remain fixed in the limit. In the dual model, large means large with respect to the natural scale $1/\alpha'$.

If we now ask that one of the secondaries have a large momentum component ($\sim \frac{1}{2}\omega q$), then we are in a situation where the large momenta span at least one space dimension (e.g., the z direction). We expect the cross-section to be dominated by events in which secondaries have large momentum components in no other direction than the z direction, and to depend on q with a power d_2 :

$$E \frac{d^3\sigma}{d^3p} \underset{q \rightarrow \infty}{\sim} q^{-d_2} f_2(\omega), \quad (1.2)$$

where $\omega = 2E/q$. We shall call d_2 the two-dimensional scaling power and we expect within the dimensional scaling picture that $d_2 > d_1$.

If, as we suppose throughout, there is a smooth transition between the two scaling behaviours, we have

$$f_1(E) \underset{E \rightarrow \infty}{\sim} E^{-\Delta} \quad (1.3a)$$

and

$$f_2(\omega) \underset{\omega \rightarrow 0}{\sim} \omega^{-\Delta}, \quad (1.3b)$$

where $\Delta = d_2 - d_1$. If we insist that there be another large momentum particle in a direction transverse to the z axis, then again we would lose in intensity. This change would be reflected by a new component d_3 , the three-dimensional power ($d_3 > d_2$), which would control the decrease with q of those events in which large secondary momenta span two spatial dimensions. A similar change should also occur when going to a configuration when three spatial dimensions are excited.

In electroproduction we are in a situation where the energy and one momentum component are large in the initial state. We see therefore that it is at the previously discussed two-dimensional level in annihilation where contact is made with electroproduction and we therefore expect d_2 to be related to the usual Bjorken scaling power. However, as we have seen above, the e^+e^- annihilation total cross-section could be dominated by the one-dimensional regime (d_1) and therefore the q dependence of $\sigma_{e^+e^- \rightarrow \text{hadrons}}$ could be different

from canonical (two-dimensional) behaviour. In particular if d_1 is sufficiently larger than d_2 the cross-section $\sigma_{e^+e^- \rightarrow \text{hadrons}}$ would decrease less rapidly than $1/q^2$ as in "canonical" models.

In Ref. 1) it was remarked that while the two-dimensional asymptotic situation is present in several phenomena (e.g., deep inelastic lepton scattering and high energy hadron-hadron processes), the one-dimensional configuration allowed by the one-photon intermediate state is effectively inaccessible in hadron-hadron processes and we find in high energy e^+e^- annihilation the first realization of such a situation ^{*)}. The dynamical models constructed up to now (quark-partons, light cone algebra, multi-peripheral models, etc.) are tailored to give the observed two-dimensional scaling and can, under certain circumstances, give rise to a three-dimensional regime. However, these models are unable to produce a one-dimensional configuration. It was argued in Ref. 1) that dual models would be good candidates to exhibit the two- to one-dimensional transition. In particular, dual models have the large number of high mass states, X , which would be necessary to build up the process $e^+e^- \rightarrow \gamma \rightarrow p + X$ where p is small.

Another way to consider this question is in terms of the quark imprisonment problem. While this problem seems to play a secondary rôle in two-dimensional processes, it presumably plays a crucial rôle in the one-dimensional configuration. Indeed, if the quark is allowed to propagate, e^+e^- annihilation will immediately become a two-dimensional process as in the usual parton-quark models. In order to study the possible rôle of one-dimensional processes, it is necessary for the quark not to propagate which leads naturally to the notion of very dense Feynman diagrams. Again we are led to dual models as a possible description of this sort of dynamics ²⁾.

At present there is no completely factorizable dual model for currents but only a plethora of phenomenological models with various degrees of partial factorization. We are thus faced with a choice of which model or models to interrogate. We would like a model which exhibits "hard structure" (e.g., Bjorken scaling and form factors decreasing as powers)

^{*)} We consider it impractical to study experimentally the contributions of single, low spin hadronic intermediate states to the hadronic processes.

while at the same time all hadrons lie on Regge trajectories. To our knowledge, the only model which has all these properties is the model of Sugawara and Ademollo and Del Giudice (SAD)³⁾. This model has a few negative features (chiefly that it does not factorize) but it possesses a large number of good phenomenological properties. We will use this model throughout the present study. It will be defined more specifically in Section 2 where we also make a few comments on other dual models for currents.

In order to study the various one-, two-, three-dimensional scaling behaviours, it is convenient to study inclusive cross-sections since a large number of degrees of freedom are then automatically summed over. Study of the four-point function (Fig. 1a), which is related to the single particle inclusive cross-section for annihilation and the total electroproduction cross-section, enables us to learn about both one- and two-dimensional scaling behaviours. To study three-dimensional scaling we need to study the six-point function (Fig. 1b). This also allows us to study the correlations between particles in the one- and two-dimensional scaling limits. We recall, for completeness, that in this language the total annihilation cross-section is given by the two-point function (Fig. 1c).

In Sections 3 and 4 we discuss the behaviour of the four- and six-point functions in the SAD model. We will find that there is indeed a difference between the one- and two-dimensional scaling behaviours given by

$$\Delta = d_2 - d_1 = -2\delta + d(0) - 1, \quad (1.4)$$

where δ is related to the asymptotic behaviour of a certain form factor [e.g., $F(q^2)_{q^2 \rightarrow \infty} \sim (q^2)^\delta$]. A similar difference is found between the two- and three-dimensional scaling behaviours^{*)}. In general the distribution of the produced hadrons is quite distinctive and different from that of the usual quark-parton model⁴⁾.

*) The presence of a three-dimensional scaling behaviour is at first sight surprising since dual models are known not to have it in purely hadronic processes. We understand its presence in terms of the way currents are introduced in the SAD model which implies the inclusion of hard structure. We return to this point in Sections 4 and 6.

The above scaling behaviours indicate that the SAD model conforms to the general picture of Ref. 1). However, we find that scaling behaviour depends on the sense of the momenta as well. Thus, if we have two fast hadrons, the amplitude has a behaviour that is specifically two-dimensional if they are moving in the same direction but not if they are moving in opposite directions. This is not unreasonable in a resonance model : particles like to move in the same direction since they can resonate, but there is no particular benefit in being at 180° as opposed to 90° .

In the following sections we shall make use of the inclusive sum rules which relate the physical inclusive cross-sections for different numbers of particles. For example, the energy conservation sum rule for annihilation gives

$$\sigma = \frac{1}{q} \int \frac{d^3p}{E} E \frac{d^3\sigma}{d^3p/E} \quad (1.5)$$

The weight of phase space associated with the one-dimensional configuration ($E \ll q$) is

$$\frac{d^3p}{E} \cdot \frac{E}{q} \sim \frac{1}{q} \quad (1.6)$$

while that associated with the two-dimensional case ($E \sim q$) is

$$\frac{d^3p}{E} \cdot \frac{E}{q} \sim q^2. \quad (1.7)$$

Thus, Eq. (1.5) tells us that Eqs. (1.1) and (1.2) give contributions to σ of order q^{-d_1-1} and q^{-d_2+2} , respectively. The one-dimensional scaling contribution dominates therefore when $\Delta > 3$ ^{*)}. In this case, the multiplicity, given by

^{*)} For $\Delta = 3$ the two contributions are of comparable size and $\sigma \sim q^{-d_2+2} \ln q$. Note that present data suggest $\Delta \approx 4-5$.

$$\langle n \rangle \sigma = \int \frac{d^3 p}{E} \frac{d^3 \sigma}{d^3 p/E} \quad (1.8)$$

is $\langle n \rangle \sim q$.

The tree diagrams in dual models are, of course, not unitary by themselves. However, if we assume that they approximate the physical amplitudes fairly well, they should satisfy the inclusive sum rules approximately. In particular, we may require that the powers of the asymptotic variables should match on both sides of the sum rules. This is the case for purely hadronic amplitudes for large s . However, this is rather trivial since the phase space analogous to Eq. (1.6) or (1.7) is constant ^{*)}. In Section 5 we will discuss whether by appropriate choices of the parameters the scaling behaviours of the tree diagrams in the SAD model can be made consistent with the sum rules in the case that one-dimensional behaviour dominates. We find that this is possible at the level of the four- and six-point functions, but is not possible for the higher point functions. The difficulty will be seen to be a general property of resonance models when only tree diagrams are considered. Indeed factorization requires higher point functions to have the same scaling behaviour as lower point functions. However, the sum rules and Eq. (1.6) require the higher point functions to be larger by powers of q if low energy secondaries make the dominant contribution. Thus tree diagrams alone cannot give a one-dimensional scaling behaviour consistent with the unitarity sum rules. Even for $\Delta < 3$ a planar tree model will eventually violate the momentum sum rules. Other contributions, such as loop diagrams, must be important at the multiparticle level. Unfortunately we are unable to actually evaluate these other diagrams and it remains an open question whether they make sizeable contributions also to the one- and two-particle distributions. In any case, it is quite possible that dimensional scaling is a general property of dual models at all levels.

^{*)} This is a result of the two-dimensional scaling dominant in these reactions. The two-dimensional scaling of Eq. (1.7), appropriate for the single-particle inclusive distribution in annihilation is different because it has spherical symmetry.

The paper will be organized as follows. In Section 2, the SAD model will be presented and a few brief comments will be made about other models. In Sections 3 and 4 the behaviour of the model for the four- and six-point functions will be calculated. In Section 5 we will discuss the results in greater detail with emphasis on their interpretation and their internal consistency. Finally in Section 6 we close with some brief comments on the general conclusions which one might reasonably seek to draw from these results.

2. - DUAL MODELS FOR CURRENTS

2.1. - General Description of the SAD Model

In the SAD model, a current amplitude is represented by an ordinary dual amplitude with two "fictitious parton" lines for each current (see Figs. 2 and 3). The amplitude has only pole singularities in the current masses (q_i^2) and the various channels (s, t) lying on the trajectories $\alpha(q_i^2)$, $\alpha(s)$, $\alpha(t)$, etc. The "trajectories" in other "channels" of the B_N amplitude are set equal to fixed parameters. The "trajectories" for "channels" containing one fictitious parton and one or more hadrons control the asymptotic behaviour of form factors. For example, the form factor of Fig. 2a is

$$F(q^2) = B_4(-\alpha(q^2), -\gamma) \underset{q^2 \rightarrow \infty}{\sim} \Gamma(-\gamma) (-q^2)^\gamma. \quad (2.1)$$

The "trajectories" for "channels" containing fictitious partons from different currents control the behaviour in the Bjorken limit. Thus, for example, the Bjorken limit of Fig. 3 is (for $q^2 = q^{*2} \rightarrow -\infty$, s/q^2 fixed)

$$A_4(s, t, q^2, q^{*2}) = B_6(-\alpha(s), -\alpha(t), \dots) \underset{q^2 \rightarrow \infty}{\sim} (-q^2)^{k_1} G(s/q^2, t; \gamma, \delta; k_1). \quad (2.2)$$

The SAD model thus represents the interaction of a current with composite hadrons (falling form factors and Regge trajectories). It is able to give Bjorken scaling without any explicit partonlike particles (particles with constant form factors) appearing as real particles. Furthermore, it has a number of properties ⁵⁾ which are also found in field theory ladder models for composite particles ⁶⁾. Firstly, the form factor coupling the current to particles of spin J_1 and J_2 lying on Regge trajectories α_1 and α_2 has the asymptotic behaviour

$$F_{J_1 J_2}(q^2) \underset{q^2 \rightarrow \infty}{\sim} (q^2)^{\gamma - J_1 - J_2}. \quad (2.3)$$

Secondly, the current amplitudes have fixed poles

$$A(s, \dots) \underset{s \rightarrow \infty}{\sim} s^{\gamma} \quad (2.4)$$

whose position is correlated with the asymptotic behaviour of form factors of particles in crossed channels to the asymptotic variable. We feel that these results are sufficient to give some feeling of confidence that the model at least qualitatively may represent the interaction of currents with composite particles even though its formulation appears quite ad hoc.

In a dual model we expect, as is the case for hadronic amplitudes, that the resonances are locally dual to the scaling functions, at least for high resonance masses. Thus, if the discontinuity of the four-point function (Fig. 1a) has the behaviour

$$\text{disc}_s A_4 \underset{q \rightarrow \infty}{\sim} q^{-d_2} f_2(\omega), \quad (2.5)$$

the residues $R_N(q^2)$ of the poles at $\alpha_s = N$ should satisfy

$$2\pi \lim_{\substack{N \rightarrow \infty \\ N/q^2 = 1-\omega \text{ fixed}}} R_N(q^2) \sim q^{-d_2} f(\omega). \quad (2.6)$$

This can be shown to be the case in the SAD model since it is an ordinary dual amplitude ⁷⁾⁻⁹⁾.

While the qualitative features of the model discussed above are quite satisfactory [for some attempts to fit data with the model, see Refs. 10] the weakness of the model is that it makes no attempt to incorporate factorization ¹¹⁾. The large number of free parameters which appear is probably a result of this. For example, in general, the parameter γ in Fig. 3 need not be the same as that in Fig. 2a. We have chosen them to be the same in order to have a minimal amount of factorization - namely on the lowest pole $\alpha(s) = 0$.

The SAD model also does not satisfy the requirements of current conservation. We shall therefore consider the simplified case of scalar currents. We do not believe that the spin of the current plays any essential rôle in the general features discussed here. The total annihilation cross-section is then related to the discontinuity of the two-point function of Fig. 1c by

$$q^2 \sigma \sim \left(\frac{1}{q^2}\right)^2 \text{disc}_{q^2} A_2. \quad (2.7)$$

Likewise the single-particle inclusive cross-section is related to the amplitude of Fig. 1a by

$$q^2 \frac{d^3\sigma}{d^3p/E} \sim \left(\frac{1}{q^2}\right)^2 \text{disc}_s A_4, \quad (2.8)$$

and similarly for multiparticle inclusive cross-sections. Since the powers of q^2 relating the cross-sections and the discontinuities are common to all the multiparticle inclusive cross-sections, we shall often neglect them in what follows and discuss interchangeably the cross-sections and the discontinuities of the amplitudes. We note that canonical two-dimensional scaling for (2.8) is

$$\text{disc}_s A_4 \sim q^{-2} f_2(\omega) \quad (2.9)$$

and correspondingly for (2.7)

$$\text{disc}_{q^2} A_2 \sim \text{const.} \quad (2.10)$$

2.2. - Other Models

We discuss briefly some other dual models for currents. Landshoff and Polkinghorne ¹²⁾ some time ago proposed a phenomenological model for the four-point function which has canonical two-dimensional scaling in electroproduction. However, this model has some serious deficiencies which we believe make it an unreliable guide. It violates badly local duality and positivity. The dominant behaviour of the residue at $\alpha_s = N$ as $q^2 \rightarrow \infty$ is $(-1)^N (-q^2)^{2\gamma+2N}$. It is thus non-positive definite and has an unphysical form factor which grows with q^2 for N large enough. Landshoff ¹³⁾ studied the continuation of the amplitude to the annihilation physical region and found it was infinite for $0 \leq \omega \leq 1$. We suspect this unphysical behaviour (which cannot be interpreted simply as a change in scaling behaviour) is related to the unphysical nature of the residues noted above.

Gremmer, Neveu and Scherk ^{14),15)} have proposed a model which has the desirable feature of incorporating a high degree of factorization ^{*)}. The planar version of their model ¹⁵⁾ is the one of interest to us since non-planar amplitudes cannot have the fixed poles associated with the two-dimensional physics associated with current algebra and probably Bjorken scaling. Bjorken scaling is obtained if the form factors are Gaussian ($F(q^2) \sim e^{-cq^2}$) and thus unphysical ^{**)} . Even in this case the dimension d_2 of Bjorken scaling is t dependent $d_2 = -\frac{1}{2}(\alpha(t) - \frac{3}{2})$ which is also undesirable. Furthermore in this model local duality is presumably violated. We have studied this model in some detail and found that the model is exponentially behaved in q^2 in the annihilation region and generally for $q^2 \neq q^{*2}$ if we use the prescriptions given by the authors ¹⁵⁾. The unpleasant phenomenological properties of the model probably stem from the fact that it is constructed in a way very similar to hadronic amplitudes and therefore is too soft to have the hard structure we desire.

^{*)} It has the same spectrum as the hadronic amplitudes except spurious states couple.

^{**)} Some authors have argued that Gaussian form factors may be a necessary consequence of the "soft" structure of the dual model ¹⁶⁾.

Recently Schwartz ¹⁷⁾ has proposed a model which has complete factorization. The amplitudes, however, are infinite and a cut-off which spoils factorization must be imposed. Nevertheless, the model seems a very interesting one to investigate. The single current amplitudes are very similar to the SAD model - for example the amplitudes for one current and three hadrons are almost identical. The four-point function with two currents is very different, however, and we have not been able to obtain the continuation to the annihilation region.

There is another class of attempts at dual models for currents in which the current amplitudes are functions of q^2 times corresponding hadron amplitudes [e.g., $B_4(-\alpha(s), -\alpha(t))$ for A_4 ; see, e.g., Ref. 18]. These have the undesirable property that the (s, t) and (u, t) terms, like the Cremmer, Neveu, Scherk model, have $d_2 = -\frac{1}{2}(\alpha(t) + c)$. We do not believe they are reliable guides in large q^2 limits. We note that Satz ¹⁹⁾ has remarked that the (s, u) term has the form $f(q^2)e^{-4\alpha'E^2}$ and thus has one dimensional behaviour. However, the function behaves like e^{q^2} and so there is no corresponding two-dimensional behaviour so we cannot study the relationship between scaling behaviours. We feel the (s, t) and (u, t) terms are the important ones to study since they have the fixed poles ¹⁸⁾ and usually other two-dimensional features.

3. - THE FOUR-POINT FUNCTION

In this section we study the behaviour of the virtual Compton amplitude of Fig. 1a. The discontinuity in s gives the single particle inclusive annihilation cross-section and the total electroproduction cross-section. We shall first discuss the two-dimensional (Bjorken) scaling limit which is $q = \sqrt{|q^2|} \rightarrow \infty$ with s/q^2 fixed. It is convenient to use the variable ω ^{*)}, where

$$s/q^2 = 1 - \omega. \quad (3.1)$$

^{*)} In our definition this is actually the ω' variable of Bloom and Gilman.

In annihilation we have $0 \leq \omega \leq 1$ and in the photon centre-of-mass frame

$$e^+e^- : \quad \omega = \frac{2E}{q} , \quad (3.2a)$$

whereas in electroproduction, $1 \leq \omega \leq \infty$, and in the photon Breit frame,

$$ep : \quad \omega = \frac{2P}{q} . \quad (3.2b)$$

3.1. - Two-Dimensional Scaling Limit

In the SAD model the virtual Compton amplitude is given by Fig. 3 which is

$$\begin{aligned} A_4(s, t, q^2, q^{*2}) &= \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dy \, x_1^{-\delta-1} (1-x_1)^{-\alpha(q^2)-1} \\ &\times x_2^{-\delta-1} (1-x_2)^{-\alpha(q^{*2})-1} y^{-\alpha(s)-1} (1-y)^{-k_1-1} \\ &\times (1-x_1 y)^{-\delta+k_1+\alpha(q^2)} (1-x_2 y)^{-\delta+k_1+\alpha(q^{*2})} \\ &\times (1-x_1 x_2 y)^{-\alpha(t)+2\delta-k_1} . \end{aligned} \quad (3.3)$$

One easily sees that γ controls the asymptotic behaviour of the form factors of the s channel resonances. The parameter δ controls the asymptotic behaviour of the coupling of one current to the t channel resonances and the resonances in the other current.

To obtain the behaviour in the Bjorken limit, we first take $\alpha(s), \alpha(q^2), \alpha(q^{*2}) \rightarrow -\infty$ where (3.3) is defined. The asymptotic behaviour is controlled by the $y \approx 1$ endpoint of integration. Writing $y = 1+z/\alpha_s$ and using the relation $\lim_{\eta \rightarrow \infty} (1-z/\eta)^2 = e^{-z}$ allows the z integral to be trivially done and gives

$$\begin{aligned}
 A_4 \sim & \Gamma(-k_1) \int_0^\infty \int_0^\infty dz_1 dz_2 z_1^{-\gamma-1} (1+z_1)^{\gamma-\delta+\alpha(t)} \\
 & \times z_2^{-\gamma-1} (1+z_2)^{\gamma-\delta+\alpha(t)} (1+z_1+z_2)^{-\alpha(t)+2\delta-k_1} \\
 & \times (-\alpha(s)-\alpha(q^2)z_1 - \alpha(q^{*2})z_2)^{k_1},
 \end{aligned} \tag{3.4}$$

where we have also written $z_i = x_i/1-x_i$. We are interested in the s discontinuity of (3.4). This can be taken easily if $\alpha(s)$ is continued to positive values, keeping $\alpha(q^2)$ and $\alpha(q^{*2})$ negative. The last factor in (3.4) then develops a phase and

$$\begin{aligned}
 \text{disc}_s A_4 \sim & \frac{\pi}{\Gamma(k_1+1)} (\alpha(s))^{k_1} \left(-\frac{\alpha(q^2)}{\alpha(s)}\right)^\delta \left(-\frac{\alpha(q^{*2})}{\alpha(s)}\right)^\delta \\
 & \times \int_0^1 \int_0^1 dy_1 dy_2 \theta(1-y_1-y_2) (y_1 y_2)^{-\gamma-1} (1-y_1-y_2)^{k_1} \\
 & \times \left(1 - \frac{\alpha(s)}{\alpha(q^2)} y_1\right)^{\gamma-\delta+\alpha(t)} \left(1 - \frac{\alpha(s)}{\alpha(q^{*2})} y_2\right)^{\gamma-\delta+\alpha(t)} \\
 & \times \left(1 - \frac{\alpha(s)}{\alpha(q^2)} y_1 - \frac{\alpha(s)}{\alpha(q^{*2})} y_2\right)^{-\alpha(t)+2\delta-k_1},
 \end{aligned} \tag{3.5}$$

where

$$y_1 = -\alpha(q^2)z_1/\alpha(s), \quad y_2 = -\alpha(q^{*2})z_2/\alpha(s)$$

In the electroproduction physical region $s/-q^2 \geq 0$ and no further continuations are necessary. The integral in (3.5) is well defined and thus the expression as it stands gives the cross-section. In e^+e^- annihilation we need to continue to $q^2 > 0$ with $\alpha(q^2) \sim q^2 + i\epsilon$ and $\alpha(q^{*2}) \sim q^2 - i\epsilon$. In this continuation $\alpha(s)/\alpha(q_1^2)$ passes from negative to positive values but remains

smaller than +1. The integrand of (3.5) is regular in the whole region of analytic continuation, and thus the integral is also, with the possible exception of the point $s/q^2 = 1 (\omega=0)$ which will be discussed below. The factor in front of the integral becomes an absolute value. Thus we have for both annihilation and electroproduction,

$$\begin{aligned} \text{disc}_s A_q(s, t, q^2 + i\epsilon, q^2 - i\epsilon) &\sim |q^2|^{k_1} |\omega - 1|^{-2\gamma + k_1} \\ &\times \frac{\pi}{\Gamma(k_1 + 1)} \int_0^1 \int_0^1 dy_1 dy_2 \theta(1 - y_1 - y_2) (y_1 y_2)^{\gamma-1} (1 - y_1 - y_2)^{k_1} \\ &\times (1 + (\omega - 1)y_1)^{\gamma - \delta + \alpha(t)} (1 + (\omega - 1)y_2)^{\gamma - \delta + \alpha(t)} \\ &\times (1 + (\omega - 1)(y_1 + y_2))^{-\alpha(t) + 2\delta - k_1}, \end{aligned} \quad (3.6)$$

which is positive definite as required. Essentially the same form has also been obtained in Ref. 9).

For the special case $k_1 = -1$ [canonical scaling, see Eq. (2.9)] the integral in (3.6) can be simplified since only its residue at $k_1 = -1$ enters. We have ^{5), 20)}

$$\begin{aligned} \text{disc}_s A_q &\sim |q^2|^{-1} |\omega - 1|^{-2\gamma - 1} \omega^{-\alpha(0) + 2\delta + 1} \\ &\times \pi \int_0^1 dv [v(1-v)]^{\gamma-1} [(1 + (\omega - 1)v)(1 + (\omega - 1)(1-v))]^{\gamma - \delta + \alpha(0)}. \end{aligned} \quad (3.7)$$

For this special case only we have the modified Gribov-Lipatov reciprocity relation ²¹⁾ as noted in Ref. 20) :

$$\text{disc}_s A_q(\omega) = -\omega^{2\delta + 1} \text{disc}_s A_q\left(\frac{1}{\omega}\right). \quad (3.8)$$

3.2. - Application to Annihilation and Electroproduction

The two-dimensional scaling behaviour is given by $q \rightarrow \omega$ with ω fixed since according to (3.2) the momentum of the hadron becomes large in this limit.

We note here the behaviour for some special values of the variables. For $\omega \rightarrow 1$ we have

$$\text{disc}_s A_q \underset{\omega \rightarrow 1}{\sim} q^{2k_1} |\omega - 1|^{-2\gamma + k_1} \quad (3.9)$$

and thus the Drell-Yan-West relation holds ^{*)}. In the Regge limit of electroproduction, $\omega \rightarrow \omega$, we have

$$\text{disc}_s A_q \underset{\omega \rightarrow \omega}{\sim} q^{2k_1} \left\{ \begin{array}{l} \omega^{\alpha(0)} \\ \omega^\delta \\ \omega^{k_1} \end{array} \right\}. \quad (3.10)$$

The first term in (3.10) comes from y_1 and y_2 finite in (3.6) and is the usual Regge behaviour. The second term comes from y_1 or $y_2 \sim 1/\omega$ and is a fixed pole of the form (2.4). The third term comes from $y_1 \sim y_2 \sim 1/\omega$ and is the current-algebra-like fixed pole.

The transition to one-dimensional scaling behaviour (E finite) which can occur in annihilation is governed by the $\omega \rightarrow 0$ behaviour of (3.6). One easily sees by examining the integrand that there is a regular contribution and in addition a singular contribution which arises when $y_1 + y_2 \sim 1$. The regular contribution is easily obtained by setting $\omega = 0$ inside the integral

$$\begin{aligned} \text{disc}_s A_q \underset{\omega \rightarrow 0}{\sim} q^{2k_1} \frac{\pi}{\Gamma(k_1+1)} \int_0^1 \int_0^1 dy_1 dy_2 \theta(1-y_1-y_2) \\ \times (y_1 y_2)^{-\gamma-1} (1-y_1)^{\gamma-\delta+d(t)} (1-y_2)^{\gamma-\delta+d(t)} \\ \times (1-y_1-y_2)^{-d(t)+2\delta} \end{aligned} \quad (3.11)$$

^{*)} This is a natural consequence of the duality relation (2.6). It is not a necessary consequence, however, since we could have

$$R_N(q^2) \underset{q^2 \rightarrow \infty}{\sim} (q^2)^{2\gamma_N} \text{ and } \gamma_N \underset{N \rightarrow \infty}{\sim} \tilde{\gamma} \neq \gamma.$$

To obtain the singular contribution we write $y_1 = u - wv$ and $y_2 = (1-u) - wv$

$$\begin{aligned}
 disc_s A_4 \underset{\omega \rightarrow 0}{\sim} q^{2k_1} \frac{\pi}{\Gamma(k_1+1)} 2^{k_1+1} \omega^{2\delta-d(t)+1} \int_0^1 du \int_0^\infty dv \\
 \times [(u-wv)(1-u-wv)]^{-\gamma-1} \\
 \times [(1-u+w(u+v))(u+w(1-u+v))]^{\gamma-\delta+d(t)} \quad (3.12) \\
 \times v^{k_1} (1+2v)^{-d(t)+2\delta-k_1}.
 \end{aligned}$$

There are again two ranges of integration to be distinguished in (3.12). The first is $u, (1-u) \gg w$, where

$$\begin{aligned}
 disc_s A_4 \underset{\omega \rightarrow 0}{\sim} q^{2k_1} \omega^{2\delta-d(t)+1} \frac{\pi}{\Gamma(k_1+1)} \int_0^1 du \int_0^\infty dv \\
 \times [u(1-u)]^{-\delta+d(t)-1} (2v)^{k_1} (1+2v)^{-d(t)+2\delta-k_1} \\
 = q^{2k_1} \omega^{2\delta-d(t)+1} \pi \frac{\Gamma^2(-\delta+d(t)) \Gamma(d(t)-2\delta-1)}{\Gamma(-2\delta+2d(t)) \Gamma(d(t)-2\delta+k_1)}. \quad (3.13)
 \end{aligned}$$

The second range of integration is $u = \epsilon w$ or $(1-u) = \epsilon w$. These two contributions are

$$\begin{aligned}
 disc_s A_4 \underset{\omega \rightarrow 0}{\sim} q^{2k_1} \omega^{\delta+1} \frac{\pi}{\Gamma(k_1+1)} 2^{k_1+2} \int_0^\infty dw \int_0^\infty dv \\
 \times (w-v)^{-\gamma-1} (1+w+v)^{\gamma-\delta+d(t)} \\
 \times v^{k_1} (1+2v)^{-d(t)+2\delta-k_1} \quad (3.14) \\
 = q^{2k_1} \omega^{\delta+1} 2\pi \frac{\Gamma(\delta-d(t)) \Gamma(-\gamma) \Gamma(-\delta-1)}{\Gamma(\delta-\gamma-d(t)) \Gamma(k_1-\delta)}.
 \end{aligned}$$

The same results can be obtained using only well-known properties of the hypergeometric function. Thus we have ^{*)}

$$\text{disc}_s A_q \underset{\omega \rightarrow 0}{\sim} q^{2k_1} \left\{ \begin{matrix} \omega^{2\delta-d(t)+1} \\ \omega^{\delta+1} \\ \omega^0 \end{matrix} \right\}. \quad (3.15)$$

For falling form factors the first term in (3.15) dominates ^{**)} and we have a change in scaling behaviour in passing from the two-dimensional configuration to the one-dimensional configuration. Thus

$$\text{disc}_s A_q \sim q^{2k_1} f_2(\omega), \quad \omega \text{ fixed},$$

(3.16)

but

$$\text{disc}_s A_q \sim q^{2k_1-2\delta+d(0)-1} f_1(E), \quad E \text{ fixed}.$$

It should be remarked that the function $f_1(E)$ in (3.16) is rather trivial being just a simple power :

$$f_1(E) = c_1 E^{2\delta-d(0)+1}. \quad (3.17)$$

However, in general the $\omega \rightarrow 0$ behaviour of $f_2(\omega)$ gives only the form of $f_1(E)$ for large E . For example the exact $f_1(E)$ might be

$$\begin{aligned} f_1(E) &= (E + E_0)^{d_2-d_1} \\ &= E^{(d_2-d_1)} \left(1 - (d_2-d_1) \frac{E_0}{E} + \dots \right). \end{aligned} \quad (3.18)$$

^{*)} For $k_1 = -1$, the last terms in (3.10) and (3.15) are absent and (3.8) connects the $\omega \rightarrow 0$ and $\omega \rightarrow \infty$ behaviours. We also note the fact that we have taken a discontinuity is essential in giving the $\omega \rightarrow 0$ behaviours.

^{**)} The necessary condition for its dominance is $\delta < \alpha(0)$ and $\delta < \frac{1}{2}(\alpha(0)-1)$.

This would correspond to contributions at fixed ω like

$$\omega^{2(d_1 - d_2)} \left(1 - (d_2 - d_1) \frac{2E_0}{q\omega} + \dots \right) \quad (3.19)$$

These are non-leading scaling contributions for fixed ω with an even greater singularity at $\omega = 0$. We have checked the non-leading scaling terms in the model and find they have no greater singularity at $\omega = 0$ thus (3.17) holds for all E in the SAD model. We should note that a one-dimensional scaling function which is not a simple power like (3.17) appears to be indicated by experiment. There is a large violation of two-dimensional scaling for small ω - the data rise as q increases as would be given by (3.19) for example.

4. - THE SIX-POINT FUNCTION

In this section we study the behaviour of the amplitude for two currents and four hadrons of Fig. 1b. The discontinuity in M^2 gives the two-particle inclusive annihilation cross-section and the single-particle inclusive electroproduction cross-section. We calculate this discontinuity in the limit $q \rightarrow \infty$ with s_1/q^2 , s_2/q^2 and M^2/q^2 fixed where with all momenta incoming $s_1 = (p_1 + q)^2$, $s_2 = (p_2 + q)^2$, $M^2 = (q + p_1 + p_2)^2$. As usual we define

$$s_1/q^2 = 1 - \omega_1, \quad s_2/q^2 = 1 - \omega_2. \quad (4.1)$$

It is convenient to use the proper frame of the photon for both annihilation and electroproduction : e.g., the centre-of-mass frame for the former and the Breit frame for the latter. We shall in both cases take the z direction as the direction of particle 1 which is outgoing for annihilation and incoming for electroproduction. With these choices of directions we have from (4.1) :

$$e^+e^- : \quad \omega_1 = \frac{2E_1}{q}, \quad \omega_2 = \frac{2E_2}{q} \quad (4.2a)$$

$$ep : \quad \omega_1 = \frac{2p_1}{q}, \quad \omega_2 = -\frac{2p_2 \cos \theta}{q}. \quad (4.2b)$$

Note in particular that ω_2 can be negative in electroproduction. In the scaling limit $q = \sqrt{|q|^2} \rightarrow \infty$, ω_i fixed ($p_i \gg m_i$):

$$\begin{aligned}
 e^+e^- : \quad s_{12} &\approx \frac{1}{2}q^2 \omega_1 \omega_2 (1 - \cos \theta) + 2m^2 \\
 M^2/q^2 &\approx 1 - \omega_1 - \omega_2 + \frac{1}{2}\omega_1 \omega_2 (1 - \cos \theta) + \frac{2m^2}{q^2} \quad (4.3a) \\
 &\approx (1 - \omega_1)(1 - \omega_2) - \frac{1}{2}\omega_1 \omega_2 (1 + \cos \theta) + \frac{2m^2}{q^2} ,
 \end{aligned}$$

$$\begin{aligned}
 ep : \quad s_{12} &\approx -\frac{1}{2}q^2 \omega_1 \omega_2 \frac{(1 - \cos \theta)}{\cos \theta} + 2m^2 \\
 M^2/q^2 &\approx 1 - \omega_1 - \omega_2 - \frac{1}{2}\omega_1 \omega_2 \frac{(1 - \cos \theta)}{\cos \theta} \quad (4.3b) \\
 &\approx (1 - \omega_1)(1 - \omega_2) - \frac{1}{2}\omega_1 \omega_2 \frac{(1 + \cos \theta)}{\cos \theta} .
 \end{aligned}$$

For finite θ , ω_1 and ω_2 , three components of p_2 are of order q so that this is a three-dimensional scaling limit for either annihilation or electroproduction. For the two-dimensional case with ω_1 and ω_2 finite we need either $\theta = 0(1/q)$ or $\theta = \pi - 0(1/q)$. In this case it is more convenient to use the variable $m_\perp^2 = m^2 + p_\perp^2$ where $p_\perp = p_2 \sin \theta$ for both reactions. We then have for $\theta \approx 0(1/q)$

$$m_\perp^2 = \frac{\omega_2}{\omega_1} (s_{12} - 2m^2) \quad \text{or} \quad s_{12} = \frac{\omega_1}{\omega_2} m_\perp^2 + 2m^2 , \quad (4.4a)$$

and for $\theta = \pi - 0(1/q)$

$$m_\perp^2 \approx q^2 \frac{\omega_2}{\omega_1} (1 - \omega_1)(1 - \omega_2) \left(1 - \frac{M^2 q^2}{s_1 s_2}\right) \quad \text{and} \quad s_{12} \approx q^2 \omega_1 \omega_2 . \quad (4.4b)$$

These results follow directly from Eqs. (4.1), (4.3) and the relations ($E \gg m$)

$$1 - \cos \theta = \frac{1 - \cos^2 \theta}{1 + \cos \theta} \approx \frac{m_\perp^2}{2E_1^2} \approx \frac{2m_\perp^2}{\omega_2^2/q^2} , \quad \text{for } \theta = 0\left(\frac{1}{q}\right) , \quad (4.5a)$$

$$1 + \cos \theta = \frac{1 - \cos^2 \theta}{1 - \cos \theta} \approx \frac{m_\perp^2}{2E_\perp^2} \approx \frac{2m_\perp^2}{\omega_2^2/q^2}, \text{ for } \theta = \pi - O(\frac{1}{q}), \quad (4.5b)$$

and hold both for annihilation and for electroproduction.

It will also be useful to have the forms of the important kinematic variables in the limit when either one or both of the energies E_1 and E_2 are held fixed as q becomes large. In the limit $q \rightarrow \infty$ ω_1 and E_2 fixed ($\omega_2 \rightarrow 0$) we have the following relations

$$\begin{aligned} e^+e^- : \quad s_{12} &\approx \omega_1 q (E_2 - p_2 \cos \theta) + 2m^2 \\ M^2/q^2 &\approx 1 - \omega_1 + O(\frac{1}{q}) \end{aligned} \quad (4.6a)$$

$$\begin{aligned} ep : \quad s_{12} &\approx -\omega_1 q (E_2 - p_2 \cos \theta) \\ M^2/q^2 &\approx 1 - \omega_1 + O(\frac{1}{q}), \end{aligned} \quad (4.6b)$$

where $E_2 - p_2 \cos \theta$ is a fixed quantity independent of q^2 for any θ . The case E_1 and ω_2 fixed is given for annihilation by Eq. (4.6a) with the interchange $1 \rightleftharpoons 2$ and is excluded for electroproduction by the kinematic constraint $\omega_1 \geq 1$. For annihilation, we have finally the case E_1 and E_2 fixed as $q \rightarrow \infty$ in which

$$\begin{aligned} s_{12} &= 2(E_1 E_2 - p_1 p_2 \cos \theta) + 2m^2 \\ M^2/q^2 &\approx 1 - \frac{2(E_1 + E_2)}{q} + O(\frac{1}{q^2}). \end{aligned} \quad (4.7)$$

Hence in this limit s_1 , s_2 and M^2 all approach $|q^2|$ whereas s_{12} remains fixed.

There are several dual diagrams corresponding to various orderings of the external lines. We study the two shown in Fig. 4. We have not studied the diagrams with asymmetric location of the currents (e.g., the top half of Fig. 4a combined with bottom half of Fig. 4b) but we believe they will vanish exponentially in the limit of interest. For a related discussion of the relationship between the diagrams with different orderings of the external lines in purely hadronic processes, we refer the reader to Ref. 7).

4.1. - The Adjacent Current Diagram

The structure of the diagram in Fig. 4a is very similar to that of Fig. 3. For simplicity we give here the expressions for the simplified case where the "trajectories" for all "channels" containing a single fictitious parton are set equal to γ . We also immediately go to the forward configuration appropriate to inclusive cross-sections. Then

$$\begin{aligned}
 & A_6^{(A)}(M^2; s_1, s_2, q^2; s_1^*, s_2^*, q^{*2}) \\
 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 du_1 dx_1 dy dx_2 du_2 u_1^{-\alpha(s_{12})-1} x_1^{-\delta-1} y^{-\alpha(M^2)-1} \\
 &\quad \times x_2^{-\delta-1} u_2^{-\alpha(s_{12}^*)-1} (1-u_1)^{-\delta-1} (1-x_1)^{-\alpha(q^2)-1} (1-y)^{-k_2-1} \\
 &\quad \times (1-x_2)^{-\alpha(q^{*2})-1} (1-u_2)^{-\delta-1} (1-u_1 x_1)^{-\alpha(s_2)+\alpha(q^2)+\delta} \\
 &\quad \times (1-x_1 y)^{-\delta+\alpha(q^2)+k_2} (1-y x_2)^{-\delta+\alpha(q^{*2})+k_2} \\
 &\quad \times (1-x_2 u_2)^{-\alpha(s_2^*)+\alpha(q^{*2})+\delta} (1-u_1 x_1 y)^{-\alpha(q^2)+\alpha(s_2)} \\
 &\quad \times (1-x_1 y x_2)^{-\alpha(0)+2\gamma-k_2} (1-y x_2 u_2)^{-\alpha(q^{*2})+\alpha(s_2^*)} \quad (4.8) \\
 &\quad \times (1-u_1 x_1 y x_2)^{-\alpha(M^2)+\alpha(0)} (1-x_1 y x_2 u_2)^{-\alpha(M^2)+\alpha(0)} \\
 &\quad \times (1-u_1 x_1 y x_2 u_2)^{-2\alpha(0)+2\alpha(M^2)}
 \end{aligned}$$

where $\alpha(m^2)$ is the spin of the external particle.

We follow the same sequence of steps as in Section 3.1. Again the asymptotic behaviour is given by $y \approx 1$. The discontinuity in M^2 can be taken as before. Finally after several changes of variables, we obtain in the limit $q^2 \rightarrow \infty$ ($s_2/q^2, s_1/q^2, M^2/q^2, s_{12}$ fixed) ^{4),22)}

$$\begin{aligned}
 \text{disc}_{M^2} A_6^{(A)} &\sim \frac{\pi}{\Gamma(k_2+1)} (\alpha(M^2))^{k_2} \left(-\frac{\alpha(q^2)}{\alpha(M^2)}\right)^{\delta} \left(-\frac{\alpha(q^{*2})}{\alpha(M^2)}\right)^{\delta} \\
 &\times \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 dy_2 dv_1 dv_2 \theta(1-y_1-y_2-v_1-v_2) (y_1 y_2)^{-\delta-1} (1-y_1-y_2-v_1-v_2)^{k_2} \\
 &\times v_1^{-\alpha(s_2)-1} v_2^{-\alpha(s_{12}^*)-1} \left(v_1 + \frac{\alpha(s_2)}{\alpha(q^2)} y_1\right)^{\alpha(s_{12})} \left(v_2 + \frac{\alpha(s_2^*)}{\alpha(q^{*2})} y_2\right)^{\alpha(s_{12}^*)} \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1\right)^{\alpha(s_{12})+\alpha(0)-\alpha(M^2)} \left(1 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{\alpha(s_{12}^*)+\alpha(0)-\alpha(M^2)} \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(q^2)} y_1 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1\right)^{\alpha(M^2)-\alpha(s_{12})} \left(1 - \frac{\alpha(M^2)}{\alpha(q^2)} y_2 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{\alpha(M^2)-\alpha(s_{12}^*)} \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{-2\alpha(0)+2\alpha(M^2)} \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(q^2)} y_1 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{-\alpha(M^2)+\alpha(0)} \quad (4.9) \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(q^{*2})} y_2 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{-\alpha(M^2)+\alpha(0)} \\
 &\times \left(1 - \frac{\alpha(M^2)}{\alpha(q^2)} y_1 - \frac{\alpha(M^2)}{\alpha(q^{*2})} y_2 - \frac{\alpha(M^2)}{\alpha(s_2)} v_1 - \frac{\alpha(M^2)}{\alpha(s_2^*)} v_2\right)^{-\alpha(0)+2\delta-k_2}.
 \end{aligned}$$

This provides the desired analytic continuation since in the physical regions of interest the integrand, like that of Eq. (3.5), is analytic. We can check the consistency of Eq. (4.9) with Eq. (3.5) for $y=\delta$ and $s \rightarrow M^2$ by the residue of the poles at $\alpha(s_{12})=\alpha(s_{12}^*)=0$ which arise for v_1 and $v_2 \approx 0$. This weak factorization requirement also requires $k_2=k_1$.

We first discuss the three-dimensional behaviour of Eq. (4.9), i.e., when ω_1, ω_2 are finite and $\theta \neq 0$ so that $s_{12} \propto q^2$. Since the amplitude depends only on the ratios $s_2/q^2, M^2/s_2$ and M^2/q^2 , which are finite in this limit [Eqs. (4.1) and (4.3)], and s_{12} , we need consider

only the s_{12} dependence ^{*}). For the limit $s_{12} \rightarrow -\infty$ the region of integration y_1 and y_2 of order $1/s_{12}$ dominates in Eq. (4.9) and the integral behaves as

$$\iiint \sim |s_{12}|^{2\gamma}. \quad (4.10)$$

In this limit this really implies a behaviour $|q^2|^{2\gamma}$ since $|s_{12}| \propto |q^2|$ and could be written as a three-dimensional scaling law

$$disc_{m^2} A_6^{(A)} \sim q^{2k_2+4\gamma} f_3(\omega_1, \omega_2, \cos \theta). \quad (4.11)$$

It should be noted, however, that this is somewhat misleading since the cut-off in s_{12} actually has a scale set by α' . This is different from the other scaling laws presented in this paper where the scale is set by q . Indeed even for fixed small q^2 in electroproduction the amplitude is again cut-off in a power behaved fashion when $m_{\perp}^2$, and hence s_{12} , becomes large. This is in contrast to the dual model of purely hadronic processes where the $m_{\perp}^2$ cut-off is exponential ^{**}), see, for example, Ref. 7). However, the usual exponential $m_{\perp}^2$ behaviour is recovered in electroproduction in the Regge limit $\omega_1 \rightarrow \infty$, $\omega_2 \rightarrow -\infty$ for particles in the central region or target fragmentation region. As we shall see below, the contribution in Eq. (4.11) to three-dimensional scaling is not the dominant one and henceforth the contribution of $A_6^{(A)}$ will be neglected in any region of phase space where $|s_{12}|$ is large.

Turning now to a discussion of the two-dimensional behaviour we consider first the limit when ω_1 , ω_2 and $m_{\perp}^2$ are all fixed as $q \rightarrow \infty$. Note that only the case where $\vec{p}_2$ is in the same direction as $\vec{p}_1$ is included, since for $\theta \approx \pi$, $|s_{12}|$ is again large even though only two dimensions are excited. Rewriting Eq. (4.9), we have

^{*}) Since $\alpha(s_{12}) > 0$ for annihilation some continuation is necessary to define the integrals. However, we do not expect this to change the behaviour discussed and we ignore this complication here.

^{**}) This suggests a modified hadronic model with power behaved $m_{\perp}^2$ cut-off constructed by inserting two "fictitious quark" lines for each hadron as is done here for the currents. The power of the $m_{\perp}^2$ fall-off will be correlated to position of fixed poles (which should be adjusted to be nonsense wrong signature) which are analogous to those in (2.4).

$$\begin{aligned}
 disc_{M^2} A_6^{(A)} &\sim |q^2|^{k_2} |1-\omega_1-\omega_2|^{-2\gamma+k_2} \frac{\pi}{\Gamma(k_2+1)} \\
 &\times \int_0^1 \int_0^1 \int_0^1 \int_0^1 dy_1 dy_2 dv_1 dv_2 \theta(1-y_1-y_2-v_1-v_2) (\gamma_1 \gamma_2)^{-\delta-1} (1-y_1-y_2-v_1-v_2)^{k_2} \\
 &\times v_1^{-d(s_{12})-1} v_2^{-d(s_{12}^*)-1} (v_1+(1-\omega_2)\gamma_1)^{d(s_{12})} (v_2+(1-\omega_2)\gamma_2)^{d(s_{12}^*)} \\
 &\times \left(1 - \frac{(1-\omega_1-\omega_2)\gamma_1}{(1-\omega_2)}\right)^{d(s_{12})+d(10)-d(u^2)} \left(1 - \frac{(1-\omega_1-\omega_2)\gamma_2}{(1-\omega_2)}\right)^{d(s_{12}^*)+d(10)-d(u^2)} \\
 &\times \left(1 - (1-\omega_1-\omega_2)(\gamma_1 + \frac{1}{(1-\omega_2)}v_1)\right)^{d(u^2)-d(s_{12})} \left(1 - (1-\omega_1-\omega_2)(\gamma_2 + \frac{1}{(1-\omega_2)}v_2)\right)^{d(u^2)-d(s_{12}^*)} \\
 &\times \left(1 - \frac{(1-\omega_1-\omega_2)}{(1-\omega_2)}(v_1+v_2)\right)^{-2d(10)+2d(u^2)} \\
 &\times \left(1 - (1-\omega_1-\omega_2)(\gamma_1 + \frac{1}{(1-\omega_2)}(v_1+v_2))\right)^{-d(u^2)+d(10)} \\
 &\times \left(1 - (1-\omega_1-\omega_2)(\gamma_2 + \frac{1}{(1-\omega_2)}(v_1+v_2))\right)^{-d(u^2)+d(10)} \quad (4.12) \\
 &\times \left(1 - (1-\omega_1-\omega_2)(\gamma_1+\gamma_2 + \frac{1}{(1-\omega_2)}(v_1+v_2))\right)^{-d(10)+2\gamma-k_2}
 \end{aligned}$$

where s_{12} is given by Eq. (4.4a).

Since $\vec{p}_1$ and $\vec{p}_2$ are in the same direction for the annihilation process, energy conservation requires that $\omega_1 + \omega_2 \leq 1$ and the integral (4.12) is well defined. For the electroproduction case recall that in the photon Breit frame we have the four-vectors $p_1^\mu = (E_1, 0, 0, p_1)$ and $q^\mu = (0, 0, 0, -q)$. Momentum conservation requires $p_2 \cos \theta \leq p_1 - q$ which, from Eq. (4.2b) gives $\omega_1 + \omega_2 \geq 2$. Since $\omega_2 \leq 0$ for $\theta \approx 0$, we have both $1 - \omega_1 - \omega_2 \leq -1$ and $(1 - \omega_1 - \omega_2)/(1 - \omega_2) \leq 0$. Thus the integral in Eq. (4.12) is well defined for both cases. We therefore have a two-dimensional scaling law of the form

$$disc_{M^2} A_6^{(A)} \sim q^{2k_2} f_2(\omega_1, \omega_2, p_\perp). \quad (4.13)$$

Recall from Eq. (4.6) that for E_2 finite, ω_1 fixed $|s_{12}| \propto q$ for any θ . Thus, in the limit E_2 fixed ($\omega_2 \rightarrow 0$) the large $|s_{12}|$ behaviour is the dominant feature in Eq. (4.12) and the contribution of this

diagram is suppressed. Hence it contributes only for $E_2 \gtrsim 0(q)$ ($\omega_2 < 0$) and $\theta \approx 0$. The general scaling behaviour is the same as that observed in light cone analysis and the parton model ²²⁾. It is instructive to consider the Mueller limit $\omega_1 \rightarrow \infty$ for electroproduction in which appropriate Regge exchanges give the target and current fragmentation regions and the hadronic central region ²³⁾. Factorization of the Regge residues gives independence of the target (current) fragmentation on the current (target). Hence this diagram contributes in the target fragmentation, hadronic central, and hole fragmentation part of the current fragmentation region - but not in the rest of the current fragmentation region where $p_2 \cos \theta \lesssim 0(q)$. This is illustrated in Fig. 5 ⁴⁾. This diagram thus reproduces the features of the naïve parton model in these regions.

A transition to one-dimensional scaling takes place as ω_1 and ω_2 tend to zero. The dominant term (for $\gamma < 0$) arises from the region of integration $y_1 + y_2 + y_1 + y_2 \approx 1$. The mechanism is very similar to that of the four-point function and the change in scaling is just the same ^{*)}

$$disc_{M^2} A_6^{(A)} \sim q^{2k_2 - 2\gamma + d(10) - 1} f_1(E_1, E_2, P_1). \quad (4.14)$$

If only one of the $\omega_i \rightarrow 0$ there is no enhancement of this form since $1 - \omega_1 - \omega_2$ does not approach 1.

4.2. - The Non-Adjacent Current Diagram

We now discuss the diagram of Fig. 4b. Again, for the simplified case discussed above, we have

^{*)} There are also other singularities like those of Eq. (3.15) as $\omega_i \rightarrow 0$. We give here only the most singular one for γ negative.

$$\begin{aligned}
 A_6^{(NA)}(M^2; s_1, s_2, q^2; s_1^*, s_2^*, q^{*2}) \\
 = \int \int \int \int \int_{00000} du, dv, dw, dv_2, du_2 u_1^{-\gamma-1} v_1^{-\alpha(s_1)-1} w^{-\alpha(M^2)-1} \\
 \times v_2^{-\alpha(s_1^*)-1} u_2^{-\gamma-1} (1-u_1)^{-\alpha(q^2)-1} (1-v_1)^{-\gamma-1} (1-w)^{-\alpha(10)-1} \\
 \times (1-v_2)^{-\gamma-1} (1-u_2)^{-\alpha(q^{*2})-1} (1-u_1 v_1)^{-\alpha(s_2)+\alpha(q^2)+\gamma} \\
 \times (1-v_1 w)^{\alpha(10)} (1-w v_2)^{\alpha(10)} (1-v_2 u_2)^{-\alpha(s_2^*)+\alpha(q^{*2})+\gamma} \\
 \times (1-u_1 v_1 w)^{-\alpha(q^2)+\alpha(s_2)} (1-v_1 w v_2)^{-k_3+2\gamma-\alpha(10)} \\
 \times (1-w v_2 u_2)^{-\alpha(q^{*2})+\alpha(s_2^*)} (1-u_1 v_1 w v_2)^{-2\gamma+\alpha(q^2)+k_3} \\
 \times (1-v_1 w v_2 u_2)^{-2\gamma+\alpha(q^{*2})+k_3} \\
 \times (1-u_1 v_1 w v_2 u_2)^{-\alpha(10)+2\gamma-k_3}
 \end{aligned} \tag{4.15}$$

The asymptotic behaviour is given by v_1, w and $v_2 \approx 1$. After doing the usual exponentiation trick, one integral can be done and the discontinuity taken much as in the previous cases. After some changes of variables we have

$$\begin{aligned}
 \text{disc}_{M^2} A_6^{(NA)} \sim |q^2|^{k_3} \frac{\pi}{\Gamma(k_3+1)} \left[|w_1-1|^{-2\gamma+k_3} \right. \\
 \times \int_0^1 \int_0^1 dy_1 dy_2 (y_1 y_2)^{-\gamma-1} (1+(w_1-1)y_1)^{\alpha(10)} (1+(w_1-1)y_2)^{\alpha(10)} \\
 \times (1+(w_1-1)(y_1+y_2))^{-\alpha(10)+2\gamma-k_3} \left. \right] \\
 \times \left[y_1, y_2 \rightarrow x_1, x_2; w_1 \rightarrow w_2 \right] \\
 \times \left\{ \frac{M^2 q^2}{s_1 s_2} - (y_1+y_2) - (x_1+x_2) + (y_1+y_2)(x_1+x_2) \right\}^{k_3} \\
 \times \theta \{ \dots \}
 \end{aligned} \tag{4.16}$$

The range of integration is shown in Fig. 6. In writing (4.16) we have given the form only for the annihilation and electroproduction physical regions. We remark that Eq. (4.16) looks rather like the product of two amplitudes of the form of Eq. (3.6).

The form (4.16) exhibits a scaling behaviour for fixed values of ω_1 , ω_2 and $M^2 q^2 / s_1 s_2$. This final combination of variables is given in the limit of fixed ω_1 , ω_2 and θ by

$$e^+e^- : \frac{M^2 q^2}{s_1 s_2} = 1 - \frac{1}{2} \frac{\omega_1 \omega_2}{(1-\omega_1)(1-\omega_2)} (1+\cos\theta) \quad (4.17a)$$

$$ep : \frac{M^2 q^2}{s_1 s_2} = 1 - \frac{1}{2} \frac{\omega_1 \omega_2}{(1-\omega_1)(1-\omega_2)} \frac{(1+\cos\theta)}{\cos\theta} . \quad (4.17b)$$

Hence for finite values of ω_1 , ω_2 this variable is confined to the range 0 to 1 and approaches 1 only when one of ω_1 , ω_2 or $(1+\cos\theta)$ vanishes ^{*}). Thus in general, (4.16) yields a three-dimensional scaling behaviour

$$\text{disc}_{M^2} A_6^{(NA)} \sim q^{2k_3} f_3'(w_1, w_2, \cos\theta) . \quad (4.18)$$

Note that this corresponds in general to an average $p_\perp$ of order $q^{1/4}$.

There is no change of scaling behaviour in passing to the two-dimensional regions $\theta = 0(1/q)$ or $\theta = \pi - 0(1/q)$ since the factors $(1-\omega_1)$ and $(1-\omega_2)$ remain far from unity even though $(M^2 q^2 / s_1 s_2) \rightarrow 1$. However, the two-dimensional regimes corresponding to either ω_1 or ω_2 of order $1/q$ are enhanced relative to the three-dimensional behaviour (4.18). For example, as $\omega_2 \rightarrow 0$ we see from Eq. (4.17) that $(M^2 q^2 / s_1 s_2) = 1 - 0(1/q)$ and from the region of integration $x_1 + x_2 = 1 + 0(1/q)$ we obtain the enhanced behaviour ^{**)}

^{*}) There is no problem in (4.17b) for $\theta \rightarrow \pi/2$ since $\omega_2 \propto \cos\theta$.

^{**)} There are again other singularities like those of Eq. (3.15). We give only the most singular term for γ negative.

$$\text{disc}_{m^2} A_6^{(NA)} \sim q^{2k_3 - 2\gamma + \alpha(0) - 1} f_2'(\omega_1, E_2, p_\perp), \quad (4.19)$$

much like that in Eq. (3.16).

There is a further enhancement as we pass to the one-dimensional regime $\omega_1 \rightarrow 0$ coming from the range of integration $y_1 + y_2 = 1 - \Theta(1/q)$ ^{*)} :

$$\text{disc}_{m^2} A_6^{(NA)} \sim q^{2k_3 + 2(-2\gamma + \alpha(0) - 1)} f_1'(E, E_2, p_\perp). \quad (4.20)$$

In summary we see that the non-adjacent current diagram yields a contribution to each of the three scaling regimes for annihilation, i.e., three-dimensional [Eq. (4.18)], two-dimensional [Eq. (4.19)] and one-dimensional [Eq. (4.20)]. The change in the q^2 power at each step is the same and the same as that observed in the other diagrams. For electroproduction only the two-dimensional and three-dimensional contributions are of interest. The two-dimensional term of Eq. (4.19) leads to particles with finite momentum in the photon Breit frame. These particles populate a finite region around the centre of the Breit frame which is $\ln q$ from the edge of the longitudinal rapidity plot. This situation is illustrated in Fig. 5. To populate the region of finite ω_2 and $\Theta \approx \pi$ (the parton fragmentation region in parton models) which lies at the edge of the rapidity plot, one must turn to the three-dimensional contribution which brings with it contributions to the region of phase space where $m_\perp^2$ is of order $|q^2|^{-4}$.

We close this section by making a small technical point. If one maintains a distinction between the two parameters γ and δ introduced in Sections 2 and 3, one finds that $\Delta \equiv d_2 - d_1 = d_3 - d_2 = -2\delta + \alpha(0) - 1$ in all cases. Recall that δ is related to the asymptotic behaviour of the coupling of a current, a t channel pole, and a pole in the other current channel whereas γ describes the behaviour of the coupling of a current, an external particle and a s channel pole. This might be important if one wanted, for phenomenological reasons, to have different values for γ and δ .

*) See previous footnote.

5. - DISCUSSION

5.1. - Physical Picture

We discuss first the physical picture for e^+e^- annihilation which emerges from the preceding calculations. The picture depends of course on the value of the parameter $\Delta \equiv d_2 - d_1$ which controls the $\omega \rightarrow 0$ singularities of the various structure functions. As we have seen the SAD model gives $\Delta = -2\delta + \alpha(0) - 1$. Introducing Eq. (1.2) into the sum rules Eqs. (1.5) and (1.8) we find

$$\sigma \sim \begin{cases} q^{2-d_2} & , \Delta < 3 \\ q^{2-d_2} \log q & , \Delta = 3 \\ q^{\Delta-d_2-1} = q^{-d_1-1} & , \Delta > 3 \end{cases} ; \langle n \rangle \sim \begin{cases} \text{const} & , \Delta < 2 \\ \log q & , \Delta = 2 \\ q^{\Delta-2} & , 2 < \Delta < 3 \\ q / \log q & , \Delta = 3 \\ q & , \Delta > 3 \end{cases} . \quad (5.1)$$

We have discussed in the introduction why the behaviour of σ changes at $\Delta = 3$ but we will review it again briefly. For any $\Delta > 0$ there is an enhancement for low energy secondaries. However, the relative phase space weight for the contribution of the low energy secondaries in the sum rule for σ is smaller by a factor of q^{-3} [see Eqs. (1.6), (1.7)] compared to the particles with $E \sim q$. Hence the events with only low energy particles in the final state make a dominant contribution to σ only for $\Delta > 3$.

As an aside we note that in the SAD model the two-point function for two currents (see Fig. 1c), which gives $\sigma_{e^+e^-}$ directly, is given by a B_4 function (see Fig. 2b). This function contains a parameter k_0 which plays the same rôle as the k_1 in the four-point function (Fig. 1a) but the two parameters need not be identified. Indeed, if we use Eq. (1.5), we find that

$$2k_0 = \begin{cases} 2k_1 + 2 & , \Delta < 3 \\ 2k_1 + \Delta - 1 & , \Delta > 3 \end{cases} . \quad (5.2)$$

Thus the freedom exists to make the two diagrams compatible. If σ is determined by slow secondaries ($\Delta \geq 3$) we can solve a difficulty that has always appeared in dual model calculations at this stage [see, e.g., Refs. 15), 17)]. The problem arises when one asks that the two-point function, calculated at the tree diagram level, scale canonically, i.e., as $1/q^2$. In dual expressions, real and imaginary parts of amplitudes are governed by the same power and for integer powers the ratio of the imaginary to the real parts of such an expression is expected to vanish. Hence one cannot have both a non-vanishing imaginary part (σ) and a finite real part. This is not the case for the inclusive distribution for which the discontinuity of the amplitude is taken in a variable (s) other than q^2 . The possibility of having a non-canonical scaling behaviour for the total e^+e^- cross-section allows one to have consistency of tree diagram dominance for the total cross-section and the inclusive distributions with the sum rules.

In order to have an idea of what final state configurations of hadrons are the dominant ones for the annihilation process, we turn to the two-particle inclusive cross-section. We first summarize the results of Section 4. For the adjacent diagram of Fig. 4a, all parameters are the same as those occurring for the single-particle calculation of Section 3 and, in particular, $k_2 = k_1$. Recall that this is a minimal factorization requirement since the four-point diagram of Fig. 3 can be obtained as a double pole in the six-point diagram of Fig. 4a. There is no such constraint for the diagram of Fig. 4b where the currents are not adjacent, since there is no way to obtain the four-point diagram as a singularity in the diagram. Therefore, the parameter k_3 in Eqs. (4.15) and those following is in principle unrelated to k_1 and we can attempt to fix it by using the sum rule

$$\frac{d^3\sigma}{d^3p_1/E_1} = \frac{1}{q-E_1} \int \frac{d^3p_2}{E_2} E_2 \frac{d^6\sigma}{d^3p_1/E_1 d^3p_2/E_2} . \quad (5.3)$$

This procedure is analogous to what was done to find the relationship between k_0 and k_1 in Eq. (5.2). Again the freedom to fix k_1 and k_3 separately will be crucial to having a self-consistent picture.

To understand physically the results, it is convenient to analyze separately the various regions of phase space in the e^+e^- centre-of-mass frame whose weights are different in the integral of Eq. (5.3).

There is first a three-dimensional region, illustrated in Fig. 7a (the quantities in parentheses are those held fixed as $q \rightarrow \infty$), where both p_1 and p_2 are of order q (ω_1 and ω_2 fixed) and θ is arbitrary. The three dimensions are spanned by large energy and large momenta in two directions. This three-dimensional configuration becomes two-dimensional when the momenta become collinear, Figs. 7b and 7c, or one of the w_i becomes small (E fixed) as in Fig. 7d. Finally, there is the one-dimensional configuration where both E_1 and E_2 are fixed as $q \rightarrow \infty$ as in Fig. 7e. To understand the relevance of each configuration in the sum rule of Eq. (5.3), we can repeat the simple argument used in Section 1 for the single particle sum rule to find the relative phase space weights of the various configurations. These factors are displayed as in Fig. 7 along with the results on the various q dependences as calculated in Section 4 for $\Delta > 0$. We have used the notation $d_3 \equiv -2k_3$ for the three-dimensional contribution of the non-adjacent current diagram (Fig. 4b) and $d_2 = -2k_2$ for the adjacent current diagram (Fig. 4a). Of course, $d_2 - d_1 = \Delta = -2\delta + \alpha(0) - 1$ as usual.

Hence the contribution of each configuration to the sum rule (5.3) is found simply by multiplying the weight times the given behaviour. By comparing the result with the expected behaviour for $E_1(d^3\sigma)/(d^3p_1)$, i.e., q^{-d_2} for fixed ω_1 , and q^{-d_1} for fixed E_1 , one readily sees that a consistent picture arises if we set

$$d_3 = \begin{cases} d_2 + 2 & , \quad \Delta < 3 \\ d_2 + \Delta - 1 & , \quad \Delta > 3 \end{cases} \quad (5.4a)$$

or, equivalently,

$$k_3 = \begin{cases} k_2 - 1 & , \quad \Delta < 3 \\ k_2 - \frac{\Delta - 1}{2} & , \quad \Delta > 3 \end{cases} \quad (5.4b)$$

Having fixed the relationships between all the parameters, we may proceed to discuss the structure of e^+e^- annihilation events. We take the cases $\Delta < 3$ and $\Delta > 3$ separately. We have already seen that for $\Delta < 3$ the total cross-section is dominated by the contribution from high energy secondaries, i.e., by the contribution of the two-dimensional scaling regime.

We can see simply by inspecting Fig. 7 that, for $\Delta < 3$ the dominant contribution to the single particle distribution for large p_1 (ω_1 fixed) arises when the other secondaries also have large momentum, i.e., for finite ω_2 . More specifically the dominant contributions come from column (b) for the adjacent current diagram and column (a) for the non-adjacent current diagram. Thus the three-dimensional regime with large p_1 contributes equally with the two-dimensional regime with small p_1 in order to produce a one-particle distribution with two-dimensional scaling⁴⁾. In summary, we find for the case $\Delta < 3$ that the total cross-section exhibits canonical scaling arising from a dominance of two-dimensional scaling behaviour in the one-particle distribution in agreement with conventional ideas (parton models, for example). In contrast to the conventional notions, however, the single-particle distribution is composed of comparable contributions from both the two-dimensional and the three-dimensional regimes of the two-particle distribution. Unlike the usual picture the events are basically spherical in nature with $\langle p_1^2 \rangle \sim q^2$.

The reason this behaviour must arise in the SAD model is easily understood by considering momentum conservation. As noted earlier the fact that the model is resonance dominated results in a dominant two-dimensional configuration for the adjacent current diagram only when the two particles are in the same direction not just collinear [i.e., only column (b) and not column (c) of Fig. 7]. To balance momentum a back-to-back configuration must be included. This arises only for the non-adjacent current diagram but again there is no special enhancement. In order to balance momentum we must have a three-dimensional configuration which is comparable on integration to the two-dimensional contribution.

For $\Delta > 3$ we have the reverse situation from that discussed above. From Eq. (5.1) we see that the total cross-section is dominated by one dimensional behaviour, i.e., by low-energy secondaries. Furthermore it follows from Fig. 7 where now $d_3 = d_2 + \Delta - 1 = d_1 + 2\Delta - 1$ that the dominant contribution to the one-particle distribution comes when the other secondaries are slow (E_2 finite). This dominance of slow secondaries is true whether we consider the one-particle distribution for a single fast particle (fixed ω_1) or for a slow one (fixed E_1). In this case, both the three-dimensional and two-dimensional regimes give insignificant contributions to the total cross-section so that both $\langle E \rangle$ and therefore $\langle p_1^2 \rangle$ are finite as $q \rightarrow \infty$.

The essential feature in the SAD model which gives rise to the behaviours discussed above is that the transitions between the two- and one-dimensional physics and between the three- and two-dimensional physics are controlled by the same parameter $\Delta = -2\delta + \alpha(0) - 1$. Thus, if we have a transition which is sufficiently large to ensure an insignificant rôle for the three-dimensional configuration, the one-dimensional configuration will dominate the two-dimensional regime. Conversely, a Δ small enough to ensure a dominant two-dimensional contribution will lead unfaillingly to a comparable three-dimensional contribution. In both cases ($\Delta > 3$ and $\Delta < 3$) the behaviour is not like the conventional (purely two-dimensional) parton model.

We have, up to now, discussed only the case of annihilation. The situation for electroproduction is completely analogous. Most of the considerations above carry over directly if we consider the situation in the Breit frame of the photon with the understanding that $\omega_1 > 1$ so that one dimensional behaviour cannot occur. The total cross-section is now given directly by the two-dimensional behaviour and thus scales canonically independent of Δ . If we consider the one-particle distribution coming from the contributions of Fig. 7, we first notice that the adjacent current diagram serves to populate the region in rapidity space from the hadron fragmentation region down to the hole fragmentation region (i.e., $\omega_2 < 0$) as illustrated in Fig. 5. In this entire region the transverse momentum is cut off at a finite value. This structure is independent of the value of Δ and corresponds to the conventional picture in this region ^{4),22)}. The non-adjacent current diagram will contribute in the photon fragmentation region ($|\omega_2| \leq 1$). However, the detailed behaviour depends on Δ . For $\Delta < 3$ one has a sizeable three-dimensional contribution in σ_{yp} leading to $\langle p_\perp^2 \rangle \propto q^2$ for particles in the photon fragmentation region ⁴⁾. Likewise, for fixed $|\omega_2| \leq 1$ and $p_\perp$ small, the single-particle distribution behaves as q^{-d_3} not q^{-d_2} where $d_3 = d_2 + 2$. For $\Delta > 3$ the configuration with slow secondaries (i.e., one-dimensional behaviours as defined in the photon Breit frame) makes the dominant contribution in the photon fragmentation region. In this case, the single particle distribution for fixed E_2 behaves as $q^{-d_3+\Delta} = q^{-d_2+1}$ and there is no significant contribution to the region at the edge of phase space ($\omega_2 > 0$, the usual parton fragmentation region). Hence we see that only if $\langle p_\perp^2 \rangle \sim |q^2|$ is the whole of the longitudinal phase space populated. For neither value of Δ do we find the conventional parton model scaling picture.

5.2. - Unitarity Constraints

Our philosophy in the above discussion has been to assume that the planar dual tree diagrams we have calculated represent the physical amplitudes to some good approximation. They then must approximately satisfy the inclusive sum rules since the physical amplitudes do. Let us see, for the annihilation case, how far this assumption can be pushed.

Consider first $\Delta > 3$ where the one-dimensional configuration is the dominant one. The n particle inclusive cross-section must then behave like $q^{-d_1+(n-1)}$. The extra powers of q are necessary to compensate the $1/q$ factors in the phase space integrals in the sum rules. There will be two distinct types of diagrams analogous to those of Section 4 contributing to this cross-section as illustrated in Fig. 8. The parameters k_i for these diagrams must by factorization be the same as those for the corresponding six-point functions. We expect, by analogy with Section 4, for Fig. 8a a behaviour enhanced by q^Δ when the energies of all hadrons are finite and for Fig. 8b a behaviour enhanced by q^Δ when the energies of all hadrons on one side of the current are finite and by $q^{2\Delta}$ when all hadrons on both sides have finite energies. The behaviour of these two diagrams for the n particle cross-section are then exactly the same as the two-particle distributions of Section 4, namely q^{-d_1} for Fig. 8a and q^{-d_1+1} for Fig. 8b. Hence, for $n > 2$, these diagrams yield too small a contribution to satisfy the sum rules when one attempts to build up the two-particle distributions.

This difficulty can be described in another, more physical fashion. The extra power of q required by the sum rules for each extra observed particle, where one dimensional behaviour is dominant, exactly measures the multiplicity since roughly q particles of finite energy are necessary to give the total energy q . At the two-particle level, the diagram of Fig. 4b exhibits the factor (q^{-d_1+1}) while the diagram in Fig. 4a does not (q^{-d_1}) . This is understandable from the standpoint of dual planar tree diagrams. In Fig. 4a, the two particles are adjacent and therefore if we fix one and integrate over the other, we measure the multiplicity of neighbouring particles only. This number is, of course, independent of q . In Fig. 4b, the second particle is not so constrained and is able to probe a multiplicity characteristic of the full multiplicity.

However, at the three-particle level, we must expect difficulty with only planar tree diagrams. Recall that planar tree diagrams represent a situation in which all the unobserved hadrons in the inclusive process can be lumped into a single resonance. Therefore the observed hadrons can form at most two distinct groups (resonances) as illustrated in Fig. 9a. Hence only two particles, one from each group, can count a multiplicity which grows as q . All the other particles in each group can count only the multiplicity of neighbours. Then when we use the energy sum rules to calculate a lower particle distribution, we can compensate at most two factors of $1/q$ in the phase space integrals. It is clear therefore that if we want to obtain, at the three or more particle inclusive level a term which can contribute to the one-dimensional configuration, we will be unable to do it within the framework of only planar tree diagrams.

This situation is quite general in any resonance dominated model. One dimensional scaling behaviour eventually becomes inconsistent with unitarity (the inclusive sum rules) in a model with planar resonance structure. This problem is clearly illustrated by the inconsistency of the factorization requirement (planar resonances) illustrated in Fig. 9a and the sum rule requirement in Fig. 9b ^{*}).

Within the framework of planar models, loop diagrams such as those of Fig. 10 must be considered in order to satisfy the sum rules since they no longer require the produced particles to be in one or two resonating groups ^{**}). This is physically reasonable when one-dimensional scaling dominates since then all particles in the final state have similar momenta.

^{*}) We remark that the sum rules can be satisfied with a scaling law

$$\frac{d^{3n} \sigma}{d^3 p_1/E_1 \dots d^3 p_n/E_n} \sim f(p_i/q^{1/3}),$$

since the phase space has constant behaviour as $q \rightarrow \infty$ in this case and the factor q^{n-1} is then absent. Such behaviours are found in statistical models - see, e.g., Ref. 24).

^{**}) Due to the lack of factorization in the SAD model, the loop contributions can, in fact, only be defined in a phenomenological manner like that of Kikkawa, Sakita and Virasoro ²⁵).

We now return to the case $\Delta < 3$. The scaling behaviours expected from Figs. 8a and 8b are consistent with the sum rule of Eq. (5.3). They correspond to one jet (Fig. 8a) or two jets (Fig. 8b) of particles with finite ω_i . Since the phase space within each jet is constant as $q \rightarrow \infty$, the n particle distribution can behave just like the two-particle distribution, as given by factorization, and still satisfy the sum rules.

Until now we have discussed only energy conservation (5.3), but momentum conservation

$$-\vec{p}_1 \frac{d^3\sigma}{d^3p_1/E_1} = \int \frac{d^3p_2}{E_2} \vec{p}_2 \frac{d^4\sigma}{d^3p_1/E_1 d^3p_2/E_2} \quad (5.5)$$

should also be considered. Equation (5.5) says simply that something must compensate the momentum of particle 1. In the two-particle inclusive cross-section with ω_1 and ω_2 fixed this compensation can occur only by including the non-adjacent diagram (Fig. 4b) since in the adjacent diagram $\vec{p}_1$ and $\vec{p}_2$ are in the same direction. When we now consider the three-particle cross-section, we easily see that there is an inconsistency. Since in a planar tree model, all further particles will appear either in the direction of $\vec{p}_1$ or $\vec{p}_2$, as discussed above, they cannot balance the momentum $\vec{p}_1 + \vec{p}_2$ unless $\vec{p}_1$ and $\vec{p}_2$ are collinear. However, this situation is not generally dominant for $\Delta < 3$ fixed due to the presence of comparable three-dimensional behaviour in the non-adjacent diagram contribution. So again we see that planar tree diagrams cannot be consistent with unitarity.

In summary, we have seen that we can have a consistent picture considering only planar tree diagrams up to the level of the two-particle inclusive cross-section. However, this picture will encounter difficulties with the inclusive sum rules for three or more particle inclusive cross-sections and more complicated diagrams must contribute at this level. For the case $\Delta > 3$, where one-dimensional scaling is dominant, the problem arises in the energy conservation sum rule and for $\Delta < 3$ there is eventually a problem with momentum conservation arising in the three-dimensional configuration. Thus planar tree diagrams and unitarity would seem to have the possibility of being roughly consistent only when purely two-dimensional scaling is dominant as is the case for the purely hadronic dual model but not for the SAD model for currents for any value of Δ .

That loop diagrams should be important for satisfying unitarity is of course not surprising since the tree diagrams are not unitary by themselves and the loops are precisely what are needed to satisfy unitarity in perturbation theory sense. While one might be tempted to ignore loop contributions in the hadronic case, a closer study reveals that the loops are crucial since the non-planar ones build up the dominant asymptotic behaviour in the cross-section (the Pomeron). Since the loops are clearly important at some stage we must in the end be careful about taking too seriously the details of the planar tree graph results [e.g., $\Delta = -2\delta + \alpha(0) - 1$] since other contributions could dominate the single particle inclusive cross-section.

We close this section with a brief discussion of how one may visualize the way in which a model with only poles, as here, succeeds in describing a transition from two- to one-dimensional scaling.

The high degeneracy of states present in the model plays a fundamental rôle. It has been recently argued ²⁶⁾ that the decay spectrum of a statistical superposition of degenerate states exhibits the type of one-dimensional behaviour discussed here with finite energy secondaries isotropically populating the available spherical phase space. This result is expected to hold for a high mass, low spin state as well as for the statistical average. The question then arises as to how it happens that a model built up from such resonances is able to produce two-dimensional behaviour in high energy hadron-hadron reactions with a sharp cut-off in the transverse direction. The answer is simply that in the hadronic case the resonances are produced in an extremely coherent (non-statistical) fashion. It is this coherence which produces the collinear structure. Indeed of the available resonances, whose number grows exponentially with the total energy, only a very small number is actually excited ²⁷⁾ and with well defined relative phases. In the one-dimensional configuration, arising in the annihilation process, this coherence is apparently absent.

One can indeed interpret the unitarity difficulties which appear in a planar tree diagram model of one-dimensional scaling, as discussed above, as a result of the fact that too much coherence still remains. For tree diagrams, the produced particles fall only into one or two groups which

are strongly coherent and have the usual two-dimensional properties. Loop diagrams or non-planar diagrams may give a reduced amount of coherence and give a one-dimensional behaviour consistent with unitarity ^{*)}.

6. - CONCLUSION

We have seen that the simple SAD model is an example of a dynamics that yields scaling behaviours which depend on the kinematic structure of the process studied. In particular, it is a realization of the recent conjecture ¹⁾ that the exponent of the power behaviour which characterizes this scaling should become more negative as one increases the number of dimensions spanned by large momenta.

The model exhibits one-, two- and three-dimensional scaling appearing, for example, in $e^+e^- \rightarrow$ slow secondaries, Bjorken scaling in deep inelastic electroproduction, and the production of particles with transverse momenta of order q in deep inelastic electroproduction, respectively. The change in the scaling behaviour between each of these configurations is determined in the model by the same parameter $\Delta = -2\delta + \alpha(0) - 1$ where $\alpha(0)$ is the leading Regge trajectory and δ is related to the asymptotic behaviour of form factors.

^{*)} Non-planar graphs are not expected to have the canonical two-dimensional scaling behaviour which in models seems always to be associated with terms with adjacent currents ^{18), 28)}. However, we remark that the large neutral to charged energy ratio apparently observed in annihilation would point in the direction of non-planar or at least non-adjacent current terms. This is because if isospin is incorporated in the usual way by Chan-Paton factors, the adjacent current terms have $I = 0$ and 1 in the current-current channel and thus have equal number of positive, neutral and negative particles produced.

For the case that the parameter Δ is less than 3, the one-dimensional configuration (all energies finite as $q \rightarrow \infty$) does not make a significant contribution to the total annihilation cross-section and the cross-section exhibits canonical (two-dimensional) scaling. Also in this case the deep inelastic electroproduction cross-section has canonical q dependence but receives a significant contribution from the three-dimensional regime where some particles have transverse momentum of order q in the photon fragmentation region. The requirement of non-dominance of particles with $\langle p_{\perp}^2 \rangle \sim |q^2|$ in electroproduction is correlated, in this model, with a non-canonical behaviour for the annihilation cross-section. In particular for the case $\Delta > 3$ the three-dimensional contribution with large transverse momentum is suppressed for both electroproduction and annihilation while the one-dimensional contribution to the annihilation process is enhanced sufficiently to make it the dominant behaviour. Hence for $\Delta > 3$, $\sigma_{e^+e^-}$ is expected to fall off more slowly in q than in the conventional picture and to be dominated by events where all secondaries are slow. The specific form for Δ in the model $[\Delta = -2\delta + \alpha(0) - 1]$ is quite reasonable for typical hadronic form factors which have $\delta \approx -1$ to -2.5 and is near the apparent experimental value $\Delta \approx 4$ to 5 .

This is certainly an interesting result but one must bear in mind the fact that at present dual models for currents are just models and not theories. In particular we have noted that, for any value of the parameter Δ , resonance models with only the planar tree diagrams included as discussed here, cannot be consistent with unitarity in the form of the inclusive sum rules beyond the level of the two-particle inclusive distribution. Thus other contributions, such as loop diagrams, which we are unable to evaluate must be important at best at the three-particle level. Also we have not yet found a simple or compelling explanation for the specific form of the parameter Δ found in this model. Hence we must consider that the specific calculation discussed above is only an illustrative and highly simplified example of a dynamics which can provide dimensional scaling behaviours. However, we do believe that this phenomenological model does succeed in reproducing an important basic feature observed in hydrodynamics which is the presence of hard structures - reflected in Bjorken scaling, power fall-off for large transverse momentum, etc. - even though all hadrons are fully Reggeized. To the extent that this feature is related to the dimensional scaling behaviour, we may anticipate that a more realistic model will produce very similar scaling behaviours.

It is interesting therefore to analyze how this hard structure has been introduced into the specific model discussed here. In the SAD model, currents are introduced through quark-antiquark pairs which are included in a dual amplitude for both the usual hadrons and these quarks. The amplitude is so defined as to possess Regge poles in ordinary hadronic channels which contain both quarks from any current and fixed poles in "channels" which contain a single quark from some current. It is the presence of these fixed poles which is the origin of the hard structure ^{*)}. It is appealing to consider this SAD model as a possible description of a model with imprisoned quarks. It was conjectured ¹⁾ that the property of dimensional scaling might be related to this notion of quark imprisonment and, seen in this light, the present calculation would seem to support that idea.

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^{*)} This relationship may in fact be a hint as to how to construct dual models with hard structure, i.e., three-dimensional scaling, for purely hadronic amplitudes - see Section 4.

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FIGURE CAPTIONS

- Figure 1 (a) Four-point function related to single-particle inclusive annihilation and total electroproduction cross-sections ;
(b) Six-point function related to two-particle inclusive annihilation and single-particle inclusive electroproduction cross-sections ;
(c) Two-point function related to total annihilation cross-section. The dotted line means a discontinuity in the indicated invariant.
- Figure 2 SAD model for :
(a) form factor, and
(b) two-point function.
- Figure 3 SAD model for four-point function.
- Figure 4 SAD model for six-point function :
(a) adjacent current diagram, and
(b) non-adjacent current diagram.
- Figure 5 Kinematic regions in electroproduction in the Mueller-Regge limit. The rapidity y is given in the photon Breit frame. The regions where the adjacent and non-adjacent current diagrams have the largest scaling exponent are doubly hatched.
- Figure 6 Integration region for Eq. (4.16).
- Figure 7 Physical picture of the two-particle inclusive annihilation cross-section in the SAD model.
- Figure 8 Planar tree diagrams contributing to the n particle inclusive annihilation cross-section for :
(a) adjacent, and
(b) non-adjacent currents.
- Figure 9 Constraints on the n particle inclusive annihilation cross-section arising from :
(a) factorization, and
(b) inclusive sum rules.
- Figure 10 Non-planar loop diagram contributing to three-particle inclusive annihilation cross-section.

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig 6

Fig.7 ω

	(a)	(b)	(c)	(d)	(e)
	3-dimensional configuration	2-dimensional configurations			1-dimensional configuration

Scaling Variables	$\omega_1, \omega_2, \theta$	ω_1, ω_2, P_L	ω_1, ω_2, P_L	ω_1, E_2, P_L or E_1, ω_2, P_L	E_1, E_2, P_L
Weight of Phase Space	q^2	q^0	q^0	q^2 for ω_i q^{-1} for E_i	q^{-1}
Adjacent Currents		q^{-d_2}	$q^{-d_2+4\gamma}$	$q^{-d_2+2\gamma}$	$q^{-d_2+\Delta}$
Nonadjacent Currents		q^{-d_3}	q^{-d_3}	$q^{-d_3+\Delta}$	$q^{-d_3+2\Delta}$

(a)

(b)

Fig. 8

(a)

(b)

Fig. 9

Fig. 10