FERMI-ROSE SUPERSYMMETRY
(SUPERAUGE SYMMETRY IN FOUR DIMENSIONS)

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1. INTRODUCTION

Fermi-Bose supersymmetry was introduced first by Wess and the author. It is a symmetry which connects particles of integral spin with particles having half-integral spin, or bosons with fermions. The possibility of defining such a symmetry was suggested to us by the existence of supergauge transformations in dual models (when formulated as two-dimensional field theories) and the name supergauge symmetry in four dimensions seemed a natural choice for it. However, the supergauge algebra has only a finite number of generators in four dimensions, so that it seems now reasonable to avoid the word gauge, which traditionally refers to groups of transformations depending upon arbitrary functions. We therefore adopt the expression Fermi-Bose supersymmetry, or simply supersymmetry, suggested recently by Salam and Strathdee.

The supersymmetry algebra is very simple. Let \( Q_\downarrow \) be a constant Majorana spinor (we may use the Majorana representation where the \( \gamma \) matrices are real; then \( Q_\downarrow \) is a Hermitian spinor). Then the algebra is

\[
\begin{align*}
\{ Q_i, \bar{Q}_j \} &= -2 (\gamma^\mu)_{ij} P_\mu, \\
\bar{Q} &= \bar{a} \gamma^0 \\
[ P_\mu, Q_i ] &= [ P_\mu, P_\lambda ] = 0,
\end{align*}
\]  

(1)

where \( P^\mu = (H, P) \) is the energy momentum operator, which generates four-dimensional translations. In a supersymmetry invariant field theory, the spinor charges are given as integrals

\[
Q_i = \int J_i^\sigma \, d^3x,
\]  

(2)

where the vector-spinor current is conserved

\[
\partial_\mu J_i^\sigma = 0.
\]  

(3)

Lorentz transformations and parity operate as isomorphisms of the algebra (1), transforming \( Q_\downarrow \) as a spinor and \( P^\mu \) as a vector. The algebra (1) is not a Lie algebra, since it contains both commutators and anticommutators (it is what mathematicians call a "graded algebra"). If one introduces parameters \( \sigma_1, \sigma_2, \ldots \), which are totally anticommuting Majorana spinors (they commute with tensors and anticommute with spinors and among themselves), the anticommutation relation (1) can be written as a commutation relation.
\[
\left[ \bar{\alpha}_1 Q, \bar{\alpha}_2 Q \right] = -2 \bar{\alpha}_1 \gamma^\mu \alpha_2 P_\mu .
\]

The supersymmetry algebra can therefore also be described as an "extended Lie algebra", with parameters belonging to a Grassmann algebra. This kind of object has been studied in the mathematical literature\(^3\).

Observe that, if one multiplies the anticommutation relation (1) by \(\gamma^0\) and takes the trace over the spinor indices, one finds an expression for the total Hamiltonian in terms of the spinor charges

\[
H = \frac{1}{4} \sum_{i=1}^{u} Q_i^2
\]

valid, in presence of interaction, for any supersymmetric theory. Similar expressions can be obtained for the components of the momentum.

The fact that supersymmetry is not an ordinary Lie algebra allows it to avoid the difficulties and no-go theorems\(^4\) which have plagued the various forms of relativistic SU(6). Actually, as we shall see, there exist non-trivial (and renormalizable) Lagrangian theories which are exactly invariant under the supersymmetry algebra.

Our motivation in introducing Fermi-Bose supersymmetry was to show the feasibility of constructing supermultiplets containing interacting particles with both integral and half integral spin. From a rather different point of view the same algebra (1) was considered independently by Volkov and Akulov\(^5\). They gave a non-linear realization of it in terms of a single spinor field (see Section 4) and suggested that it may be relevant as a description of the properties of the neutrino. Their non-linear Lagrangian is non-renormalizable.

In Ref. 1), an algebra larger than (1) was described, which contains also Lorentz transformations, dilatations, conformal and chiral transformations. In that algebra a second set of spinor charges occurs, which, together with the \(Q_i\), forms an eight-component conformal Majorana spinor\(^6\). That larger algebra was later abandoned\(^7\), in order to avoid the problems arising from scale and conformal anomalies. Anyway, if one is interested in the formal construction of Lagrangians invariant under the larger algebra, one need only observe that a Lagrangian invariant under (1) and under the conformal algebra is automatically invariant under the larger algebra.
2. SUPERMULTIPLETS AND LAGRANGIANS

The simplest supermultiplet consists of a scalar field $A$, a pseudoscalar field $B$, a Majorana spinor $\psi$, and two auxiliary fields $\mathcal{F}$ and $\mathcal{G}$. Writing

$$\delta A = \left[ \bar{\alpha} \partial, A \right] \quad \text{etc.}$$

for an infinitesimal supertransformation, one has

$$\begin{cases}
\delta A = i \bar{\alpha} \psi \\
\delta B = i \bar{\alpha} \gamma_5 \psi \\
\delta \psi = \partial_\mu (A - \gamma_5 \mathcal{B}) \gamma_\mu \alpha + (\mathcal{F} + \gamma_5 \mathcal{G}) \alpha \\
\delta \mathcal{F} = i \bar{\alpha} \gamma_\mu \mathcal{G} \psi \\
\delta \mathcal{G} = i \bar{\alpha} \gamma_5 \gamma_\mu \partial \psi.
\end{cases}$$

The commutator of two supertransformations is a translation. For instance

$$\delta_2 \delta_1 A = i \bar{\alpha}_1 \delta_2 \psi = i \bar{\alpha}_1 \gamma_\mu \alpha_2 \partial_\mu A + \ldots$$

where the dots denote terms symmetric in 1 and 2. Therefore

$$[\delta_2, \delta_1] A = 2i \bar{\alpha}_1 \gamma_\mu \alpha_2 \partial_\mu A.$$  

With this supermultiplet, Wess and the author \(^7\) constructed the first non-trivial supersymmetric model, with Lagrangian

$$L = -\frac{1}{2} \left[ (\partial_\mu A)^2 + (\partial_\mu B)^2 + i \bar{\psi} \gamma_\mu \partial_\mu \psi - F^2 - G^2 \right] + m (FA + GB - \frac{i}{2} \bar{\psi} \psi) + g \left( FA^2 - FB^2 + 2GAB - i \bar{\psi} \psi A + i \bar{\psi} \gamma_5 \psi B \right).$$

The various terms of the Lagrangian, that is the kinetic term, the term proportional to $m$ and that proportional to $g$, each change by a divergence under (7). Therefore the action integral is invariant. The auxiliary fields $\mathcal{F}$ and $\mathcal{G}$ can be eliminated by using their own equations of motion.
\[ F + m A + g (A^2 - B^2) = 0 \]
\[ G + m B + 2g A B = 0 \]

and the Lagrangian takes the more familiar form
\[ L = -\frac{1}{2} \left[(\partial^\mu A)^2 + (\partial^\mu B)^2 + i \bar{\psi} \gamma^\mu \partial_\mu \psi + m^2 A^2 + m^2 B^2 + i m \bar{\psi} \gamma^\mu \psi \right] \]
\[ -g m A (A^2 + B^2) - \frac{g}{2} (A^2 + B^2)^2 - i g \bar{\psi} (A - \gamma_5 B) \psi . \]

Observe that, as a consequence of supersymmetry invariance, the scalar, the pseudoscalar and the spinor have the same mass, and all the couplings are expressed in terms of the single coupling constant \( g \). One can easily verify that the supercurrent
\[ J^\mu = \gamma^\lambda \gamma^\mu (A - \gamma_5 B) \gamma^\lambda \psi - (F + \gamma_5 G) \gamma^\mu \psi \]

is conserved as a consequence of the equations of motion.

This model has been studied in great detail by Ferrara, Iliopoulos and the author\(^3\). It can be regularized in a supersymmetric way by introducing higher order derivatives in the kinetic term of the Lagrangian. The Ward identities corresponding to the conservation of the supercurrent \( J^\mu \) can be written and used to prove that renormalization does not spoil the relations among masses and coupling constants due to supersymmetry. The model is found to be less divergent than the generic theory with the same kind of couplings. In particular, only one renormalization constant is required, the single wave function renormalization constant \( Z \) common to all fields. The renormalized mass and coupling constant are given by
\[ m_r = Z \ m_0 \quad , \quad g_r = Z^{3/2} \ g_0 . \]

The Callan-Symanzik equations take a particularly simple form for this model, the functions \( \beta \) and \( \gamma \) being proportional to each other. As a consequence one can argue that the function \( \beta(g_r) \) cannot vanish except at the origin and that the effective coupling constant increases indefinitely with energy.

Supersymmetry can be broken softly, by adding to the Lagrangian a term proportional to the field \( A \). Just as in the analogous case of the
model, the renormalization program can still be carried out. The masses of the various fields of the multiplet are now no longer equal. Instead one finds, in the tree approximation, the mass relation

\[ m_A^2 + m_B^2 = 2 m_\psi^2. \] (15)

In higher orders this relation is corrected by finite terms.

The fact that in the supersymmetric model cancellations of divergences occur, which make it less divergent than the generic theory of its kind, leads one to ask whether a supersymmetric theory might not be renormalizable even if it does not appear to be so by simple power counting. To answer this question, Lang and Wess\(^9\) have replaced, in the Lagrangian (10), the renormalizable interaction proportional to \(g\) with the interaction

\[ \int \left\{ F A^3 - G B^3 + 3 G A^2 B - 3 F A B^2 - \frac{3}{2} i (A^2 - B^2) \bar{\psi} \psi + 3 i A B \bar{\psi} \gamma_5 \psi \right\}. \] (16)

This interaction is supersymmetric but non-renormalizable by power counting. For instance, if one eliminates the auxiliary fields \(F\) and \(G\), it gives rise to the interaction

\[ \int \left\{ -m (A^4 - B^4) - \frac{3}{2} i (A^2 - B^2) \bar{\psi} \psi + 3 i A B \bar{\psi} \gamma_5 \psi \right\} - \frac{1}{2} \int (A^2 + B^2)^3. \] (17)

In the one-loop approximation this model exhibits a number of cancellations of divergences. For instance, the two-point function is finite, and so are the sixth and higher point functions. However, the four-point function is logarithmically divergent. To cancel this divergence one must introduce a new counter term, proportional to \(x^2\) times

\[ (A^2 + B^2) \left( F^2 + G^2 - (\partial_x A)^2 - (\partial_x B)^2 - i \bar{\psi} \gamma^x \partial_x \psi \right) \]

\[ - i (A F + B G) \bar{\psi} \psi + i (A G - B F) \bar{\psi} \gamma_5 \psi \]

\[ + \left\{ \frac{1}{2} \bar{\psi} \gamma_5 \gamma^x \psi - i (A \partial_x B - B \partial_x A) \right\} \bar{\psi} \gamma_5 \gamma^x \psi. \] (13)

This new interaction, in turn, generates new divergences and it does not seem that a finite number of counter terms is sufficient. While this question is being investigated, it appears that, in the particular model considered by
Lang and Wess supersymmetry, in spite of the compensation of divergences it gives rise to, is not sufficient to render the theory renormalizable.

3. **GAUGE INVARIANCE AND SUPERSYMMETRY**

The existence of Lagrangian theories which are both gauge invariant and supersymmetric was first shown by Wess and the author\(^{10}\).

Their model makes use of a supermultiplet consisting of a vector field \( v_\mu \), a Majorana spinor \( \lambda \) and an auxiliary field \( D \), transforming as

\[
\begin{align*}
\delta v_\mu &= i \bar{\alpha} \gamma_\mu \lambda \\
\delta \lambda &= -\frac{1}{4} v_\mu \gamma^\nu \gamma^\alpha \gamma_\alpha + D \gamma_\alpha \\
\delta D &= i \bar{\alpha} \gamma_\alpha \gamma^\nu \partial_\nu \lambda, \quad \gamma_\mu \gamma^\nu = \partial_\mu \nu - \partial_\nu \nu\end{align*}
\]

under a supersymmetry transformation. The commutator of two transformations (19) is a translation accompanied by a gauge transformation, but, by enlarging the multiplet, one could arrange that it be exactly a translation\(^{10}\).

This supermultiplet is put in interaction with a complex multiplet, or a pair of real multiplets, of the kind discussed in the previous section. Using real fields the Lagrangian can be written as

\[
L = -\frac{1}{4} v_\mu v^\mu - \frac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \\
- \frac{1}{2} \sum_{i=1}^2 \left[(\partial_\mu A_i)^2 + (\partial_\mu B_i)^2 + i \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + m^2 (A_i^2 + B_i^2) + i m \bar{\psi}_i \gamma^5 \psi_i \right] \\
+ g \left[D (A_1 B_2 - A_2 B_1) - v^\mu (A_1 \partial_\mu A_2 - A_2 \partial_\mu A_1 + B_1 \partial_\mu B_2 - B_2 \partial_\mu B_1 - i \bar{\psi}_1 \gamma_5 \psi_2) \right] \\
- i \bar{\lambda} \left\{ (A_1 + \gamma_5 B_1) \psi_2 - (A_2 + \gamma_5 B_2) \psi_1 \right\} \\
- \frac{1}{2} v_\mu^2 (A_1^2 + A_2^2 + B_1^2 + B_2^2)
\]

where the fields \( F_i \) and \( G_i \) \((i = 1, 2)\) have already been eliminated. The gauge transformation rotates the fields with the subscripts 1 and 2 into each other and changes \( v_\mu \) by a four-gradient. The fields \( \lambda \) and \( D \) are gauge invariant. The Lagrangian (20) can be considered as a sort of supersymmetric extension of the quantum electrodynamics of scalars, pseudoscalars, and spinors. Observe that all couplings are expressed in terms of the single coupling constant \( g \), as a consequence of the supersymmetry of the Lagrangian. The Lagrangian (20) has been shown to be renormalizable in the one-loop
approximation in a manner consistent with gauge invariance and supersymmetry. A preliminary investigation of higher orders supports this conclusion.

4. **SPONTANEOUS SYMMETRY BREAKING**

The model described in the previous section, with a small modification, gives an example of spontaneous breaking of supersymmetry, with the corresponding emergence of a "Goldstone" spinor. This has been shown by Fayet and Iliopoulos\(^{11}\) who have added to the Lagrangian (20) a parity violating, but gauge and supersymmetry invariant term \( \xi D / g \). Upon elimination of the field \( D \) this gives rise to a term

\[
-\frac{\xi}{2} (A_1 B_2 - A_2 B_1).
\]

The mass matrix for the fields \( A_1 \) and \( B_1 \) \((i = 1, 2)\) can be diagonalized by introducing the new fields

\[
\begin{align*}
a_1 &= \frac{1}{\sqrt{2}} (A_1 - B_2) \\
a_2 &= \frac{1}{\sqrt{2}} (B_1 + A_2) \\
b_1 &= \frac{1}{\sqrt{2}} (A_1 + B_2) \\
b_2 &= \frac{1}{\sqrt{2}} (-B_1 + A_2).
\end{align*}
\]

This results in the potential (tree approximation)

\[
\frac{1}{2} \left( m^2 - \xi \right) (a_1^2 + a_2^2) + \frac{1}{2} \left( m^2 + \xi \right) (b_1^2 + b_2^2) + \frac{g^2}{\xi} \left( a_1^2 + a_2^2 - b_1^2 - b_2^2 \right).
\]

At this point one sees already that supersymmetry is spontaneously broken, since the masses of the fields in a given supermultiplet are no longer equal. For \( |\xi| < m^2 \) the fields \( a_i \) and \( b_i \) have vanishing vacuum expectation value, while \( <D> = -\xi / g \) and, from (19)

\[
\delta \lambda = -\frac{\xi}{\alpha} Y_\alpha + \cdots
\]

where the dots denote terms containing other fields. The field \( \lambda \) is a Goldstone Fermion (germion). For \( |\xi| > m^2 \) one of the quadratic terms in (23) has a negative coefficient. Now the gauge invariance is also spontaneously broken and the vector field acquires a mass given by \( m_v^2 = 2(\xi - m^2) \) (Higgs mechanism).
The Goldstone fermion is now a linear combination $\lambda'$ of the fields $\lambda, \gamma_5 \gamma_1$ and $\gamma_2$. It transforms as

$$\delta \lambda' = -\frac{m}{2} \sqrt{2} \frac{1}{3-m^2} \gamma_5 \alpha + \cdots.$$  \hfill (25)

The Goldstone fermions are massless spinors arising from the spontaneous breaking of the supersymmetry corresponding to the conservation law (3), just as Goldstone bosons arise when chiral symmetry is spontaneously broken. In analogy with that case, one may ask whether non-linear realizations of supersymmetry exist. The first non-linear realization was given by Volkov and Akulov$^5$. They use a single Majorana spinor $X$ transforming as

$$\delta X = \frac{1}{a} \alpha + ia (\bar{\alpha} \gamma^r \chi) \partial_r \chi$$ \hfill (26)

where $a$ is a (universal) constant. Volkov and Akulov have also given an invariant non-linear action describing the self-interaction of the spinor $X$, as well as its interaction with other fields. They suggest that their theory is a description of the (?) neutrino. A different non-linear realization was found by the author. It is

$$\delta X = \frac{1}{a} \alpha + ia (\bar{\alpha} \gamma^r \chi) \partial_r \chi + ia (\bar{\alpha} \gamma_5 \gamma^r \chi) \gamma_5 \partial_r \chi.$$ \hfill (27)

Just as for chiral symmetry, one may inquire about the connection between linear and non-linear realizations. It is interesting that there exist functions of the field $X$ transforming as in (27) which transform linearly as in (7).

5. SUPERSPACE AND SUPERFIELDS

Supersymmetry representations have been studied with various techniques$^{12,13}$. Salam and Strathdee$^{14}$ have introduced the very interesting concept of superfield and have described supersymmetry transformations as operations on superfields. The concept of superfield has been extended by Ferrara, Wess and the author$^{15}$. A general review of the technique with applications is given by Salam and Strathdee$^{16}$. Superspace was considered by Volkov and Akulov$^5$ in their work on non-linear realizations.

The idea is very simple. Consider a space (superspace) whose points are labelled by co-ordinates $x^\mu, \theta, \bar{\theta}$, where $x^\mu$ are the usual space-time co-ordinates, $\theta$ is a (totally anticommuting) two-component spinor and
its complex conjugate. If one wishes, one can arrange $\Psi$ and $\bar{\Psi}$ into a single four-component Majorana spinor

$$\left( \begin{array}{c} \Psi \\ \bar{\Psi} \end{array} \right)$$

using a (complex) representation of the $\gamma$ matrices in which $\gamma_5$ is diagonal. The parameters $\alpha$ of a supersymmetry transformation now take the form

$$\alpha = \left( \begin{array}{c} \frac{\Psi}{\bar{\Psi}} \\ \frac{\bar{\Psi}}{\Psi} \end{array} \right).$$

(28)

Supersymmetry transformations are geometrical transformations in superspace

$$\begin{align*}
\delta x_\mu &= i \theta \sigma_\mu \bar{x} - i \bar{x} \sigma_\mu \bar{\theta} \\
\delta \theta &= \bar{x} \\
\delta \bar{\theta} &= \bar{x} 
\end{align*}$$

(29)

It is easy to see that the commutator of two such transformations is a translation. Furthermore (29) and translations leave invariant the differential form

$$\omega_\mu = dx_\mu + i \bar{\theta} \sigma_\mu d\theta - i \theta \sigma_\mu d\bar{\theta}$$

(30)

and if one adjoins Lorentz transformations, one still has that the "line element"

$$\omega_\mu \omega^\mu$$

(31)

is invariant. A superfield is a field in superspace, $V(x, \theta, \bar{\theta})$. If one expands it in $\theta$ and $\bar{\theta}$, the power series terminates after a finite number of terms, since the square of each component of $\theta$ and $\bar{\theta}$ vanishes

$$V(x, \theta, \bar{\theta}) = C(x) + i \theta \chi(x) - i \bar{\theta} \bar{\chi}(x) + \ldots + \theta \bar{\theta} \bar{\theta} \frac{1}{2} D(x).$$

(32)

Therefore a superfield corresponds to a finite supermultiplet of ordinary fields. A superfield is taken to transform as a scalar in superspace under (29)

$$V'(x', \theta', \bar{\theta}') = V(x', \theta', \bar{\theta}')$$

(33)

and can have spinor or vector indices which determine how it transforms under Lorentz transformations. From (33) and the expansion (32), one can derive the transformation property of the fields $C, X, \text{etc.}$ of the
supermultiplet. Observe that, if one introduces the new coordinates

$$Z_\mu = x_\mu + i \partial \sigma_\mu \bar{\Theta},$$

(34)

(29) gives

$$\delta Z_\mu = 2i \partial \sigma_\mu \bar{S},$$

(35)

which does not contain $\bar{\Theta}$. Therefore, it is consistent to require that a superfield be a function only of $z_u$ and $\Theta$, $S(z, \Theta)$, or that it satisfy

$$\mathcal{D} S = 0$$

(36)

where

$$\mathcal{D} = -\frac{\partial}{\partial \Theta} \bigg|_x - \frac{\partial}{\partial \Theta} \bigg|_z - i \Theta \sigma_\mu \partial_\mu.$$  

(37)

$\mathcal{D}$ is a covariant derivative [under (29)] and so is

$$\mathcal{D} = \frac{\partial}{\partial \Theta} \bigg|_x + i \sigma_\mu \partial_\mu.$$  

(38)

A superfield satisfying (36) is called left-handed, one satisfying the covariant constraint

$$\mathcal{D} S = 0$$

(39)

is called right-handed. A left-handed superfield (together with its right-handed complex conjugate) corresponds to the multiplet described in Section 2.

One can develop a geometry of superspace. For instance, the larger supersymmetry group of Ref. 1) can be characterized as containing those transformations in superspace which multiply the line element (31) by a rescaling factor. Volkov and Soroka have developed a description of curved superspace which combines gravitational theory with the interactions of particles of spin $\frac{3}{2}$, 1, and $\frac{1}{2}$. Can a theory of this kind, because of the compensation of divergences due to supersymmetry, provide a renormalizable description of gravitational interactions?
6. YANG-MILLS GAUGES AND SUPERSYMMETRY

Using the technique of superfields, Salam and Strathdee\textsuperscript{19}) and Ferrara and the author\textsuperscript{20}) have succeeded in constructing Lagrangian theories which are both supersymmetric and invariant under non-Abelian gauge transformations. We skip the technical details and give only the results. The simplest supersymmetric and gauge invariant theory is the ordinary Yang-Mills theory of vectors in interaction with a multiplet of Majorana spinors belonging to the regular (adjoint) representation of the internal symmetry group. For instance, for SU(N), using \( N \times N \) matrix notation

\[
L = \mathcal{T}_2 \left( -\frac{1}{4} \mathcal{V}_{\mu}^{\nu} \mathcal{V}_{\nu}^{\mu} - \frac{i}{2} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \right) + \frac{1}{2} \mathcal{D}^2
\]

That this theory is supersymmetric can be seen most easily by checking that the supercurrent

\[
J^\mu = -\frac{1}{4} \mathcal{T}_2 \left( \mathcal{V}_{\nu}^{\rho} \left[ \gamma^\nu, \gamma^\rho \right] \gamma^\mu \right)
\]

is conserved as a consequence of the equations of motion.

The multiplet \( v, \lambda, D \) can be coupled to "matter multiplets" \( A, B, \psi, F, G \) belonging to any representation of the internal group. For the case of a multiplet in the regular representation, the Lagrangian is given by (40) plus

\[
\mathcal{T}_2 \left\{ -\frac{1}{2} \left[ (\mathcal{D}_{\mu} A)^2 + (\mathcal{D}_{\nu} B)^2 + i \mathcal{A}_{\mu} \mathcal{A}_{\nu} \psi - F^2 + G^2 \right] + m \left( F A + G B - \frac{i}{2} \mathcal{A} \psi \right) + i g D \left[ A, B \right] + i g \mathcal{A} \left[ A + \kappa B, \psi \right] \right\}
\]

Theories of this kind contain scalar and pseudoscalar fields. Since all coupling constants, including the self-interaction of the scalars and the pseudoscalar, are expressed in terms of the single Yang-Mills coupling \( g \), they will be asymptotically free provided the Callan-Symanzik function \( \beta \) is negative (and provided the theories are renormalizable in accordance with supersymmetry!). For (40) plus \( n \) matter multiplets like (42), it turns out to be

\[
\beta = -\frac{g^3}{16\pi^2} (3 - n) N
\]

where \( N \) refers to SU(N).
If one adds (40) plus (42) for $m = 0$, eliminates the field $D$, and combines the two Majorana fields $\lambda$ and $\psi$ into a complex spinor

$$\varphi = \frac{i}{\nu_2} (\lambda + i \psi),$$

one obtains the Lagrangian

$$\mathcal{L} = \frac{1}{4} \nu_{\mu} \nu_{\nu} - \frac{1}{2} (D_{\mu} A)^2 - \frac{1}{2} (D_{\mu} B)^2 - i \overline{\varphi} D_{\mu} \varphi$$

$$- i g \overline{\varphi} [A + \gamma_5 B, \varphi] - \frac{g^2}{4} (i [A, B])^2 \right\}$$

which is invariant under the fermion number transformation

$$\varphi \rightarrow e^{i \omega} \varphi, \quad \varphi^* \rightarrow e^{-i \omega} \varphi^*.$$  

The conserved supercurrent for (45) is

$$J^\mu = \mathcal{L} \left[ - \frac{1}{4} \nu_{\rho} \left[ \gamma_\mu, \gamma_{\rho} \right] \gamma^\rho \varphi + i g \left[ A, B \right] y_5 y^\nu \varphi - i y^A D_\lambda (A - \gamma_5 B) y^\nu \right].$$

One can also obtain a $V + A$ scheme with fermion number. One need only combine two multiplets $v_{(1)}^{u, (1)}$ and $v_{(2)}^{u, (2)}$ described by Lagrangians like (40). The total Lagrangian can be rewritten in terms of the complex spinor

$$\varphi = \frac{i}{2} \left( 1 - i \gamma_5 \right) \lambda^{(u)} + \frac{i}{2} \left( 1 + i \gamma_5 \right) \lambda^{(a)}$$

and of the vector and axial vector fields

$$\nu_{\mu} = \frac{i}{\nu_2} \left( \nu_{\mu}^{(u)} + \nu_{\mu}^{(a)} \right), \quad a_{\mu} = \frac{i}{\nu_2} \left( \nu_{\mu}^{(u)} - \nu_{\mu}^{(a)} \right)$$

and is again invariant under a transformation like (46).

The examples described in this section show that the point has been reached where one can attempt to construct realistic supersymmetric models. The main difficulty at present is that one does not have a device for generating masses without spoiling the renormalizability of the theory. The Fayet-Iliopoulos trick described in Section 4 can be applied only to an invariant Abelian subgroup, otherwise a term in the Lagrangian proportional to one of the components of the field $D$ would spoil explicitly gauge invariance and consequently renormalizability. Perhaps consideration of the effective potential in higher orders will provide a solution to this problem.
7. NON-TRIVIAL MIXING OF INTERNAL SYMMETRY AND SUPERSYMMETRY

The combination of internal symmetry and supersymmetry described in the previous section can be considered as trivial, because it consists essentially in attributing all fields of a supermultiplet to the same representation of the internal symmetry group (it was not trivial that this could be done locally). A more interesting possibility was suggested by Salam and Strathdee \cite{13} and by Wess and the author \cite{23}. It consists in an actual combination of the two symmetries to a new algebra.

Imagine the algebra (1) written in two-component notation, by using a representation of the \( \gamma \) matrices in which \( \gamma_5 \) is diagonal. The supercharges will consist of a two-component spinor and its conjugate. One can give these two-component spinors an internal symmetry index \( i \) or \( j \) and write the algebra as

\[
\begin{align*}
\{ Q_{\alpha}^i \, , \, Q_{\beta}^j \} &= \{ \overline{Q}_{\dot{\alpha}}^i \, , \, \overline{Q}_{\dot{\beta}}^j \} = 0 , \\
\{ Q_{\alpha}^i \, , \, \overline{Q}_{\dot{\beta}}^j \} &= 2 \delta^i_j \, (\sigma_\mu)_{\alpha\dot{\beta}} \, P^\mu , \\
[ Q_{\alpha}^i , P_\mu ] &= [ P_{\lambda} , P_\mu ] = 0 .
\end{align*}
\]

(50)

For instance, the indices \( i \) and \( j \) could refer to the \( N \) dimensional representation of \( SU(N) \) (upper indices to the complex conjugate). The representations of this algebra can be studied by the method of superfields \cite{24}, in a superspace labelled by \( x , \overline{\phi}_a^i , \overline{\phi}_a^j \). A simpler technique has been suggested by Salam and Strathdee \cite{13}. Since the momentum operator commutes with \( Q_{\alpha}^i \), one can go to the rest frame

\[
P^\mu \rightarrow (m, 0, 0, 0).
\]

The algebra (50) becomes then simply an algebra of creation and destruction operators. As an example we give, for the case of \( SU(2) \), the multiplet corresponding to a left-handed superfield, satisfying

\[
\overline{\mathcal{D}}_{\alpha}^i \, S = 0 ,
\]

(51)

which can be taken to be a function \( S(z, \overline{\nu}_{\alpha i}) \) independent of \( \overline{\phi}_a^j \). The quantum numbers \((I,J) = (\)isospin,spin\) for the fields of the multiplet are \((0,0)\), \((\frac{1}{2},\frac{1}{2})\), \((1,0)\), \((0,1)\), \((\frac{1}{2},\frac{3}{2})\), \((0,0)\), as easily seen by expanding in \( \overline{\nu}_{\alpha i} \) or by using the technique of creation and destruction operators.
The symmetry described by (50), or other similar generalizations, seems well worth exploring, by any available techniques. Work along these lines is presently being done by several groups.

8. CONCLUSION

What is the future of supersymmetry? Three lines of development come immediately to mind. The first would use supersymmetry as a way of classifying hadronic states and their interactions, a kind of more general version of relativistic SU(6) connecting bosons and fermions and free of the theoretical contradictions which were the basic difficulty afflicting all relativistic versions of SU(6). The Lagrangian model described in Section 2 gives an extremely simplified version of this kind of theory. The inclusion of internal symmetries could be effected either by introducing them as an additional invariance group commuting with the supersymmetry or by enlarging the algebra as indicated in Section 7. Various enlargements of that kind are possible. For instance, the indices i, j occurring in (50) can be extended to include, besides the physical SU(N), a colour SU(N).

The second line of development is suggested by the gauge invariant model of Section 3. It is very tempting to interpret the supermultiplet (19) as containing the photon and the (electron) neutrino. The non-Abelian generalizations of the model of Section 3, given in Section 6, are the first step towards a description of weak, electromagnetic, and possibly strong interactions. Perhaps the spinors which are part of the same supermultiplets as the Yang-Mills vectors should be interpreted as leptons with the Yang-Mills vectors describing the photon and a number of intermediate bosons.

The third line of development could be the construction of a generalized (possibly renormalizable) theory of gravitation, as indicated at the end of Section 5.
REFERENCES


2) A. Neveu and J.H. Schwarz, Nuclear Phys. B31, 86 (1971);
   P. Ramond, Phys. Rev. D3, 2415 (1971);
   Y. Aharonov, A. Casher and L. Susskind, Phys. Letters 35B, 512 (1971);

3) F.A. Berezin and G.I. Katz, Mathemat. Sbornik (USSR) 82, 343 (1970),
   English Translation Vol. 11.

4) See e.g., L. O'Raifeartaigh, Phys. Rev. Letters 14, 575 (1965);


6) A conformal spinor is the same as the entity called a twistor by
   Penrose, and there are interesting similarities between the larger
   algebra of Ref. 1 and twistors. For twistor theory see, e.g.,


8) J. Iliopoulos and B. Zumino, Nuclear Phys. B75, 310 (1974);
   S. Ferrara, J. Iliopoulos and B. Zumino, Nuclear Phys. B77, 413 (1974);
   see also: H.S. Tsao, Brandeis University Preprint (1974).

9) W. Lang and J. Wess, University of Karlsruhe Preprint (1974), submitted
   to Nuclear Phys. B.

10) J. Wess and B. Zumino, CERN Preprint TH.1857 (1974), to be published in

    Letters B. Goldstone spinors were discussed earlier from the point
    of view of supersymmetry by:
    A. Salam and J. Strathdee, Phys. Letters 49B, 465 (1974); and by
    J. Iliopoulos and the author in the first paper of Ref. 8).


13) A. Salam and J. Strathdee, Trieste Preprint IC/74/16 (1974), to be
    published in Nuclear Phys. B.


15) S. Ferrara, J. Wess and B. Zumino, CERN Preprint TH.1863 (1974), to be
    published in Phys. Letters B.

16) A. Salam and J. Strathdee, Trieste Preprint IC/74/42 (1974), submitted
    to Phys. Rev.


19) A. Salam and J. Strathdee, Trieste Preprint IC/74/36 (1974).

21) The supercurrents (41) and (47) were worked out by the author in collaboration with W. Bardeen.

