CM-P00060494

Ref.TH.1907-CERN

DERIVATION OF THE EFFECTIVE POTENTIAL

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A B S T R A C T

A simple derivation of the effective potential for single and composite operators is given. Green's functions under general constraints are considered and compact expressions for derivatives of the effective potential are given.

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It is well known that the absolute minimum of the effective potential $V(\varphi_c)$ of a field theory determines the true ground state of the theory. This fact made $V(\varphi_c)$ very useful particularly in studies of spontaneous symmetry breaking. The potential $V(\varphi_c)$ has been calculated exactly in the one loop approximation by summing an infinite number of diagrams ^{1),2)}. Also Jackiw ³⁾ derived a general formula for it which has the advantage of summing all relevant graphs to a given order in the loop expansion, each order being represented by a finite number of graphs.

In these calculations $V(\varphi_c)$ was obtained after the introduction of the Legendre transformation ⁴⁾ $\Gamma[\Phi] = W[J] - \int d^4x J(x) \Phi(x)$, where $J(x)$ is an external source, $W[J]$ is defined in terms of the vacuum to vacuum amplitude $\langle 0^+ | 0^- \rangle_J$ in the presence of J , by the relation $\langle 0^+ | 0^- \rangle_J = e^{iW[J]}$, and $\Phi(x) \equiv \delta W[J] / \delta J(x)$. In the above definition of $\Gamma[\Phi]$, J is to be expressed in terms of Φ by inverting the relation which defines $\Phi(x)$. Therefore $\Gamma[\Phi]$ is an implicit function of $\Phi(x)$, and this fact makes the derivation of the explicit expression for $V(\varphi_c)$ rather complicated ³⁾.

We want in this paper to discuss a short-cut method of obtaining the explicit expression for the effective potential, without introducing external sources, but introducing instead δ -functions in the functional integrals to express constraints on the fields. In comparison with previous methods our derivation is simpler, direct and emphasizes the special role of the four-momentum zero mode of the field. To illustrate the power of our method we present a simple derivation of the effective potentials $V(\varphi_c)$ and $V(\varphi_c, G)$ as functions of the vacuum expectation value φ_c of the field, and the propagator G . We show the connection of this approach and the usual one by the method of stationary phase. Also we consider Green's functions under the constraint that the vacuum expectation values of some sets of fields are fixed. Using this result we finally give the expressions for the first and the second derivative of the effective potential $V(\varphi_c)$ with respect to the vacuum expectation value of the field, and we show that the functional differential equation for the effective action can be reduced to our expression for the first derivative of the effective potential.

Let $\varphi(x)$ be a scalar field. Following Goldstone ⁵⁾ we write

$$\varphi(x) = \varphi_0 + \varphi'(x) \quad (1)$$

where φ_0 is the zero four-momentum ($p_\mu = 0$) component of the field, and $\varphi'(x)$ is the rest part of the field, satisfying the relation

$$\int d^4x \varphi'(x) = 0 \quad (2)$$

The separation of the $\underline{p} = 0$ mode of the field is a basic feature of Bogoliubov's theory of superfluidity⁶⁾. Also special treatment of the $p_\mu = 0$ mode of a field has been considered by Goldstone in connection with field theories with superconductor solutions⁵⁾.

Let $\mathcal{L}(\varphi)$ be the Lagrangian of the theory. The effective potential $V(\varphi_c)$ can be regarded as the vacuum energy density in the vacuum state in which the expectation value of the field is φ_c ⁷⁾. Since in our case energy momentum conservation does not allow $\varphi'(x)$ to develop a vacuum expectation value, in the functional integral language the constraint $\langle 0|\varphi|0 \rangle = \varphi_c$ is equivalent to the introduction of $\delta(\varphi_0 - \varphi_c)$ in the functional integral. As we shall show later this leads to the correct definition of $V(\varphi_c)$. Thus we get

$$\begin{aligned} \langle 0|S|0 \rangle &\equiv e^{-i \underline{Q} V(\varphi_c)} \\ &= \text{const.} \int e^{i \int \mathcal{L}(\varphi) d^4x} \delta(\varphi_0 - \varphi_c) [d\varphi] \\ &= \text{const.} \int e^{i \int \mathcal{L}(\varphi_c + \varphi') d^4x} [d\varphi'] \end{aligned} \quad (3)$$

where the constant in front does not depend on φ_c , and $\underline{Q} \equiv \int d^4x$. Equation (3) is a compact expression for $V(\varphi_c)$.

To proceed in the explicit calculation of $V(\varphi_c)$ let us write

$$\mathcal{L}_{free}(\varphi', \varphi_c) = \frac{1}{2} \partial_\mu \varphi' \partial_\mu \varphi' - \frac{1}{2} B(\varphi_c) \varphi'^2 \quad (4)$$

$$\begin{aligned} \int \mathcal{L}_{int}(\varphi', \varphi_c) d^4x &\equiv \int d^4x \left[\mathcal{L}(\varphi' + \varphi_c) - \mathcal{L}_{free}(\varphi', \varphi_c) - \left(\frac{\partial \mathcal{L}}{\partial \varphi'} \right)_{\varphi'=0} \varphi' - \mathcal{L}(\varphi_c) \right] \\ &= \int d^4x \left[\mathcal{L}(\varphi' + \varphi_c) - \mathcal{L}_{free}(\varphi', \varphi_c) - \mathcal{L}(\varphi_c) \right] \end{aligned} \quad (5)$$

where $-\frac{1}{2}B(\varphi_c)$ is the coefficient of $(\varphi')^2$ in $\mathcal{L}(\varphi' + \varphi_c)$ and the last equality of Eq. (5) is a consequence of Eq. (2). Then Eq. (3) becomes

$$\begin{aligned} e^{-i \mathcal{O} V(\varphi_c)} &= \text{const } e^{i \mathcal{O} \mathcal{L}(\varphi_c)} \int e^{i \int \mathcal{L}_{free}(\varphi', \varphi_c) d^4x} [d\varphi'] \\ &\times \frac{\int e^{i \int d^4x [\mathcal{L}_{free}(\varphi', \varphi_c) + \mathcal{L}_{int}(\varphi', \varphi_c)]} [d\varphi']}{\int e^{i \int \mathcal{L}_{free}(\varphi', \varphi_c) d^4x} [d\varphi']} \end{aligned} \quad (6)$$

The first two factors on the right-hand side of Eq. (6) give the well-known contribution of the tree graphs and the one loop graphs, respectively, to the effective potential ^{1), 2)}. The last factor gives ⁸⁾

$$\frac{\int e^{i \int [\mathcal{L}_{free}(\varphi', \varphi_c) + \mathcal{L}_{int}(\varphi', \varphi_c)] d^4x} [d\varphi']}{\int e^{i \int \mathcal{L}_{free}(\varphi', \varphi_c) d^4x} [d\varphi']} = \left\langle T e^{i \int \mathcal{L}_{int}(\varphi', \varphi_c) d^4x} \right\rangle \quad (7)$$

The calculation of the last expression proceeds in the usual way, except that only one-particle irreducible diagrams are present, because the field φ' does not contain the $p_\mu = 0$ mode. We can also argue that the effective potential, as expressed by Eq. (3), gets contributions only from one-particle irreducible diagrams, because by definition $\varphi'(x)$ has zero vacuum expectation value. Therefore, the explicit expression for $V(\varphi_c)$ we have obtained in a simple way is identical to Jackiw's expression ³⁾.

For simplicity we have considered a theory containing a single scalar field. The generalization of our arguments to a theory containing a set of fields is straightforward.

To see the connection between our approach and the usual one, in which a Legendre transform is introduced, we replace $\delta(\varphi_0 - \varphi_c)$ in Eq. (3) by its integral representation. Remembering that $\varphi_0 = 1/\Omega \int d^4x \varphi(x)$ we get

$$\begin{aligned} e^{-i\Omega V(\varphi_c)} &= C \int_{-\infty}^{\infty} dJ \int [d\varphi] e^{i\int d^4x [\mathcal{L}(\varphi) + \varphi J] - i\varphi_c J \Omega} \\ &= C' \int_{-\infty}^{\infty} dJ e^{i(W[J] - \varphi_c J) \Omega} \end{aligned} \quad (8)$$

where C and C' are constants and $W[J]$ is the vacuum energy density with an opposite sign in the presence of a constant external source J . When $\Omega \rightarrow \infty$ the dominant contribution to the integral of Eq. (8) comes from the stationary point of the phase of the integrand. Therefore, we get (apart from additive constant terms)

$$V(\varphi_c) = (\varphi_c J - W'[J])_{\varphi_c} = \frac{\partial W'}{\partial J} \quad (9)$$

The above expression is the usual definition of the effective potential.

We can generalize our approach to calculate the effective potential $V(f_{ic})$ in the vacuum state in which the composite objects $f_i[\varphi(x)\psi(x)\bar{\psi}(x)\dots]$, $i = 1, \dots, m$, have the vacuum expectation values f_{ic} . The f_i 's can be arbitrary functions of the fields $\varphi(x), \psi(x), \bar{\psi}(x), \dots$. Proceeding as before we get

$$\begin{aligned} e^{-i\Omega V(f_{ic})} &= C \int e^{i\int d^4x \mathcal{L}(\varphi, \psi, \bar{\psi}, \dots)} \prod_i \delta(f_i - f_{ic}) [d\varphi][d\psi][d\bar{\psi}] \dots \\ &= C' \int \prod_i dJ_i \int e^{i\int d^4x [\mathcal{L} + \sum_i f_i J_i] - i\sum_i f_{ic} J_i \Omega} [d\varphi][d\psi][d\bar{\psi}] \dots \end{aligned}$$

$$= C' \int \prod_i dJ_i e^{i(W'[J_i] - \sum_i f_{ic} J_i) \underline{Q}} \quad (10)$$

where C, C' and C'' are constant factors, and $f_{i0}(\varphi, \psi, \bar{\psi}, \dots) = 1/\underline{Q} \int d^4x f_i(\varphi, \psi, \bar{\psi}, \dots)$. As before in the limit $\underline{Q} \rightarrow \infty$ the dominant contribution to the last integral of Eq. (10) comes from the stationary point of the phase of the integrand and we get the multiple Legendre transform

$$V(f_{ic}) = \left(\sum_i f_{ic} J_i - W'[J_i] \right)_{f_{ic} = \frac{\partial W'}{\partial J_i}} \quad (11)$$

The relations $f_{ic} = \delta W' / \delta J_i$ are invertible to express J_i 's in terms of f_{ic} 's if $W'[J_i]$ satisfies the condition

$$\det \left(\frac{\partial^2 W'}{\partial J_i \partial J_j} \right) \neq 0 \quad (12)$$

Equation (10) can be generalized to Green's functions. Consider an n -point connected Green's function $G(x_1, x_2, \dots, x_n, f_{1c}, \dots, f_{mc})$, which is the usual connected Green's function under the constraint that the vacuum expectation value of the function $f_i[\varphi(x) \dots \psi(x) \dots \bar{\psi}(x) \dots]$, $i = 1, 2, \dots, m$ is f_{ic} . We get

$$\begin{aligned} & G(x_1, x_2, \dots, x_n, f_{1c}, f_{2c}, \dots, f_{mc}) \\ &= \frac{\int \varphi(x_1) \varphi(x_2) \dots \psi(x_n) \dots \bar{\psi}(x_n) e^{i \int d^4x \mathcal{L}(\varphi, \psi, \bar{\psi}, \dots)} \prod_i \delta(f_{i0} - f_{ic}) [d\varphi] [d\psi] [d\bar{\psi}] \dots}{\int e^{i \int d^4x \mathcal{L}(\varphi, \psi, \bar{\psi}, \dots)} \prod_i \delta(f_{i0} - f_{ic}) [d\varphi] [d\psi] [d\bar{\psi}] \dots} \end{aligned} \quad (13)$$

Proceeding as above we obtain

$$\begin{aligned}
 & G(x_1, x_2, \dots, x_n, f_{ic}, f_{ic}, \dots, f_{mc}) \\
 &= \frac{\int G(x_1, x_2, \dots, x_n, J_1, J_2, \dots, J_m) e^{i(W[J_i] - \sum_{i=1}^m f_{ic} J_i)} dJ_1 dJ_2 \dots dJ_m}{\int e^{i(W[J_i] - \sum_{i=1}^m f_{ic} J_i)} dJ_1 dJ_2 \dots dJ_m} \quad (14) \\
 &= G(x_1, x_2, \dots, x_n, J_1, J_2, \dots, J_m) \Big|_{\frac{\partial W}{\partial J_i} = f_{ic}} \\
 &\equiv \langle 0 | \varphi(x_1) \varphi(x_2) \dots \psi(x_l) \psi(x_{l+1}) \dots \bar{\psi}(x_{n-1}) \bar{\psi}(x_n) | 0 \rangle_{J_i = \frac{\partial V(f_{ic})}{\partial f_{ic}}}
 \end{aligned}$$

where $G(x_1, x_2, \dots, x_n, J_1, J_2, \dots, J_m)$ is the Green's function in the presence of the external constant sources J_i , coupled to $f_i(\varphi, \dots, \psi, \dots, \bar{\psi}, \dots)$, and $W[J]$ is the vacuum energy density with an opposite sign again for constant sources J_i , coupled to $f_i(\varphi, \dots, \psi, \dots, \bar{\psi}, \dots)$. Equation (14) will be used later. The formula given by Eq. (14) for $f_i = \varphi$, $n = 2$ and $m = 1$ was essentially used by Goldstone⁵⁾.

A further generalization consists in introducing constraints on operators of the form $f_i[\varphi(x_1)\varphi(x_2)\dots\psi(x_l)\dots\bar{\psi}(x_n)]$. Consider the effective potential $V(G_{ic})$ under the constraints

$$\langle 0 | T f_i[\varphi(x_1)\varphi(x_2)\dots\psi(x_l)\dots\bar{\psi}(x_n)] | 0 \rangle = G_{ic}(x_1, x_2, \dots, x_l, x_n) \quad (15)$$

Previous arguments suggest that $V(G_{ic})$ is given by

$$\begin{aligned}
 e^{-i \int V(G_{ic})} &= C \int e^{i \int \mathcal{L}[\varphi(x), \psi(x), \bar{\psi}(x), \dots]} d^4x \\
 &\times \prod_i \prod_{x=x_i} \delta \left[\frac{1}{i} \int f_i(\varphi(x), \psi(x), \bar{\psi}(x)) d^4x - G_{ic} \right] [d\varphi][d\psi][d\bar{\psi}] \dots
 \end{aligned} \quad (16)$$

where $x = 1/n(x_1 + x_2 + \dots + x_n)$. Indeed stationary phase arguments imply that the expression $V(G_{ic})$ defined by Eq. (16) is the effective potential as usually defined.

As an application of Eq. (16) we shall calculate the effective potential under the constraints

$$\langle 0 | \varphi(x) | 0 \rangle = \varphi_c \quad (17)$$

$$\langle 0 | T \varphi'(x_1) \varphi'(x_2) | 0 \rangle = G(x_1 - x_2) \quad (18)$$

we get

$$\begin{aligned} & e^{-i \underline{Q} V(\varphi_c, G)} \\ &= C \int e^{i \int \mathcal{L}[\varphi(x)] dx} \delta\left[\frac{1}{\underline{Q}} \int \varphi(x) dx - \varphi_c\right] \prod_{x=x_1}^{x_2} \delta\left[\frac{1}{\underline{Q}} \int \varphi'(x) \varphi'(x_2) dx - G(x_1 - x_2)\right] [d\varphi] \\ &= C \int e^{i \int \mathcal{L}[\varphi'(x) + \varphi_c] dx} \prod_y \delta\left[\frac{1}{\underline{Q}} \int \varphi'(x + \frac{y}{2}) \varphi'(x - \frac{y}{2}) dx - G(y)\right] [d\varphi'] \quad (19) \\ &= C e^{i \underline{Q} \mathcal{L}(\varphi_c)} \int e^{i \int dx^4 [\mathcal{L}_{free}(\varphi', \varphi_c) + \mathcal{L}_{int}(\varphi', \varphi_c)]} \prod_y \delta\left[\frac{1}{\underline{Q}} \int \varphi'(x + \frac{y}{2}) \varphi'(x - \frac{y}{2}) dx - G(y)\right] [d\varphi'] \end{aligned}$$

where $\mathcal{L}_{int}$ is defined by Eq. (5) and C is a constant. If we write

$$\int dx^4 \mathcal{L}_{free}(\varphi', \varphi_c) = \frac{1}{2} \int \varphi'(x + \frac{y}{2}) i \underline{Q}^{-1}(y, \varphi_c) \varphi'(x - \frac{y}{2}) dy dx, \quad (20)$$

Equation (19) gives

$$V(\varphi_c, G) = -\mathcal{L}(\varphi_c) - \frac{i}{2} \int \mathcal{D}^{-1}(y, \varphi_c) G(y) d^4y + V_2(\varphi_c, G), \quad (21)$$

$$\begin{aligned} V_2(\varphi_c, G) &= -\frac{1}{i0} \ln \int e^{i \int \mathcal{L}_{int}(\varphi, \varphi_c) d^4x} \prod_y \delta \left[\frac{1}{0} \int \varphi(x+\frac{y}{2}) \varphi(x-\frac{y}{2}) d^4x - G(y) \right] [d\varphi] + C \\ &= -\frac{1}{i0} \ln \int [dJ] e^{-i0 \int G(y) J(y) d^4y} \int e^{i \int \varphi(x+\frac{y}{2}) \varphi(x-\frac{y}{2}) J(y) d^4x d^4y} e^{i \int \mathcal{L}_{int}(\varphi, \varphi_c) d^4x} [d\varphi] + C' \\ &= -\frac{1}{i0} \ln \int [dJ] e^{-i0 \int [G(k) J(k) - \frac{i}{2} \ln J(k)] \frac{d^4k}{(2\pi)^4}} \langle T e^{i \int \mathcal{L}_{int}(\varphi, \varphi_c) d^4x} \rangle + C'' \quad (22) \end{aligned}$$

where C' and C'' are constants, $G(k)$ and $J(k)$ are the Fourier transforms of $G(y)$ and $J(y)$, respectively, and the propagator is set equal to $i/2J(k)$. Apart from a constant additive term the phase $\int [G(k)J(k) - i/2 \ln J(k)] d^4k$ becomes real after the Wick rotation, because in that case causality implies that the propagator G becomes pure imaginary (in our convention). Therefore, as $0 \rightarrow \infty$ the method of stationary phase implies $G(-k) = i/2J(k)$ and the effective potential is given by

$$\begin{aligned} V(\varphi_c, G) &= -\mathcal{L}(\varphi_c) - \frac{i}{2} \int \mathcal{D}^{-1}(-k) G(k) \frac{d^4k}{(2\pi)^4} - \frac{i}{2} \int \ln G(k) \frac{d^4k}{(2\pi)^4} \\ &\quad - \frac{1}{i0} \langle T e^{i \int \mathcal{L}_{int}(\varphi, \varphi_c) d^4x} \rangle + \text{const.} \quad (23) \end{aligned}$$

where in the perturbation expansion of the time ordered product the propagator $G(k)$ should be used.

The presence of the δ -function removes the two-particle reducible graphs from $V_2(\varphi_c, G)$. To show this consider the diagram of Fig. 1, where the four lines are to be contracted. The amplitude for this graph is of the form

$$\begin{aligned}
 & \int \int \int \varphi'(x_1) \varphi'(x_2) F_1(x_1, x_2) d^4x_1 d^4x_2 \prod_y \delta \left[\frac{1}{\Omega} \int \varphi'(x + \frac{y}{2}) \varphi'(x - \frac{y}{2}) d^4x - G(y) \right] \int \int \varphi'(y_1) \varphi'(y_2) F_2(y_1, y_2) d^4y_1 d^4y_2 [d\varphi] \\
 & = \Omega^2 \int \int G(x) F_1(x) dx \int \int G(y) F_2(y) dy \prod_y \delta \left[\frac{1}{\Omega} \int \varphi'(x + \frac{y}{2}) \varphi'(x - \frac{y}{2}) d^4x - G(y) \right] [d\varphi]
 \end{aligned} \tag{24}$$

where $F_1(x)$ and $F_2(y)$ are the Feynman amplitudes for the left and the right hand side blobs of Fig. 1, which are calculated in the usual way with propagator G . The last expression of Eq. (24) is represented graphically by Fig. 2. Therefore, we see that the presence of the δ -function in $V_2(\varphi_c, G)$ prevents those contractions which give rise to two-particle reducible diagrams. This means that the effective potential $V(\varphi_c, G)$ is given by Eq. (23) where the two-particle reducible graphs of

$$\langle T e^{i \int \mathcal{L}_{int}(\varphi', \varphi_c) d^4x} \rangle$$

should be excluded. Of course this term does not contain one-particle reducible diagrams, since the field $\varphi'(x)$ is used. These results were expected since our definition of $V(\varphi_c, G)$ coincides with the usual one involving a double Legendre transform, and it is known that this transform eliminates the one- and the two-particle reducible diagrams.

It is easy to show that expression (23) with the modifications $\frac{1}{2} \rightarrow -1$, $\int d^4k \rightarrow \text{Tr} \int d^4k$, and $\varphi_c = 0$ gives the effective potential in the fermion case. Here the trace refers to the spinor indices.

Returning to $V(\varphi_c)$ we observe that Eq. (3) allows the calculation of the expressions $d^n V(\varphi_c) / d^n \varphi_c$. For example, we obtain

$$\frac{dV(\varphi_c)}{d\varphi_c} = - \frac{1}{\Omega} \frac{\int \int \frac{d\mathcal{L}(\varphi' + \varphi_c)}{d\varphi_c} d^4y e^{i \int \mathcal{L}(\varphi' + \varphi_c) d^4x} [d\varphi']}{\int e^{i \int \mathcal{L}(\varphi' + \varphi_c) d^4x} [d\varphi']}$$

$$\begin{aligned}
 &= - \left\langle \frac{d\mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c} \right\rangle \\
 &= -g_0(\varphi_c) - \sum_{n=2} g_n(\varphi_c) \langle \varphi'^n(0) \rangle
 \end{aligned} \tag{25}$$

where

$$g_m(\varphi_c) = \frac{1}{m!} \left. \frac{d^{m+1} \mathcal{L}(\varphi' + \varphi_c)}{d^{m+1} \varphi_c} \right|_{\varphi' = 0} \tag{26}$$

Also we get

$$\begin{aligned}
 \frac{d^2 V(\varphi_c)}{d^2 \varphi_c} &= - \left\langle \frac{d^2 \mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c^2} \right\rangle - i \int d^4 x \left\langle \frac{d\mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c} \frac{d\mathcal{L}[\varphi'(x) + \varphi_c]}{d\varphi_c} \right\rangle \\
 &\quad + i \mathcal{O} \left(\left\langle \frac{d\mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c} \right\rangle^2 \right) \\
 &= - \left\langle \frac{d^2 \mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c^2} \right\rangle - i \int d^4 x \left\langle \frac{d\mathcal{L}[\varphi'(0) + \varphi_c]}{d\varphi_c} \frac{d\mathcal{L}[\varphi'(x) + \varphi_c]}{d\varphi_c} \right\rangle_{\text{con.}}
 \end{aligned} \tag{27}$$

where in the last term of the above expression only the connected diagrams are taken. Equations (25) and (27) are compact expressions for $dV(\varphi_c)/d\varphi_c$ and $d^2 V(\varphi_c)/d^2 \varphi_c$, respectively.

As an example, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi - \frac{1}{2} \mu^2 \varphi^2 - \frac{1}{4} \lambda \varphi^4 \tag{28}$$

We get

$$\frac{dV(\varphi_c)}{d\varphi_c} = \mu^2 \langle \varphi \rangle + \lambda \langle \varphi^3 \rangle \tag{29}$$

$$\begin{aligned} \frac{d^2 V(\varphi_c)}{d\varphi_c^2} &= \mu^2 + 3\lambda \langle \varphi^2 \rangle - i\lambda^2 \left[\int d^4x \langle \varphi^3(x) \varphi^3(x) \rangle - \underline{0}(\langle \varphi^3 \rangle^2) \right] \\ &= \mu^2 + 3\lambda \langle \varphi^2 \rangle - i\lambda^2 \int d^4x \langle \varphi^3(x) \varphi^3(x) \rangle_{\text{con.}} \end{aligned} \quad (30)$$

where $\varphi = \varphi' + \varphi_c$. Using Eq. (14) we can write Eq. (29) in the form

$$\begin{aligned} \frac{dV(\varphi_c)}{d\varphi_c} &= \mu^2 \varphi_c + \lambda G(0,0,0, \frac{dV(\varphi_c)}{d\varphi_c}) \\ &= \mu^2 \varphi_c + \lambda \langle 0 | \varphi^3 | 0 \rangle_J = \frac{dV}{d\varphi_c} \end{aligned} \quad (31)$$

where $G[0,0,0, dV(\varphi_c)/d\varphi_c]$ and $\langle 0 | \varphi^3 | 0 \rangle_J = dV/d\varphi_c$ are defined in Eq. (14). The above equation is a consistency condition for $dV/d\varphi_c$, and can be used iteratively to get an expansion for $dV/d\varphi_c$. At the extrema of the effective potential we get from Eq. (31)

$$\mu^2 \varphi_c + \lambda \langle 0 | \varphi^3 | 0 \rangle_{J=0} = 0 \quad (32)$$

Therefore, if the field φ has non-zero (zero) vacuum expectation value then φ^3 should also have non-zero (zero) vacuum expectation value, with an opposite sign.

It can be shown that the functional differential equation for the effective action ^{9),10)} can be reduced to our Eq. (25). From Eqs. (14) and (25) we get

$$\begin{aligned} \frac{dV(\varphi_c)}{d\varphi_c} &= -\frac{1}{\underline{0}} \left\langle \int d^4x \frac{d\mathcal{L}[\varphi(x)]}{d\varphi(x)} \right\rangle \\ &= -\frac{1}{\underline{0}} \frac{\int [d\varphi] \int \frac{\partial \mathcal{L}[\varphi(x)]}{\partial \varphi(x)} d^4x e^{i \int d^4x \{ \mathcal{L}[\varphi(x)] + \zeta(x) \varphi(x) \}}}{\int [d\varphi] e^{i \int d^4x \{ \mathcal{L}[\varphi(x)] + \zeta(x) \varphi(x) \}}} \Bigg|_{\zeta = \frac{dV}{d\varphi_c}} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{\underline{\partial}} \frac{\int d^4x \left\langle T \left(\frac{\partial \mathcal{L}[\varphi(x)]}{\partial \varphi(x)} e^{i \int d^4x' \{ \mathcal{L}_{int}[\varphi(x')] + \zeta(x') \varphi(x') \}} \right) \right\rangle}{\left\langle T e^{i \int d^4x' \{ \mathcal{L}_{int}[\varphi(x')] + \zeta(x') \varphi(x') \}} \right\rangle} \Bigg|_{\zeta = \frac{dV}{d\varphi_c}} \\
 &= \frac{1}{\underline{\partial}} \int d^4x e^{-iW[\zeta]} \mathcal{F} \left(\frac{1}{i} \frac{\delta}{\delta \zeta(x)} \right) e^{iW[\zeta]} \Bigg|_{\zeta = \frac{dV}{d\varphi_c}}
 \end{aligned} \tag{33}$$

where the operator $\mathcal{F}(\varphi)$ is defined by

$$\mathcal{F}(\varphi) = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} \tag{34}$$

The last expression in (33) was obtained by integrating the equation preceding Eq. (7) of Ref. 9). Equation (33) becomes

$$\frac{dV(\varphi_c)}{d\varphi_c} = \frac{1}{\underline{\partial}} \int d^4x \mathcal{F} \left\{ \frac{\delta W[\zeta]}{\delta \zeta(x)} + \frac{1}{i} \frac{\delta}{\delta \zeta(x)} \right\} \Bigg|_{\zeta = \frac{dV}{d\varphi_c}} \tag{35}$$

Allowing for the moment φ_c to be x dependent we get

$$\varphi_c(x) = \frac{\delta W[\zeta]}{\delta \zeta(x)} \tag{36}$$

$$\frac{\delta}{\delta \zeta(x)} = \int d^4x' \frac{\delta \varphi_c(x')}{\delta \zeta(x)} \frac{\delta}{\delta \varphi_c(x')} = \int d^4x' G(x, x' | \varphi_c) \frac{\delta}{\delta \varphi_c(x')} \tag{37}$$

where $G(x, x' | \varphi_c)$ is the inverse of the propagator. From Eqs. (35)-(37) we get

$$\frac{dV(\varphi_c)}{d\varphi_c} = \mathcal{F} \left\{ \varphi_c(x) + \frac{1}{i} \int d^4x' G(x, x' | \varphi_c) \frac{\delta}{\delta \varphi_c(x')} \right\} \Bigg|_{\varphi_c = \text{const.}} \tag{38}$$

The basic functional differential equation (9) for the effective action $\Gamma[\varphi_c]$ of Ref. 10) is easily transformed to Eq. (38) by observing that ¹⁾

$$\Gamma[\varphi_c] = \int d^4x \left\{ -V[\varphi_c(x)] + \frac{1}{2} [\partial_\mu \varphi_c(x)]^2 Z[\varphi_c(x)] + \dots \right\} \quad (39)$$

which implies

$$\left. \frac{\delta \Gamma[\varphi_c]}{\delta \varphi_c(x)} \right|_{\varphi_c = \text{const.}} = - \frac{dV(\varphi_c)}{d\varphi_c} \quad (40)$$

We have checked the equivalence of Eq. (9) of Ref. 10) and our Eq. (25) for some specific interactions. Eq. (25) allows the explicit calculation of $dV(\varphi_c)/d\varphi_c$, while it is not clear how this can be done, if we use Eq. (9) of Ref. 10). Also this equation implies that $d^n V(\varphi_c)/d^n \varphi_c$ can be calculated by applying directly the Feynman rules, which does not seem to be the case for $V(\varphi_c)$.

After this work has been completed we received a paper by J.M. Cornwall, R. Jackiw and E. Tomboulis [MIT preprint No 419 (1974)], where expression (23) for $V(\varphi_c, G)$ is derived by a different method.

ACKNOWLEDGMENTS

We thank Professor S. Coleman for an interesting discussion. Also we thank Professor J. Goldstone for interesting discussions and reading the manuscript.

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FIGURE CAPTIONS

Fig. 1 Graphical representation of the left-hand side of Eq. (24).

Fig. 2 Graphical representation of the right-hand side of Eq. (24).

Fig. 1

Fig. 2