THE CALCULATION OF INDEFINITE
INTEGRALS OF B-SPLINES

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ABSTRACT

The purpose of this paper is to give a useful formula for the indefinite integral of a B-spline. We show that the formula does not introduce any redundant knots, and, when used in conjunction with Cox's algorithm (1972), provides a very convenient method for computing the indefinite integral.

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1. Introduction

Spline functions were introduced in 1946 by I.J. Schoenberg (1946) and are piecewise polynomials satisfying certain derivative continuity requirements. Recently they have become very popular as a class of approximating functions, mainly because of their flexibility and smoothness properties. At the same time, Schoenberg introduced what he called "Fundamental Splines" which have more recently been renamed as basic or B-splines. These spline functions are zero, except over a small number of intervals, consequently for computational purposes they give rise to matrices of band structure (Reid, 1967). They also possess the very valuable property that any spline function can be represented as a linear combination of B-splines.

It is sometimes necessary to evaluate the indefinite integral of a B-spline, e.g. if a function $f(x)$ is approximated by a linear combination of B-splines: $\sum_i \lambda_i M_i(x)$, then an estimate of the indefinite integral $\int_a^x f(\xi) d\xi$ requires knowledge of $\int_a^x M_i(\xi) d\xi$, for all i . The purpose of this paper is to give a useful formula for the indefinite integral of a B-spline. In Section 2 we define spline functions and B-splines. In Section 3 we derive the main result of this paper, as a theorem, and in Section 4 we discuss the implementation of the formula. Finally, as an example, we present a simple geometrical description of the formula.

2. Definitions

The function $S(x)$ is a spline of degree k if it has the following properties

- (i) There exists a sequence of real numbers, called knots

$$x_1 < x_2 < \dots < x_{N-1} < x_N \quad (2.1)$$

such that on each of the intervals $[x_i, x_{i+1}]$, $i=1, \dots, N-1$, $S(x)$ is a polynomial of degree at most k . Usually these piecewise polynomials are different on each interval.

(ii) The piecewise polynomials are joined together so that $S(x)$ and its first $k-1$ derivatives are continuous throughout the range of x .

B-splines

For the definition of B-splines we make use of the function

$$g_{k+1}(x; y) = (y-x)_+^k = \begin{cases} (y-x)^k & y \geq x \\ 0 & y < x \end{cases} \quad (2.2)$$

We now denote a sequence of $k+2$ knots by

$$x_i < x_{i+1} < \dots < x_{i+k} < x_{i+k+1} \quad (2.3)$$

and define the B-spline $M_{k+1,i}(x)$ to be the $(k+1)$ st divided difference of $g_{k+1}(x; y)$, with respect to the variable y , based on the knots $x_i, x_{i+1}, \dots, x_{i+k+1}$. This divided difference is conventionally denoted by $g_{k+1}(x; x_i, \dots, x_{i+k+1})$, (Steffenson, 1927), and hence

$$M_{k+1,i}(x) = g_{k+1}(x; x_i, \dots, x_{i+k+1}) \quad (2.4)$$

The function $M_{k+1,i}(x)$ is a spline of degree k , (or order $k+1$), with the $k+2$ knots $x_i, x_{i+1}, \dots, x_{i+k+1}$; it is non-zero only over the interval (x_i, x_{i+k+1}) for the following reasons. By definition $M_{k+1,i}(x) = 0$, for $x \geq x_{i+k+1}$ and for $x \leq x_i$, $M_{k+1,i}(x)$ is the $(k+1)$ st divided difference of a polynomial of degree k , and therefore vanishes identically.

The notation $M_{k+1,i}(x)$ for a B-spline of degree k is now standard. However, to simplify subscripts we prefer to work, for the remainder of this paper, with B-splines of degree $k-1$, i.e. $M_{k,i}(x)$. Further we use the notation $g_{k+1}(x; x_i, \dots, x_{i+p})$ for the p th divided difference of the

function (2.2), based on the knots $x_i, \dots, x_{i+p}$.

3. The Indefinite Integral of a B-spline

In this section we prove that the formula (3.2) is true for the indefinite integral of a B-spline, which is the main result of this paper. The method of proof requires the sequence of knots to continue beyond x_{i+k} , if necessary by the introduction of artificial knots.

Theorem 1

The indefinite integral of the B-spline $M_{k,i}(x)$: $\int_{x_i}^x M_{k,i}(\xi) d\xi$, for all x in the interval $[x_i, x_{i+k})$ is given by the formula

$$\begin{aligned} \int_{x_i}^x M_{k,i}(\xi) d\xi = k^{-1} & \left\{ \sum_{r=0}^{q-1} (x - x_{i+r}) M_{k-r, i+r}(x) \right. \\ & + \sum_{r=0}^{q-1} (x_{i+k+1+r} - x) M_{k+1+r-q, i+q}(x) \Big\} \\ & + \int_{x_i}^x M_{k, i+q}(\xi) d\xi \end{aligned} \quad (3.1)$$

for any integer q in the range $1 \leq q \leq k-1$. In particular if we let q be the integer j defined by the inequalities $x_{i+j-1} \leq x < x_{i+j}$ then

$$\int_{x_i}^x M_{k,i}(\xi) d\xi = k^{-1} \sum_{r=0}^{j-1} (x - x_{i+r}) M_{k-r, i+r}(x). \quad (3.2)$$

Proof

We prove by induction that equation (3.1) is true. Consider first the case when $q=1$.

Using a property of divided differences we can write

$$M_{k+1,i}(x) = \frac{g_{k+1}(x; x_{i+1}, \dots, x_{i+k+1}) - g_{k+1}(x; x_i, \dots, x_{i+k})}{x_{i+k+1} - x_i} \quad (3.3)$$

Using (2.2) we differentiate (3.3), with respect to x , and obtain the formula

$$\frac{d}{dx} \cdot M_{k+1,i}(x) = \left(\frac{k}{x_{i+k+1} - x_i} \right) \left(-g_k(x; x_{i+1}, \dots, x_{i+k+1}) + g_k(x; x_i, \dots, x_{i+k}) \right) \quad (3.4)$$

$$= \left(\frac{k}{x_{i+k+1} - x_i} \right) \left(M_{k,i}(x) - M_{k,i+1}(x) \right) \quad (3.5)$$

Integrating both sides with respect to x , we obtain

$$M_{k+1,i}(x) = \left(\frac{k}{x_{i+k+1} - x_i} \right) \left(\int_{x_i}^x M_{k,i}(\xi) d\xi - \int_{x_i}^x M_{k,i+1}(\xi) d\xi \right) \quad (3.6)$$

and rearrangement gives

$$\int_{x_i}^x M_{k,i}(\xi) d\xi = \left(\frac{x_{i+k+1} - x_i}{k} \right) M_{k+1,i}(x) + \int_{x_i}^x M_{k,i+1}(\xi) d\xi \quad (3.7)$$

We now make use of a slightly modified version of the recurrence relation (De Boor 1972) to replace $M_{k+1,i}(x)$ by the expression

$$M_{k+1,i}(x) = \frac{(x - x_i) M_{k,i}(x) + (x_{i+k+1} - x) M_{k,i+1}(x)}{x_{i+k+1} - x_i} \quad (3.8).$$

Thus from (3.7) we obtain the equation

$$\int_{x_i}^x M_{k,i}(\xi) d\xi = k^{-1} \left[(x - x_i) M_{k,i}(x) + (x_{i+k+1} - x) M_{k,i+1}(x) \right] + \int_{x_i}^x M_{k,i+1}(\xi) d\xi \quad (3.9)$$

which proves equation (3.1) in the case $q=1$.

We now assume that equation (3.1) is valid for $q=s$ and prove that equation (3.1) is true for $q=s+1$. Hence under the inductive hypothesis we have, from (3.1), the equation

$$\begin{aligned}
 \int_{x_i}^x M_{k,i}(\xi) d\xi &= k^{-1} \left\{ \sum_{r=0}^{s-1} (x-x_{i+r}) M_{k-r,i+r}(x) \right. \\
 &\quad \left. \sum_{r=0}^{s-1} (x_{i+k+1+r}-x) M_{k+1+r-s,i+s}(x) \right\} \\
 &\quad + \int_{x_i}^x M_{k,i+s}(\xi) d\xi
 \end{aligned} \tag{3.10}$$

Except for the lower limits of integration, we replace i by $i+s$ in (3.9), and obtain the expression

$$\begin{aligned}
 \int_{x_i}^x M_{k,i+s}(\xi) d\xi &= k^{-1} \left[(x-x_{i+s}) M_{k,i+s}(x) \right. \\
 &\quad \left. + (x_{i+k+s+1}-x) M_{k,i+s+1}(x) \right] \\
 &\quad + \int_{x_i}^x M_{k,i+s+1}(\xi) d\xi
 \end{aligned} \tag{3.11}$$

which will replace the integral in (3.10). Moreover we obtain a sum to replace the second sum of equation (3.10) in the following way.

Equation (3.8) gives the identity

$$\begin{aligned}
 (x_{i+k+1+r}-x_{i+s}) M_{k+1+r-s,i+s}(x) &= (x-x_{i+s}) M_{k+r-s,i+s}(x) \\
 &\quad + (x_{i+k+1+r}-x) M_{k+r-s,i+s+1}(x)
 \end{aligned} \tag{3.12}$$

which can be rearranged in the form

$$(x_{i+k+l+r}-x) M_{k+l+r-s,i+s}(x) = (x-x_{i+s}) \left[M_{k+r-s,i+s}(x) - M_{k+l+r-s,i+s}(x) \right] \\ + (x_{i+k+l+r}-x) M_{k+r-s,i+s+1}(x) \quad (3.13)$$

We now sum from $r=0$ to $s-1$, noting the cancellation between the terms in square brackets, to obtain the equation

$$\sum_{r=0}^{s-1} (x_{i+k+l+r}-x) M_{k+l+r-s,i+s}(x) = (x-x_{i+s}) M_{k-s,i+s}(x) \\ - (x-x_{i+s}) M_{k,i+s}(x) + \sum_{r=0}^{s-1} (x_{i+k+l+r}-x) M_{k+r-s,i+s+1}(x) \quad (3.14)$$

Expressions (3.11) and (3.14) are substituted in equation (3.10) to obtain

$$\int_{x_i}^x M_{k,i}(\xi) d\xi = k^{-1} \left\{ \sum_{r=0}^{s-1} (x-x_{i+r}) M_{k-r,i+r}(x) \right. \\ + (x-x_{i+s}) M_{k-s,i+s}(x) - (x-x_{i+s}) M_{k,i+s}(x) \\ + \sum_{r=0}^{s-1} (x_{i+k+l+r}-x) M_{k+r-s,i+s+1}(x) \\ \left. + (x-x_{i+s}) M_{k,i+s}(x) + (x_{i+k+s+1}-x) M_{k,i+s+1}(x) \right\} \\ + \int_{x_i}^x M_{k,i+s+1}(\xi) d\xi \quad (3.15)$$

$$\begin{aligned}
& -k^{-1} \left\{ \sum_{r=0}^s (x - x_{i+r}) M_{k-r, i+r}(x) \right. \\
& \quad \left. + \sum_{r=0}^s (x_{i+k+1+r} - x) M_{k+r-s, i+s+1}(x) \right\} \\
& \quad + \int_{x_i}^x M_{k, i+s+1}(\xi) d\xi \quad (3.16)
\end{aligned}$$

which proves equation (3.1) for $a=s+1$. Consequently, since the theorem is true for $q=1$, we have proved from our inductive hypothesis that equation (3.1) holds for any q in the range $1 \leq q \leq k-1$. In particular if we let q be the integer j defined by the inequalities $x_{i+j-1} \leq x < x_{i+j}$ then since the following equations hold

$$M_{k, i+j}(x) = 0 = M_{k+1+r-j, i+j}(x), \quad r=0, \dots, j-1$$

we deduce from (3.1) that

$$\int_{x_i}^x M_{k, i}(\xi) d\xi = k^{-1} \sum_{r=0}^{j-1} (x - x_{i+r}) M_{k-r, i+r}(x)$$

which proves equation (3.2).

This concludes the proof of Theorem 1.

4. Discussion

We first tried to perform the indefinite integral $\int_{x_i}^x M_{k, i}(\xi) d\xi$ by using formulae (3.7). However this has the disadvantage that the term $(x_{i+k+1} - x_i) M_{k+1, i}(x)$, introduces the point x_{i+k+1} , which is outside the interval $[x_i, x_{i+k}]$. However, by using a slightly modified version of the recurrence relation (de Boor, 1972), we have obtained formula (3.2), which does not introduce any extra points, depending only on the knots $x_i, x_{i+1}, \dots, x_{i+k}$.

To compute the indefinite integral $\int_{x_1}^x M_{k,i}(\xi) d\xi$, using formula (3.2), it is necessary to calculate the j B-splines $\{M_{k-r,i+r}(x)\}$, $r=0,\dots,j-1$, evaluated at the point x . We can obtain all of these calculations, very easily, by one application of the algorithm proposed by Cox (1972). Cox has shown this algorithm to be numerically stable for evaluating B-splines. Moreover since $(x-x_{i+r})$ and $M_{k-r,i+r}(x)$ are positive for all $r=0,\dots,j-1$, there is no cancellation in evaluating the right hand side of (3.2). Consequently, when used in conjunction with Cox's algorithm, formula (3.2) provides a very efficient method for obtaining the indefinite integral $\int_{x_1}^x M_{k,i}(\xi) d\xi$.

Another interesting feature of formula (3.2) is that it has a simple geometric description. We show this by means of an example. Suppose we wish to evaluate the indefinite integral $\int_{x_1}^x M_{3,1}(\xi) d\xi$, for a value of x in the interval $(x_1, x_4]$. We will consider the three possibilities:

- (i) $x_1 < x < x_2$
- (ii) $x_2 \leq x < x_3$ and
- (iii) $x_3 \leq x \leq x_4$.

(i) $x_1 < x < x_2$

From (3.2) we have that

$$\int_{x_1}^x M_{3,1}(\xi) d\xi = \frac{1}{3} (x-x_1) M_{3,1}(x) .$$

The term $(x-x_1) M_{3,1}(x)$ is the area of the rectangle in figure 1, and the integral $\int_{x_1}^x M_{3,1}(\xi) d\xi$ is $1/3$ of this area.

(ii) $x_2 \leq x < x_3$

From (3.2) we have that

$$\int_{x_1}^x M_{3,1}(\xi) d\xi = \frac{1}{3} \left[(x-x_1) M_{3,1}(x) + (x-x_2) M_{2,2}(x) \right] .$$

The terms $(x-x_1) M_{3,1}(x)$ and $(x-x_2) M_{2,2}(x)$ are the areas of the rectangles in figure 2, shaded with vertical and horizontal lines respectively. The integral $\int_{x_1}^x M_{3,1}(\xi) d\xi$ is $1/3$ of the sum of these shaded areas.

(iii) $x_3 \leq x < x_4$

From (3.2) we have that

$$\int_{x_1}^x M_{3,1}(\xi) d\xi = \frac{1}{3} \left[(x-x_1) M_{3,1}(x) + (x-x_2) M_{2,2}(x) + (x-x_3) M_{1,3}(x) \right] .$$

The terms $(x-x_1) M_{3,1}(x)$, $(x-x_2) M_{2,2}(x)$, and $(x-x_3) M_{1,3}(x)$ are the areas of the rectangles in figure 3, shaded with vertical, horizontal and diagonal lines respectively. The integral $\int_{x_1}^x M_{3,1}(\xi) d\xi$ is $1/3$ of the sum of these shaded areas. The maximum heights of $M_{2,2}(x)$ and $M_{1,3}(x)$ are equal because $(x_4-x_3) = (x_3-x_2)$. Note that similar geometric descriptions hold for all values of k .

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References

- Cox, M.G. (1972). "The Numerical Evaluation of B-splines". J. Inst. Maths. Applics. 10, 134-149.
- De Boor, C. (1972). "On calculating with B-splines". J. App. Theory, 6, 50-62.
- Reid, J.K. (1967). "A note on the least squares solution of a band system of linear equations by Householder reductions". Comp. J. 10, 188-189.
- Schoenberg, I.J. (1946). "Contributions to the problem of approximation of equidistant data by analytic functions". Q. Appl. Maths. 4, 45-99.
- Steffenson, J.F. (1927). "Interpolation". Chelsea Publishing Co. New York.

FIGURE 1

FIGURE 2

FIGURE 3