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ELECTRON AND MUON $(g-2)/2$ FROM VACUUM POLARIZATION INSERTIONS

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A B S T R A C T

We introduce a new method of calculating the lepton anomalous magnetic moment, combining the features of the dispersive and the mass operator approach.

We apply it to the study of the contributions to the electron and muon anomalies coming from QED vertex graphs containing vacuum polarization insertions. Complete analytic results up to sixth order are given, together with partial results for the muon eight-order mass-dependent contributions.

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1. INTRODUCTION

The existing measurements of the anomalous magnetic moment (anomaly) of the electron are already sufficiently accurate to test the size of the sixth-order (α^3) electrodynamic (QED) corrections and a next generation of experiments is planned to go even further. For the muon anomaly, the measurements now in preparation should be accurate enough to observe sixth-order electrodynamic corrections as well as strong interactions effects. [A thorough discussion of the experimental and theoretical situation, as well as full reference to previous works, can be found in Rich and Wesley¹⁾ and Lautrup et al.²⁾ respectively.]

Complete numerical evaluations of the α^3 purely electrodynamic corrections exist for both the electron and the muon: all such calculations are however affected by numerical uncertainty and furthermore some discrepancies exist among them [besides Ref. 2, see Proc. Third Colloquium on Advanced Computing Methods in Theoretical Physics³⁾ for more recent developments.] For a better use of the planned high-precision experiments it would be highly desirable to rely on a fully analytic evaluation of sixth-order electron and muon anomalies, no matter how difficult this undertaking seems to be.

This paper is a step in this direction: in it we describe the analytic calculation of the contributions to the sixth-order electron and muon g-factors coming from all the electrodynamic contributions which contain vacuum polarization (vac. pol.) effects. They constitute only a part of the whole α^3 electrodynamic corrections; the reason why we consider vac. pol. contributions separately is only technical, no physical meaning being attached to them alone. Let us recall, however, that in the muon case this class of contributions gives, together with the not yet analytically computed light-light corrections due to an electron loop, the full "mass-dependent" sixth-order anomaly.

From a technical point of view, the analytic calculations are partly a matter of efficient algebraic organization (each step is to be made in such a way as to prevent uncontrolled explosion of the number of terms), but mainly of ability in finding a possible way to perform the multiple integrations occurring in the relevant Feynman graphs. Typically, after some trivial integrations, one is faced with multiple integrands containing a number of different square roots of quadratic forms, such that the rationalization of these integrands by means of suitable changes of variables is usually not trivial -- and often impossible.

Various analytic methods have been developed and successfully applied to the calculation of some partial contributions to the sixth-order radiative corrections of lepton anomalies, such as mass operator approach⁴⁾, hyperspherical integrations⁵⁾, and dispersion relations. As in our previous works, we use in this paper dispersive

techniques. This method, in the traditional version, consists in using the analytic properties of the Pauli form factor $F_2(t)$ to write its real part as Hilbert's transform of the imaginary part, the anomaly being $F_2(0)$, the value of $F_2(t)$ at $t = 0$. The imaginary part, in turn, is obtained from any considered vertex graph by means of the Cutkosky cutting rules, i.e. by replacing some of the Feynman propagators by the corresponding mass shell δ -functions. In this way, it turns out that the resulting expressions are easier to integrate on the loop momenta than the complete Feynman graph expressions and the number of integrations is rapidly reduced. The integration variables, furthermore, have a direct physical meaning, namely energies and angles of physical particles' phase spaces, so that, in dealing with them, one can rely not only on mathematics but also on analogy with other cases and physical intuition. All this can be of great help in performing the calculations. A general description of the method, together with its application to the study of the fourth order vertex, is given in Barbieri et al.⁶⁾. It was also used, for the analytic evaluation of some simple sixth-order contributions⁷⁾.

This dispersive method, however, has a serious drawback: in order to compute the anomaly $F_2(0)$, namely a purely static quantity, one has to know the full imaginary part of the magnetic moment form factor $\text{Im } F_2(t)$ for any value of the momentum transfer t . It is well known, on the other hand, that, given the Feynman amplitude, say $\bar{u}(p_2)M_\mu[p = \frac{1}{2}(p_1 + p_2), \Delta = p_1 - p_2]u(p_2)$ corresponding to an electromagnetic vertex diagram, its contribution to the anomaly depends only on the amplitude $M_\mu(p, 0)$ and on its first derivative with respect to Δ , $\partial M_\mu(p, \Delta)/\partial \Delta_\nu$ at zero external photon four-momentum Δ and $p = p_1 = p_2$. To deal with $M_\mu(p, 0)$ and $\partial M_\mu(p, 0)/\partial \Delta_\nu$ means to deal with amplitudes depending on one external momentum only, p , which have therefore the structure of self-mass amplitudes, not with vertex amplitudes dependent on two-momenta. This feature is, for example, embedded in the mass operator approach [reference to Schwinger's original work can be found in Ref. 4], where one deals with the propagation of a lepton in an external static magnetic field H , and its self energy is evaluated to first order in H .

Starting from these observations, we have tried to reformulate the dispersive approach, avoiding the treatment of vertex amplitudes, but dealing only with self-mass-like amplitudes from the very beginning. This turns out to be feasible, even in a rather straightforward way. Given the original vertex amplitude $M_\mu(p, \Delta)$ we first take the static limit to get the anomaly in terms of $M_\mu(p, 0)$ and $\partial M_\mu(p, 0)/\partial \Delta_\nu$. Then we look for a function of the scalar variable p^2 only, therefore having the structure of a self-mass amplitude, such that it gives the anomaly when computed at $p^2 = -M^2$ (the lepton squared mass), and satisfies unsubtracted dispersion relations with the discontinuity given by the cutting rules applied

to the corresponding self-mass-like graph. Starting from this point the calculation proceeds as usual. In particular, all the angular integrations are straightforward in the p rest frame, because no external direction is present, and they do not generate any new square roots (the dangerous obstacles in this game!) besides the usual ones appearing in working out the phase spaces. This is, to some extent, the key point that allows one to overcome the difficulties one would encounter in the usual dispersive approach for the vertex form factor.

The paper is organized as follows. In Section 2 the notations are established and all the main results are given. They concern the contribution of the lepton anomalies from all the vertex diagrams in fourth and sixth order containing vacuum polarization subgraphs, and some partial contributions from a set of eighth-order diagrams. The main ingredient to get all these results is the analytic calculation of the contributions to the anomaly, $K^{(2)}(a/M^2)$, $K^{(4)}(a/M^2)$ in our notation, coming from the second- and fourth-order vertex diagrams with one photon propagator $-i\delta_{\mu\nu}/k^2$ replaced in all possible ways by a "massive photon" propagator $-i\delta_{\mu\nu}/(k^2 + a)$. This is done in Section 3 after a detailed description of the new dispersive method we use. In Section 4 the results for the electron and muon anomalies are derived. The most technical points in the calculation of the fourth-order function $K^{(4)}(a/M^2)$ are given in Section 5.

2. NOTATIONS AND MAIN RESULTS

2.1 Notations and previous results for fourth- and sixth-order anomalies

For the sake of a plain introduction of the notation and of the results obtained in this paper, let us consider first the simple case of the fourth-order graph, Fig. 1a, corresponding to the insertion of second-order vac. pol., Fig. 1b, into the second-order vertex, Fig. 1c. Note that we indicate by M the mass of the lepton whose anomaly is concerned and by m the mass of the lepton responsible for the vac. pol. Both M and m can take the values of the masses m_e , m_μ of the known leptons, electron and muon, for a total of four possibilities, Fig. 2.

The vac. pol. tensor $\Pi_{\mu\nu}(k)$ can, in general, be written as

$$\Pi_{\mu\nu} = i(k_\mu k_\nu - k^2 \delta_{\mu\nu}) \Pi(-k^2) \quad (2.1)$$

where the invariant function $\Pi(-k^2)$ in the usual perturbation theory of QED can be expanded as

$$\Pi(-k^2) = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \Pi^{(2n)}(-k^2), \quad (2.2)$$

The function $\Pi(-k^2)$, and thus the $\Pi^{(2n)}(-k^2)$'s, all satisfy subtracted dispersion relations. For the $\Pi^{(2)}(-k^2)$ of Fig. 1b, for instance, we have

$$\Pi^{(2)}(-k^2) = -k^2 \int_{4m^2}^{\infty} \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m^2}\right) \frac{1}{k^2 + a} . \quad (2.3)$$

We write the spectral function as $\phi^{(2)}(a/m^2)$ to emphasize that it depends only on the dimensionless variable (a/m^2) . Its well-known explicit expression is

$$\varphi^{(2)}(b) = \frac{1}{3} \sqrt{1 - \frac{4}{b}} \frac{b+2}{b} . \quad (2.4)$$

The graph of Fig. 1a is obtained by replacing the photon propagator $-i\delta_{\mu\nu}/k^2$ in Fig. 1c by

$$-i \frac{\delta_{\mu\rho}}{k^2} i (k_\rho k_\sigma - k^2 \delta_{\rho\sigma}) \Pi^{(2)}(-k^2) (-i) \frac{\delta_{\sigma\nu}}{k^2} \quad (2.5)$$

Dropping the gauge term $k_\rho k_\sigma$, which anyhow cancels out in the final result, the replacement becomes

$$-i \frac{\delta_{\mu\nu}}{k^2} \longrightarrow \int_{4m^2}^{\infty} \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m^2}\right) \frac{-i\delta_{\mu\nu}}{k^2 + a} . \quad (2.6)$$

Let us call $(\alpha/\pi)^2 A^{(2,2)}(m/M)$ the fourth-order contribution from Fig. 1a, the superscripts referring to the insertion of the second-order vac. pol. into the second-order vertex. One has

$$A^{(2,2)}\left(\frac{m}{M}\right) = \int_{4m^2}^{\infty} \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m^2}\right) K^{(2)}\left(\frac{a}{M^2}\right) , \quad (2.7)$$

where $(\alpha/\pi)K^{(2)}(a/M^2)$ can be regarded as the contribution to the anomaly from Fig. 1d, the dotted line representing the propagator $-i\delta_{\mu\nu}/(k^2 + a)$ of a "heavy photon" of squared mass a . In terms of $A^{(2,2)}(m/M)$, the fourth-order contribution to the electron anomaly $a_e(\text{vac. pol.})^{4\text{th}}$, corresponding to the graphs of Figs. 2a and b, is

$$a_e(\text{vac. pol.})^{4\text{th}} = \left(\frac{\alpha}{\pi}\right)^2 \left[A^{(2,2)}(1) + A^{(2,2)}\left(\frac{m_e}{m_e}\right) \right] \quad (2.8)$$

Similarly, the fourth-order muon anomaly $a_\mu(\text{vac. pol.})^{4\text{th}}$ from the graphs of Figs. 2c and d is

$$a_\mu(\text{vac. pol.})^{4\text{th}} = \left(\frac{\alpha}{\pi}\right)^2 \left[A^{(2,2)}(1) + A^{(2,2)}\left(\frac{m_e}{m_\mu}\right) \right] . \quad (2.9)$$

All the quantities introduced till now are known. For $K^{(2)}(b)$ see Ref. 2 and Section 3. $A^{(2,2)}(m/M)$ is also known exactly⁸⁾, as a function of m/M . The values of interest here are²⁾

$$A^{(2,2)}(1) = \frac{119}{36} - 2\zeta(2) , \quad \zeta(2) = \frac{\pi^2}{6} , \quad (2.10)$$

$$A^{(2,1)}\left(\frac{m_\mu}{m_e}\right) \simeq \frac{1}{45} \left(\frac{m_e}{m_\mu}\right)^2, \quad (2.11)$$

$$A^{(2,2)}\left(\frac{m_e}{m_\mu}\right) \simeq \frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36} + \frac{3}{2} \zeta(2) \frac{m_e}{m_\mu}. \quad (2.12)$$

As $m_e \ll m_\mu$, terms of higher order in m_e/m_μ are ignored on the right-hand sides of Eqs. (2.11) and (2.12) [and will be ignored in the following]. Due to the smallness of $A(m_\mu/m_e)$ instead of Eq. (2.9) it is customary to write

$$\begin{aligned} a_\mu &= a_e + (a_\mu - a_e) \\ (a_\mu - a_e)^{4k} &= \left(\frac{\alpha}{\pi}\right)^{2k} A^{(2,2)}\left(\frac{m_e}{m_\mu}\right) = \left(\frac{\alpha}{\pi}\right)^{2k} \left[\frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36} + \frac{3}{2} \zeta(2) \frac{m_e}{m_\mu} \right]^{2k} \end{aligned} \quad (2.13)$$

Generalization of the above formulae to sixth-order anomalies is straightforward, with a minor complication in the notation in order to account properly for the "mixed" vac. pol. graphs. Let us define the following quantities:

- i) $\phi^{(4)}(a/m^2)$, the spectral function of the fourth-order vac. pol. amplitude due to the graphs of Fig. 3, known from the classical work of Källén and Sabry⁹⁾ (the calculation has been repeated in Ref. 10, with a different approach to infrared divergences);
- ii) $(\alpha/\pi)^2 L^{(4)}(a/M^2)$, corresponding to one half of the contribution to the anomaly coming from the graphs of Fig. 4, which are obtained by replacing in all possible ways, in the fourth-order vertex graphs without vac. pol. insertions, one of the photon propagators $-i\delta_{\mu\nu}/k^2$ by a "massive photon" propagator $-i\delta_{\mu\nu}/(k^2 + a)$;
- iii) $(\alpha/\pi)^2 K^{(4)}(a/M^2)$, corresponding to one half of the anomaly from all the graphs of Figs. 4 and 5 together. The graphs of Figs. 4 and 5 are numbered as in Ref. 2. The normalization is chosen in order to have

$$K^{(4)}(0) = \frac{197}{144} + \frac{1}{2} \zeta(2) + \frac{3}{4} \zeta(3) - 3 \zeta(2) \log 2, \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}, \quad (2.14)$$

which is the classical value of the fourth-order anomaly¹¹⁾. The need for introducing a notation which distinguishes between the graphs of Fig. 4 and those of Fig. 5 comes from the presence of the "mixed" vac. pol. graphs, drawn in Fig. 8. No such problem exists for the fourth-order vertex diagrams, and only for convenience of later use we define $L^{(2)}(a/M^2) = K^{(2)}(a/M^2)$.

In terms of the above quantities, we define further ($p, q = 2, 4$),

$$\begin{aligned} A^{(p,q)}\left(\frac{m}{M}\right) &= \frac{1}{2} \int_{4m^2}^{\infty} \frac{da}{a} \varphi^{(p)}\left(\frac{a}{m^2}\right) L^{(q)}\left(\frac{a}{M^2}\right), \\ B^{(p,q)}\left(\frac{m}{M}\right) &= \frac{1}{2} \int_{4m^2}^{\infty} \frac{da}{a} \varphi^{(p)}\left(\frac{a}{m^2}\right) K^{(q)}\left(\frac{a}{M^2}\right), \end{aligned} \quad (2.15)$$

so that, by generalizing the replacement rule (2.6), the whole sixth-order anomalies $a_e(\text{vac. pol.})^{6\text{th}}$, $a_\mu(\text{vac. pol.})^{6\text{th}}$ of the electron and the muon, due to all possible vac. pol. insertions, are

$$\begin{aligned} a_e(\text{vac. pol.})^{6\text{th}} &= \left(\frac{\alpha}{\pi}\right)^3 \left[A^{(4,2)}(1) + A^{(2,4)}(1) + A^{(4,2)}\left(\frac{m_\mu}{m_e}\right) + B^{(2,4)}\left(\frac{m_\mu}{m_e}\right) \right], \\ a_\mu(\text{vac. pol.})^{6\text{th}} &= \left(\frac{\alpha}{\pi}\right)^3 \left[A^{(4,2)}(1) + A^{(2,4)}(1) + A^{(4,2)}\left(\frac{m_e}{m_\mu}\right) + B^{(2,4)}\left(\frac{m_e}{m_\mu}\right) \right]. \end{aligned} \quad (2.16)$$

Again $A^{(4,2)}(m_\mu/m_e)$ and $B^{(2,4)}(m_\mu/m_e)$ are very small, so that one can write

$$(a_\mu - a_e)_{\text{vac. pol.}}^{6\text{th}} \simeq \left(\frac{\alpha}{\pi}\right)^3 \left[A^{(4,2)}\left(\frac{m_e}{m_\mu}\right) + B^{(2,4)}\left(\frac{m_e}{m_\mu}\right) \right]. \quad (2.17)$$

In these equations $A^{(4,2)}(1)$ accounts for the contribution coming from the graphs of Fig. 6 with $m = M$, which is already known in the literature¹²⁾

$$\begin{aligned} A^{(4,2)}(1) &= \frac{269}{81} - \frac{434}{135} \zeta(2) + \frac{61}{18} \zeta(3) - \frac{14}{15} \zeta^2(2) + \frac{32}{3} a_4 - \frac{8}{3} \zeta(2) \log 2 \\ &\quad - \frac{8}{3} \zeta(2) \log^2 2 + \frac{4}{9} \log^4 2, \quad a_4 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^4} = 0.517479061. \end{aligned} \quad (2.18)$$

The contribution $A^{(4,2)}(m_e/m_\mu)$ to the muon anomaly comes from the graphs of Fig. 6 with $M = m_\mu$, $m = m_e$, and is also already known¹³⁾

$$A^{(4,2)}\left(\frac{m_e}{m_\mu}\right) \simeq \frac{2}{9} \log^2 \frac{m_e}{m_\mu} - \frac{73}{108} \log \frac{m_\mu}{m_e} + \frac{1}{2} \zeta(3) + \frac{2}{9} \zeta(2) + \frac{91}{162}, \quad (2.19)$$

The knowledge of $K^{(2)}(b)$ and of $\phi^{(4)}(b)$ allows one to compute easily $A^{(4,2)}(m_\mu/m_e)$. In leading order of m_e/m_μ we have

$$A^{(4,2)}\left(\frac{m_\mu}{m_e}\right) \simeq \frac{1}{3} \frac{m_e^2}{m_\mu^2} \int_4^{\infty} \frac{db}{b^2} \varphi^{(4)}(b) = \frac{1}{3} \frac{m_e^2}{m_\mu^2} \Pi^{(4)'}(0) = \frac{41}{486} \frac{m_e^2}{m_\mu^2} \quad (2.20)$$

2.2 New results on sixth- and eighth-order anomalies

The explicit analytic calculation of the functions $K^{(4)}(b)$ and $L^{(4)}(b)$ is the bulk of the present paper. It enables us to obtain the analytic values of all the other constants appearing in Eqs. (2.16), namely:

i) the contribution from all the graphs of Fig. 7 with $M = m$,

$$A^{(2,4)}(1) = \frac{227}{72} + \frac{31}{9} \zeta(2) - 8 \zeta(2) \log 2 - \frac{35}{8} \zeta(3) + 2 \zeta^2(2) \quad (2.21)$$

[analytic results for all the graphs of Fig. 7 separately are given elsewhere^{7,14}]; the contributions from graphs 9, 10, 15-18 of Fig. 7 have also been evaluated analytically in the second paper of Ref. 4];

ii) the "mass-dependent" contributions to the muon (electron) anomaly from the graphs of Figs. 7 and 8 together, with $M = m_\mu$, $m = m_e$, and $M = m_e$, $m = m_\mu$, respectively,

$$\begin{aligned} B^{(2,4)}\left(\frac{m_e}{m_\mu}\right) &= \frac{2861}{648} - \frac{77}{9} \zeta(2) + 10 \zeta(2) \log 2 - \frac{7}{2} \zeta(3) + 3 C_4 \\ &\quad + \left(\frac{197}{108} + \frac{2}{3} \zeta(2) - 4 \zeta(2) \log 2 + \zeta(3) \right) \log \frac{m_\mu}{m_e} \\ C_4 &= \frac{11}{18} \zeta^2(2) - \frac{4}{9} \zeta(2) \log^2 2 - \frac{1}{27} \log^4 2 - \frac{8}{9} a_4 = 0.83376809 \end{aligned} \quad (2.22)$$

$$B^{(2,4)}\left(\frac{m_\mu}{m_e}\right) = -\frac{2}{3} \frac{m_e^2}{m_\mu^2} \left[-\frac{2689}{5400} + \frac{2}{5} \zeta(2) + \frac{23}{90} \log \frac{m_\mu}{m_e} \right] \quad (2.23)$$

(Graphs 19 and 20 of Fig. 8 with $M = m_\mu$ and $m = m_e$ have been evaluated analytically elsewhere¹⁵); complete results for all graphs separately are given in Refs. 7 and 14). Agreement with existing numerical results^{2,3} is always good. By inserting Eqs. (2.21) and (2.22) into Eq. (2.16), the explicit analytic expressions of all sixth-order vac. pol. contributions to electron and muon anomalies are obtained. (Terms proportional to m_e/m_μ are neglected.)

$$\begin{aligned} a_e(\text{vac. pol.})^{6H} &\simeq \left(\frac{\alpha}{\pi}\right)^3 \left[A^{(4,2)}(1) + A^{(2,4)}(1) \right] = \\ &= \left(\frac{\alpha}{\pi}\right)^3 \left[\frac{4195}{648} + \frac{31}{135} \zeta(2) - \frac{71}{72} \zeta(3) - \frac{32}{3} \zeta(2) \log 2 + \frac{16}{15} \zeta^2(2) + \frac{32}{3} a_4 \right. \\ &\quad \left. - \frac{8}{3} \zeta(2) \log^2 2 + \frac{4}{9} \log^4 2 \right] = -0.09474 \left(\frac{\alpha}{\pi}\right)^3 \end{aligned} \quad (2.24)$$

$$\begin{aligned} (a_\mu - a_e)(\text{vac. pol.})^{6H} &\simeq \left(\frac{\alpha}{\pi}\right)^3 \left[A^{(4,2)}\left(\frac{m_e}{m_\mu}\right) + B^{(2,4)}\left(\frac{m_e}{m_\mu}\right) \right] \simeq \\ &\simeq \left(\frac{\alpha}{\pi}\right)^3 \left[\frac{1075}{216} - \frac{25}{3} \zeta(2) + 10 \zeta(2) \log 2 - 3 \zeta(3) + 3 C_4 + \left(\frac{31}{27} + \frac{2}{3} \zeta(2) - \right. \right. \\ &\quad \left. \left. - 4 \zeta(2) \log 2 + \zeta(3) \right) \log \frac{m_\mu}{m_e} + \frac{2}{9} \log^2 \frac{m_\mu}{m_e} \right] = \\ &= \left(\frac{\alpha}{\pi}\right)^3 \left[1.56602 - 1.1139 \log \frac{m_\mu}{m_e} + \frac{2}{9} \log^2 \frac{m_\mu}{m_e} \right] = 1.94404 \left(\frac{\alpha}{\pi}\right)^3. \end{aligned} \quad (2.25)$$

The knowledge of $K^{(4)}(b)$ permits also to give some speculations and some partial results on eighth- and higher-order contributions to the muon anomalies. From all the eighth-order muon vertex diagrams, except for

- i) diagrams with an electron-loop light-light subgraph (see, for example, Fig. 12a),
- ii) "mixed vac. pol." vertex diagrams, i.e. diagrams with an electron loop containing a muon loop (see, for example, Fig. 12b),

we get the following contribution to the muon anomaly, in agreement with existing numerical estimates¹⁶⁾,

$$\begin{aligned} \left(\frac{\alpha}{\pi}\right)^6 A_1^{(2)}\left(\frac{m_\mu}{m_e}\right) &= \left(\frac{\alpha}{\pi}\right)^6 \left[\frac{4}{27} \log^2 \frac{m_\mu}{m_e} + \left(\frac{71}{54} + \frac{2}{3} \zeta(2) - 4 \zeta(3) \log 2 + \zeta(3) \right) \log^2 \frac{m_\mu}{m_e} + \right. \\ &+ \left(\frac{8813}{864} - \frac{97}{6} \zeta(2) + 17 \zeta(2) \log 2 - \frac{31}{4} \zeta(3) + 6 C_4 + 2 K_0^{(6)} \right) \log \frac{m_\mu}{m_e} + (\text{unknown const}) \Big] \\ &= \left(\frac{\alpha}{\pi}\right)^6 \left[4.5317 + 10.663 K_0^{(6)} + (\text{unknown const}) \right]. \end{aligned} \quad (2.26)$$

Here $K_0^{(6)}$ is the "mass-independent" sixth-order anomaly, at present known only numerically. Taking for $K_0^{(6)}$ the numerical theoretical value¹⁷⁾ $K_0^{(6)} = 1.17$ one has

$$A_1^{(8)}\left(\frac{m_\mu}{m_e}\right) = 17.0 + (\text{unknown const}). \quad (2.27)$$

3. A NEW METHOD TO COMPUTE THE ANOMALY

3.1 Description of the method

This subsection is devoted to the description of a new method to extract from any vertex diagram its contribution to the lepton anomalies. The method, especially developed to compute the "magnetic moments" $K^{(4)}(b)$, $L^{(4)}(b)$ introduced in the previous section, seems rather powerful and can possibly be used to evaluate other sixth-order graphs too. To show how it works, when introducing it, we work out explicitly, at the same time, the second-order function $K^{(2)}(a/M^2)$, i.e. the anomaly due to the lowest-order vertex with a "heavy photon" of squared mass a (Fig. 1d). We will give also the results and the essential points of the calculation for $K^{(4)}(b)$, $L^{(4)}(b)$, confining more technical details to Section 5.

Let us start from the off-mass shell Feynman amplitude $(-e)M_\mu(p, \Delta)$ for any vertex graph in momentum representation for leptons with momenta $p + \frac{1}{2}\Delta$, $p - \frac{1}{2}\Delta$, and a photon of momentum Δ , $\Delta^2 = -t$, and Lorentz index μ . Expand $M_\mu(p, \Delta)$ around $\Delta = 0$:

$$M_\mu(p, \Delta) \simeq M_\mu(p, 0) + \Delta_\nu \frac{\partial}{\partial \Delta_\nu} M_\mu(p, \Delta) \Big|_{\Delta=0} \equiv V_\mu(p) + \Delta_\nu T_{\nu\mu}(p). \quad (3.1)$$

The contribution of this vertex amplitude to the magnetic moment anomaly is then given by

$$a = \frac{1}{12M^2} \text{Tr} \left[G_\mu V_\mu(p) + H_{\mu\nu} T_{\nu\mu}(p) \right]_{p^2 = -M^2}, \quad (3.2)$$

where M is the lepton mass and

$$G_\mu = M^2 \gamma_\mu + 3iM p_\mu + 4 \not{p} p_\mu, \quad (3.3)$$

$$H_{\mu\nu} = -\frac{i}{4} (-i\not{p} + M)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)(-i\not{p} + M).$$

To check Eqs. (3.2) and (3.3) consider the most general off-mass-shell vertex amplitude $M_\mu(p, \Delta)$, written in terms of its twelve independent invariant amplitudes, and project out of it the anomaly through the formula

$$a \equiv \lim_{t \rightarrow 0} F_2(t) = \lim_{t \rightarrow 0} \frac{M^2}{t(t-4M^2)} \text{Tr} \left[(-\gamma_\mu + 2i \frac{t+2M^2}{t-4M^2} \frac{p_\mu}{M}) (-i\not{p} - \frac{i}{2} \not{\Delta} + M) M_\mu(p, \Delta) \cdot (-i\not{p} + \frac{i}{2} \not{\Delta} + M) \right], \quad (3.4)$$

with $\Delta^2 = -t$, $(p\Delta) = 0$, $p^2 = (t/4) - M^2$, already used in previous work⁶⁾. The right-hand sides of Eqs. (3.2) and (3.4) are then found to coincide. The idea of looking at the anomaly, even within the free particle Feynman graph approach, as a limiting value of a suitable self-mass-like amplitude was already proposed by Peterman¹⁸⁾, and used by Kahana and Polkinghorne¹⁹⁾ to recover the lowest-order Schwinger result. An essentially equivalent procedure is also used by Levine and Roskies⁵⁾.

The connection between Eq. (3.2) and operator mass approach is manifest (see in particular Carroll and Yao²⁰⁾, where the anomaly is obtained, in the operator-mass approach, and for numerical integration purposes, without introducing auxiliary parameters).

For all the graphs we consider -- Figs. 1d, 4 and 5 -- taking as integration loop variables the momenta of the internal photons, Δ_μ appears explicitly only in the lepton propagators. It is easily seen that, for those graphs, the differentiation with respect to Δ_ν , appearing in the definition of $T_{\nu\mu}(p)$, Eq. (3.1), can be exchanged with the loop integration. One then has, for a lepton line of momentum, say, $q + \frac{1}{2}\Delta$,

$$\frac{\partial}{\partial \Delta_\nu} \frac{-i}{i\not{q} + \frac{i}{2}\not{\Delta} + M} \Big|_{\Delta_\nu=0} = \frac{1}{2} \frac{-i}{i\not{q} + M} \gamma_\nu \frac{-i}{i\not{q} + M}. \quad (3.5)$$

Considering, for instance, the lowest-order vertex amplitude $M_\mu^{(2)}(p, \Delta)$, Fig. 9a, the corresponding $V_\mu(p)$ is depicted in Fig. 9b, and, on account of Eq. (3.5), $T_{\nu\mu}(p)$ can be represented by the two graphs of Fig. 9c. Note the relative sign between the two "graphs" for $T_{\nu\mu}(p)$, preventing simple interpretation in terms of Compton scattering amplitudes.

As far as the loop integrations are concerned, Eq. (3.2) has the form

$$a = \int \prod_i \left\{ \frac{id^4 q_i}{(2\pi)^4} \right\} \left. \frac{1}{2} (q_i, p) \right|_{p^2 = -M^2}, \quad (3.6)$$

where $f(q_i; p)$ is a scalar function of p and the loop momenta q_i . In the explicit case of the graph of Fig. 1d, working out the algebra and remembering that we call $K^{(2)}(a/M^2)$ the magnetic moment coming from it, the result is

$$K^{(2)}\left(\frac{a}{M^2}\right) = \frac{i}{(2\pi)^2} \int \frac{d^4 k}{[(p-k)^2 + M^2]^2} \frac{1}{k^2 + a} \frac{A}{k^2 + A} \cdot \quad (3.7)$$

$$\frac{4}{3M^2} \left[4(pk)^2 + M^2 k^2 + 3M^2(pk) \right] \Big|_{p^2 = M^2}.$$

In the integrand, to guarantee finiteness of all terms separately, a regulator function $A/(k^2 + A)$ has been introduced and the limit $A \rightarrow \infty$ is left understood. Note that, once the $\Delta \rightarrow 0$ limit is taken, in the general expression (3.6) and so in the particular one (3.7) the integrand, besides the integration loop momenta, depends only on one external momentum p , just as in the case of a self-mass amplitude. As a main difference with respect to a true self-mass, we have here the possible appearance of squared Feynman denominators. In particular it is clear that, just as in the mass-operator approach, vertex graphs topologically equivalent to the same self-mass-like graphs, once the $\Delta \rightarrow 0$ limit is taken, can be grouped and treated together on the same footing, with considerable unification and simplification. Explicitly, graphs 9, 15, 16 of Fig. 4 correspond to graph a of Fig. 10, graphs 10, 17, 18 of Fig. 4 correspond to graph b of Fig. 10, graphs 7, 11, 14 of Fig. 4 correspond to graph c of Fig. 10 and finally graphs 8, 12, 13 of Fig. 4 correspond to graph d of Fig. 10.

We want now to consider the anomaly as given by Eq. (3.6) in general, or Eq. (3.7) in the particular example, as the value at $p^2 = -M^2$ of a suitable function of p^2 satisfying an unsubtracted dispersion relation in this variable. Dispersion relations are usually easily established for Feynman graph amplitudes, for which squared denominators do not play an essential role. As a step towards the standard case, write any squared denominator as

$$\frac{1}{(q^2 + M^2)^2} \equiv - \frac{\partial}{\partial \mu^2} \frac{1}{q^2 + \mu^2 - i\epsilon} \Big|_{\mu^2 = M^2},$$

exchange the derivative with the loop integrations (this turns out to be allowed in all the cases we have explicitly worked out) and consider the amplitude in which $1/(q^2 + M^2 - i\epsilon)^2$ is replaced by $1/(q^2 + \mu^2 - i\epsilon)$. Once a dispersion relation for the new amplitude is established, by differentiating it with respect to μ^2 at $\mu^2 = M^2$, the quantity we are interested in is recovered. More precisely, we look for a function $G(u, \mu_1^2)$, which corresponds to the amplitude for a suitable Feynman graph G with the topological structure of a self-mass (no squared propagators allowed), with external squared invariant mass u , and internal masses μ_1 , such that:

- i) applying to it a suitable polynomial in the derivatives over the "internal masses" $P(\partial/\partial\mu_1^2)$, it reproduces the anomaly as given by Eq. (3.6) through

$$a = P\left(\frac{\partial}{\partial\mu_1^2}\right) G(u, \mu_1^2) \Big|_{\substack{\mu_1^2 = M^2 \\ u = M^2}} ; \quad (3.8)$$

- ii) it satisfies in the variable u an unsubtracted dispersion relation

$$G(u, \mu_1^2) = \frac{1}{\pi} \int_{v_0}^{\infty} \frac{dv}{v-u} \text{Im} G(v, \mu_1^2) , \quad (3.9)$$

with $\text{Im} G(v, \mu_1^2)$ given by the Cutkosky-Veltman cutting rules applied to the corresponding self-mass-like graph [see 't Hooft and Veltman²¹⁾ for a simple derivation of the rules]. From Eqs. (3.8) and (3.9) we then have

$$a = P\left(\frac{\partial}{\partial\mu_1^2}\right) \frac{1}{\pi} \int_{v_0(\mu_1^2)}^{\infty} \frac{dv}{v-M^2} \text{Im} G(v, \mu_1^2) .$$

Even though, in general, the threshold v_0 will depend on the μ_1^2 , the discontinuity vanishes at the threshold values, and the properties of $\text{Im} G(v, \mu_1^2)$ allow us to write

$$a = \frac{1}{\pi} \int_{v_0(M^2)}^{\infty} \frac{dv}{v-M^2} P\left(\frac{\partial}{\partial\mu_1^2}\right) \text{Im} G(v, \mu_1^2) \Big|_{\mu_1^2 = M^2} \equiv \int_{v_0}^{\infty} \frac{dv}{v-M^2} g(v) \quad (3.10)$$

This equation is our starting point in the explicit calculations.

3.2 Calculation of $K^{(2)}(a/M^2)$

To see how to construct the function $G(u, \mu_1^2)$ of Eqs. (3.8) to (3.10), consider the particular case of $K^{(2)}(a/M^2)$, Eq. (3.7). Generalization to more general cases will be clearly straightforward. We can write

$$K^{(2)}\left(\frac{a}{M^2}\right) = \frac{4}{3M^2} \frac{\partial}{\partial\mu^2} \left[(4p_\rho p_\sigma + M^2 \delta_{\rho\sigma}) W_{\rho\sigma}(p, \mu^2) + 3M^2 p_\rho Y_\rho(p, \mu^2) \right] \Big|_{p^2 = -M^2} \quad (3.11)$$

where

$$\{W_{\rho\sigma}(p, \mu^2); Y_\rho(p, \mu^2)\} \equiv -\frac{i}{(2\pi)^2} \int \frac{d^4k}{k^2+a} \left(\frac{1}{(p-k)^2+\mu^2} - \frac{1}{(p-k)^2+R^2} \right) \frac{A}{k^2+A} \{k_\rho k_\sigma; k_\rho\} \quad (3.12)$$

The term with the mass R guarantees convergence, but will not otherwise play any role in the following. On invariance grounds

$$\begin{aligned} W_{\rho\sigma}(p, \mu^2) &= M^2 \alpha_1(-p^2, \mu^2) \delta_{\rho\sigma} + \alpha_2(-p^2, \mu^2) p_\rho p_\sigma , \\ Y_\rho(p, \mu^2) &= \alpha_3(-p^2, \mu^2) p_\rho . \end{aligned} \quad (3.13)$$

Standard considerations (by means of Feynman parameters, for instance) show that the $\alpha_i(u, \mu^2)$ vanish as $u \rightarrow \infty$ and are free from kinematical singularities, so that they admit unsubtracted dispersion relations in u , with the discontinuity given by

the usual cutting rules. To obtain a function $G(u, \mu^2)$ satisfying the requirements (i) and (ii) of the previous subsection, it is sufficient to substitute Eq. (3.13) into Eq. (3.11), imposing $p^2 = -M^2$ only in contracting Lorentz indices, keeping, however, the dependence on $-p^2 = u$. One gets

$$G(u, \mu^2) = 4 M^2 \left[\alpha_2(u, \mu^2) - \alpha_3(u, \mu^2) \right] \quad (3.14)$$

(note the disappearance of $\alpha_1(u, \mu^2)$, the only one diverging for $A \rightarrow \infty$; from now on, A -regularization can be ignored) and

$$K^{(2)}\left(\frac{a}{M^2}\right) = \frac{\partial}{\partial \mu^2} G(M^2, \mu^2) \Big|_{\mu^2=M^2} = \frac{1}{\pi} \int_{(M+\bar{a})^2}^{\infty} \frac{dv}{v-M^2} \frac{\partial}{\partial \mu^2} \text{Im} G(v, \mu^2) \Big|_{\mu^2=M^2} \quad (3.15)$$

The calculation of $\text{Im} G(v, \mu^2)$, μ^2 -differentiation and the v -integration are straightforward (details can be found in the contribution to Ref. 3 by E. Remiddi).

The final result is

$$K^{(2)}(b) = \frac{1}{2} - b + \left(\frac{1}{2}b^2 - b\right) \log b + \frac{\frac{1}{2}b^3 - 2b^2 + b}{\sqrt{b(b-4)}} \log \frac{\sqrt{b} - \sqrt{b-4}}{\sqrt{b} + \sqrt{b-4}} \quad (3.16)$$

At $b = 0$ (massless photon) one reproduces the Schwinger result $K^{(2)}(0) = \frac{1}{2}$, whereas, for large b , one has the asymptotic behaviour

$$K^{(2)}(b) \xrightarrow{b \rightarrow \infty} K_{\infty}^{(2)}(b) = \frac{1}{3b} \quad (3.17)$$

As expected on general grounds, for any $b > 0$, $K^{(2)}(b)$ is a real function. It develops a cut along the negative real axis $b < 0$ with the imaginary part

$$\text{Im} K^{(2)}(b) = \pi \theta(-b) \left\{ \frac{1}{2}b^2 - b + \frac{\frac{1}{2}b^3 - 2b^2 + b}{\sqrt{b(b-4)}} \right\} \quad (3.18)$$

As $K^{(2)}(b)$ vanishes at infinity, it satisfies the dispersion relation

$$K^{(2)}(b) = \frac{1}{\pi} \int_{-\infty}^0 \frac{dc}{c-b} \text{Im} K^{(2)}(c) = \int_0^{\infty} \frac{dc}{c+b} \left(-\frac{1}{2}c^2 - c + \frac{\frac{1}{2}c^3 + 2c^2 + c}{\sqrt{c(c+4)}} \right) \quad (3.19)$$

By introducing the integration variable y through $c = (1-y)^2/y$ and then exchanging $(1-y) \leftrightarrow y$ the above equation takes the remarkably simple form

$$K^{(2)}(b) = \int_0^1 dy \frac{y^2(1-y)}{y^2 + (1-y)b} \quad (3.20)$$

which is often encountered in the literature²⁾.

3.3 Results for $K^{(4)}(b)$ and $L^{(4)}(b)$

The explicit analytic expression of $K^{(4)}(a/M^2)$, corresponding to one half of the "magnetic moment" due to all graphs of Figs. 4 and 5 together is

$$\begin{aligned}
K^{(4)}(b) = & -\frac{139}{144} + \frac{115}{72}b + \left(\frac{19}{12} - \frac{7}{36}b + \frac{23}{144}b^2 + \frac{1}{b-4}\right) \log b + \\
& + \left(-\frac{4}{3} + \frac{127}{36}b - \frac{115}{72}b^2 + \frac{23}{144}b^3\right) \frac{\log y}{\sqrt{b(b-4)}} + \left(\frac{9}{4} + \frac{5}{24}b - \frac{1}{2}b^2 - \frac{2}{b}\right) \zeta(2) \\
& + \frac{5}{96}b^2 \log^2 b + \left(-\frac{1}{2}b + \frac{17}{24}b^2 - \frac{7}{48}b^3\right) \frac{\log y}{\sqrt{b(b-4)}} \log b \\
& + \left(\frac{19}{24} + \frac{53}{48}b - \frac{29}{96}b^2 - \frac{1}{3b} + \frac{2}{b-4}\right) \log^2 y \\
& + \left(-2b + \frac{17}{6}b^2 - \frac{7}{12}b^3\right) \frac{1}{\sqrt{b(b-4)}} D_p(b) \\
& + \left(\frac{13}{3} - \frac{7}{6}b + \frac{1}{4}b^2 - \frac{1}{6}b^3 - \frac{4}{b-4}\right) \frac{D_m(b)}{\sqrt{b(b-4)}} + \left(\frac{1}{2} - \frac{7}{6}b + \frac{1}{2}b^2\right) T(b)
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
y &= \frac{\sqrt{b} - \sqrt{b-4}}{\sqrt{b} + \sqrt{b-4}} \\
D_p(b) &= Li_2(y) + \log y \log(1-y) - \frac{1}{4} \log^2 y - \zeta(2) \\
D_m(b) &= Li_2(-y) + \frac{1}{4} \log^2 y + \frac{1}{2} \zeta(2) \\
T(b) &= -6 Li_3(y) - 3 Li_3(-y) + \log^2 y \log(1-y) + \frac{1}{2} (\log^2 y + 6 \zeta(2)) \log(1+y) \\
&\quad + 2 \log y (Li_2(-y) + 2 Li_2(y)) .
\end{aligned} \tag{3.22}$$

The functions $Li_2(y)$, $Li_3(y)$ are the dilogarithm and trilogarithm defined through

$$Li_2(y) = - \int_0^y \frac{dt}{t} \log(1-t), \quad Li_3(y) = \int_0^y \frac{dt}{t} Li_2(t) .$$

The function $L^{(4)}(a/M^2)$, differs from $K^{(4)}(a/M^2)$, because it does not account for the contributions of the graphs of Fig. 5, containing vac. pol. insertions. For their difference, we find

$$\begin{aligned}
K^{(4)}(b) - L^{(4)}(b) = & \frac{35}{36} + \frac{8}{9}b + \left(\frac{4}{3} - \frac{1}{9}b - \frac{5}{18}b^2\right) \log b + \left(-\frac{4}{3} + \frac{19}{9}b + \frac{4}{9}b^2 - \right. \\
& - \frac{5}{18}b^3) \frac{\log y}{\sqrt{b(b-4)}} + \left(1 + \frac{1}{3}b - \frac{1}{6}b^2 - \frac{2}{b}\right) \zeta(2) + \left(\frac{1}{2} + \frac{1}{6}b - \frac{1}{12}b^2 - \right. \\
& \left. - \frac{1}{3b}\right) \log^2 y + \left(\frac{16}{3} - \frac{4}{3}b - \frac{4}{3}b^2 + \frac{1}{3}b^3\right) \frac{1}{\sqrt{b(b-4)}} D_m(b) .
\end{aligned} \tag{3.23}$$

From Eq. (3.21) one finds

$$K^{(4)}(0) = \frac{197}{144} + \frac{1}{2} \zeta(2) - 3 \zeta(2) \log 2 + \frac{3}{4} \zeta(3) \quad (3.24)$$

which is the classical fourth-order anomaly of the electron. This result, although expected, is not a simple consistency check. Actually, let us indicate by $K_2^{(4)}(a_1/M^2, a_2/M^2)$ the complete fourth-order anomaly obtained giving squared masses a_1, a_2 to the two internal photons of all the fourth-order graphs. The usual fourth-order anomaly a^{4th} is then defined as

$$a^{4th} = \lim_{\lambda^2 \rightarrow 0} K_2^{(4)}\left(\frac{\lambda^2}{M^2}, \frac{\lambda^2}{M^2}\right)$$

whereas

$$K^{(4)}(0) \equiv \lim_{a \rightarrow 0} K^{(4)}\left(\frac{a}{M^2}\right) = \lim_{a \rightarrow 0} \left[\lim_{\lambda^2 \rightarrow 0} K_2^{(4)}\left(\frac{a}{M^2}, \frac{\lambda^2}{M^2}\right) \right].$$

The equality $a^{4th} = K^{(4)}(0)$, found to be true *a posteriori* is then an indication of uniformity of the zero photon mass limit. From Eq. (3.25) one finds also the asymptotic behaviour

$$K^{(4)}(b) \xrightarrow{b \rightarrow \infty} K_\infty^{(4)}(b) \equiv -\frac{1}{b} \left(\frac{23}{36} \log b + 2\zeta(2) - \frac{223}{54} \right). \quad (3.25)$$

As expected on general grounds, the functions $K^{(4)}(b)$ and $L^{(4)}(b)$ are real for any real $b \geq 0$. To check this, note that as b runs from $+\infty$ to 4, the variable y moves correspondingly from 0 to 1 and then in the lower part of the unit circle till $y = -1$, when b ranges from 4 to 0 with a small positive imaginary part. (See Fig. 11.) For $b \geq 4$, i.e. $0 \leq y \leq 1$, the reality of Eqs. (3.21) and (3.23) is obvious. For $0 < b < 4$, writing $y = e^{i\phi}$, $-\pi < \phi < 0$, the reality is verified on account of the following relations

$$\text{Im} \log(1 - e^{i\varphi}) = -\frac{1}{2}(\pi - \varphi); \quad \text{Im} \log(1 + e^{i\varphi}) = \frac{1}{2}\varphi;$$

$$\text{Re} \text{Li}_2(e^{i\varphi}) = \frac{\pi^2}{6} - \frac{\pi}{2}\varphi + \frac{1}{4}\varphi^2; \quad \text{Re} \text{Li}_2(-e^{i\varphi}) = -\frac{\pi^2}{12} + \frac{1}{4}\varphi^2;$$

$$\text{Im} \text{Li}_3(e^{i\varphi}) = \frac{\pi^2}{6}\varphi - \frac{\pi}{4}\varphi^2 + \frac{1}{12}\varphi^3; \quad \text{Im} \text{Li}_3(-e^{i\varphi}) = -\frac{\pi^2}{12}\varphi + \frac{1}{12}\varphi^3.$$

The above formulae, valid for $|\phi| \leq \pi$, are most easily proved by checking them at a particular value and comparing the derivatives of both sides, repeatedly when needed. In particular, for $0 \leq b < 4$, $D_p(b)$, $D_m(b)$ are purely imaginary, while $T(b)$ is purely real.

We want to mention here that Eq. (3.21) follows, in our approach, by summing the results from all the possible ways of cutting the graphs contributing to it. As shown in Section 5, some cuts are not analytic in b for all values of b . A

two-particle cut, in particular, is analytic only for $0 \leq b \leq 2$ and develops an anomalous threshold at $b = 2$. In the total sum, however, those anomalous singularities cancel, and the result, as we have seen, is real for any $b \geq 0$. This means, in particular, that no "anomalous discontinuity" is to be computed, as we can evaluate the full $K^{(4)}(b)$ or $L^{(4)}(b)$ functions in the "normal region" $0 \leq b < 2$ and then take their simple analytic continuation^{*)} to $b \geq 2$.

4. ELECTRON AND MUON ANOMALIES FROM VAC. POL. CORRECTIONS

4.1 Sixth-order electron anomaly

According to Section 2, the electron anomaly due to the graphs of Fig. 7, with $m = M$, is

$$A^{(2,4)}(1) = 2 \int_4^\infty \frac{db}{b} \varphi^{(2)}(b) L^{(4)}(b), \quad (4.1)$$

with $\phi^{(2)}(b)$ and $L^{(4)}(b)$ given in Eqs. (2.4), (3.21) and (3.23). [The factor 2 in front of the integral comes from the normalization of $L^{(4)}(b)$ we have chosen.] By expressing everything in terms of the variable y , Eq. (3.22), one obtains an integral from 0 to 1 in y , the integrand consisting of logarithms, dilogarithms and trilogarithms, introduced in Section 3, times rational functions of y . All these integrals can be performed using the techniques and the results of Ref. 6. Equation (2.21) is then obtained.

From the sixth-order electron vertex graphs with a second-order muon vac. pol. correction inserted (the graphs of Figs. 7 and 8 with $M = m_e$ and $m = m_\mu$), one has

$$B^{(2,4)}\left(\frac{m_\mu}{m_e}\right) = 2 \int_{4m_\mu^2}^\infty \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m_\mu^2}\right) K^{(4)}\left(\frac{a}{m_e^2}\right). \quad (4.2)$$

Since $m_\mu^2 \gg m_e^2$, one can write

$$B^{(2,4)}\left(\frac{m_\mu}{m_e}\right) \sim 2 \int_{4m_\mu^2}^\infty \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m_\mu^2}\right) K_\infty^{(4)}\left(\frac{a}{m_e^2}\right), \quad (4.3)$$

where $K_\infty^{(4)}(b)$ is the leading asymptotic term of $K^{(4)}(b)$ and is given in Eq. (3.25). The result of Eq. (2.23) is easily obtained. As expected, $B^{(2,4)}(m_\mu/m_e)$ is negligible in comparing the theory with the experimental results.

4.2 Sixth-order muon anomaly

According to Section 2, the contribution to the muon anomaly from all the graphs of Figs. 7 and 8 with $M = m_\mu$ and $m = m_e$, is

$$B^{(2,4)}\left(\frac{m_e}{m_\mu}\right) = 2 \int_{4m_e^2}^\infty \frac{da}{a} \varphi^{(2)}\left(\frac{a}{m_e^2}\right) K^{(4)}\left(\frac{a}{m_\mu^2}\right). \quad (4.4)$$

*) A discussion with J. Bros on this point is gratefully acknowledged.

From Eq. (2.4) one has

$$\varphi_{\infty}^{(2)}(b) \equiv \lim_{b \rightarrow \infty} \varphi^{(2)}(b) = \frac{1}{3}$$

and, on account of $m_e^2 \ll m_{\mu}^2$, Eq. (4.4) can be written as

$$\begin{aligned} B^{(2,4)}\left(\frac{m_e}{m_{\mu}}\right) &= 2 \int_{4m_e^2}^{\infty} \frac{da}{a} \left[\varphi_{\infty}^{(2)}\left(\frac{a}{m_e^2}\right) + \varphi^{(2)}\left(\frac{a}{m_e^2}\right) - \varphi_{\infty}^{(2)}\left(\frac{a}{m_{\mu}^2}\right) \right] K^{(4)}\left(\frac{a}{m_{\mu}^2}\right) \\ &\simeq \frac{2}{3} \int_{4m_e^2}^{\infty} \frac{da}{a} K^{(4)}\left(\frac{a}{m_{\mu}^2}\right) + 2K^{(4)}(0) \int_{4m_e^2}^{\infty} \frac{db}{b} \left[\varphi^{(2)}(b) - \varphi_{\infty}^{(2)}(b) \right]. \end{aligned} \quad (4.5)$$

An integration by parts gives (always for $m_e^2 \ll m_{\mu}^2$)

$$\begin{aligned} \int_{4m_e^2}^{\infty} \frac{da}{a} K^{(4)}\left(\frac{a}{m_{\mu}^2}\right) &\simeq -\log 4 K^{(4)}(0) - \int_{4m_e^2}^{\infty} da \log \frac{a}{m_e^2} \frac{d}{da} K^{(4)}\left(\frac{a}{m_{\mu}^2}\right) \\ &\cong -\log 4 K^{(4)}(0) - \int_0^{\infty} db (\log b + 2 \log \frac{m_{\mu}}{m_e}) \frac{d}{db} K^{(4)}(b) \\ &= K^{(4)}(0) (2 \log \frac{m_{\mu}}{m_e} - \log 4) + K_1^{(4)} \end{aligned} \quad (4.6)$$

where $K^{(4)}(0)$ is given in Eq. (3.28) and

$$K_1^{(4)} \equiv - \int_0^{\infty} db \log b \frac{d}{db} K^{(4)}(b). \quad (4.7)$$

To evaluate $K_1^{(4)}$, we use the explicit expression Eq. (3.21), introduce the variable y of Eq. (3.22) so that $\int_0^{\infty} db \rightarrow \int_0^{-1} dy$ and then change $y \rightarrow -y$, obtaining a sum of terms of the standard type discussed in Ref. 6. The result is

$$K_1^{(4)} = \frac{641}{72} - 12\zeta(2) + 10\zeta(2)\log 2 - 4\zeta(3) + \frac{9}{2}c_4 = -0.490888 \quad (4.8)$$

with c_4 given in Eq. 2.18. Finally

$$\int_4^{\infty} \frac{db}{b} \left[\varphi^{(2)}(b) - \varphi_{\infty}^{(2)}(b) \right] = -\frac{5}{9} + \frac{1}{3} \log 4, \quad (4.9)$$

so that, putting together Eqs. (4.5), (4.6), (4.8) and (4.9), the final result of Eq. (2.22) is recovered.

4.3 Partial results on the eighth-order muon anomaly

In this subsection we show that the previous formulae can be easily generalized and used to obtain some information on eighth-order (and eventually higher) contributions to the mass-dependent muon anomaly due to vac. pol. insertions.

We recall that, at present, the vac. pol. amplitude is explicitly known up to fourth-order⁹⁾ only, but some information on higher orders is given by the

renormalization group. More precisely, consider the vac. pol. due to one kind of lepton only (say the electron). In this case, the amplitude depends on the dimensionless ratio $z = (\text{squared momentum transfer/squared mass of the lepton})$. Let $\Pi(z)$, $\pi\phi(z)$ be the real and imaginary parts of the vac. pol. amplitude, and define $\Pi_\infty(z)$, $\phi_\infty(z)$ as their asymptotic parts, with

$$\begin{aligned} \lim_{z \rightarrow \infty} [\Pi(z) - \Pi_\infty(z)] &= 0 \\ \lim_{z \rightarrow \infty} [\phi(z) - \phi_\infty(z)] &= 0 \end{aligned} \quad (4.10)$$

All above functions are to be regarded as (formal) power series in the fine structure constant α . Present knowledge gives for the total amplitude (proper and improper)

$$\begin{aligned} \Pi_\infty(z) &= \left(\frac{\alpha}{\pi}\right) \left[\frac{5}{9} - \frac{1}{3} \log z \right] \\ &+ \left(\frac{\alpha}{\pi}\right)^2 \left[-\frac{65}{648} - \zeta(3) + \frac{13}{108} \log z - \frac{1}{9} \log^2 z \right] \\ &+ \left(\frac{\alpha}{\pi}\right)^3 \left[(\text{unknown const}) + \left(\frac{1549}{2592} - \zeta(3)\right) \log z - \frac{5}{216} \log^2 z - \frac{1}{27} \log^3 z \right] + \dots \end{aligned} \quad (4.11)$$

and

$$\phi_\infty(z) \equiv \sum_{n=0}^{\infty} c_n \log^n z \quad (4.12)$$

with

$$\begin{aligned} c_0 &= \frac{1}{3} \frac{\alpha}{\pi} - \frac{13}{108} \left(\frac{\alpha}{\pi}\right)^2 + \left(-\frac{1549}{2592} - \frac{2}{9} \zeta(2) + \zeta(3)\right) \left(\frac{\alpha}{\pi}\right)^3 + \dots \\ c_1 &= \frac{2}{9} \left(\frac{\alpha}{\pi}\right)^2 + \frac{5}{108} \left(\frac{\alpha}{\pi}\right)^3 + \dots ; \quad c_2 = \frac{1}{9} \left(\frac{\alpha}{\pi}\right)^3 + \dots \end{aligned} \quad (4.13)$$

[Above results with reference to previous work can be found, for instance, in de Rafael and Rosner²²⁾].

We will need also the explicit expression of the quantity

$$\gamma \equiv \int_{z_0}^{\infty} \frac{dz}{z} [\phi(z) - \phi_\infty(z)] \quad (4.14)$$

Starting from the dispersion relation connecting $\Pi(z)$ with $\phi(z)$, and taking the limit of large z one finds

$$\begin{aligned} \Pi(z) &= z \int_{z_0}^{\infty} \frac{dx}{x(x-z)} \phi(x) = z \int_{z_0}^{\infty} \frac{dx}{x(x-z)} [\phi_\infty(x) + (\phi(x) - \phi_\infty(x))] \\ &\simeq z \int_{z_0}^{\infty} \frac{dx}{x(x-z)} \phi_\infty(x) - \int_{z_0}^{\infty} \frac{dx}{x} [\phi(x) - \phi_\infty(x)] \end{aligned}$$

or

$$\gamma = \lim_{z \rightarrow \infty} \left[z \int_{z_0}^{\infty} \frac{dx}{x(x-z)} \varphi_{\infty}(x) - \Pi_{\infty}(z) \right] \quad (4.15)$$

On account of Eqs. (4.11), (4.12) and (4.13), performing the integration (up to the order we consider, the threshold z_0 is 4), one has

$$\gamma = \left[-\frac{5}{9} + \frac{1}{3} \log 4 \right] \left(\frac{\alpha}{\pi} \right) + \left[\frac{65}{648} - \frac{2}{9} \zeta(2) + \zeta(3) - \frac{13}{108} \log 4 + \frac{1}{9} \log^2 4 \right] \left(\frac{\alpha}{\pi} \right)^2 + \dots \quad (4.16)$$

The renormalization group gives the sixth-order coefficients of $\Pi_{\infty}(z)$, $\phi_{\infty}(z)$ once the fourth-order coefficients are known. It does not give the sixth-order constant term in Eq. (4.11), which in turn fixes the sixth-order constant in Eq. (4.16) too, for whose knowledge a direct calculation would be needed. So much for the vac. pol. amplitude.

Concerning the lepton vertex, define

$$K\left(\frac{a}{M^2}\right) \equiv \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n n K^{(2n)}\left(\frac{a}{M^2}\right), \quad (4.17)$$

where M is the mass of the lepton, and $K(a/M^2)$ the contribution to its anomaly obtained by giving a mass $\sqrt{a}$ to one of the photons in all the n possible ways and in all graphs. The first coefficients $K^{(2)}(b)$, $K^{(4)}(b)$ are given in Eqs. (3.20) and (3.21); higher orders are unknown. The mass-dependent contributions to the muon g -factor, from the insertion into a single photon line of all electron vac. pol. graphs accounted for by $\phi(z)$, are

$$\begin{aligned} A_1\left(\frac{m_e}{m_{\mu}}\right) &= \int_{4m_e^2}^{\infty} \frac{da}{a} \varphi\left(\frac{a}{m_e^2}\right) K\left(\frac{a}{m_{\mu}^2}\right) \\ &= \int_{4m_e^2}^{\infty} \frac{da}{a} \left[\varphi_{\infty}\left(\frac{a}{m_e^2}\right) + \left(\varphi\left(\frac{a}{m_e^2}\right) - \varphi_{\infty}\left(\frac{a}{m_e^2}\right) \right) \right] K\left(\frac{a}{m_{\mu}^2}\right) \\ &\simeq \sum_{n=0}^{\infty} c_n \int_{4m_e^2}^{\infty} \frac{da}{a} \log^n \frac{a}{m_e^2} K\left(\frac{a}{m_{\mu}^2}\right) + K(0) \gamma, \end{aligned} \quad (4.18)$$

where use is made of Eq. (4.12). By partial integration, as a straightforward generalization of Eq. (4.5), one has ($n \geq 0$)

$$\int_{4m_e^2}^{\infty} \frac{da}{a} \log^n \frac{a}{m_e^2} K\left(\frac{a}{m_{\mu}^2}\right) \simeq \frac{1}{n+1} K_0 \left[\log^{n+1} \frac{m_{\mu}^2}{m_e^2} - \log^{n+1} 4 \right] + \frac{1}{n+1} \sum_{s=1}^{n+1} \binom{n+1}{s} K_s \log^{n+1-s} \left(\frac{m_{\mu}^2}{m_e^2} \right) \quad (4.19)$$

where

$$K_s \equiv - \int_0^\infty db \log^s b \frac{d}{db} K(b) . \quad (4.20)$$

K_0 is directly related to the mass independent anomaly

$$K_0 \equiv K(0) = K_0^{(2)} \left(\frac{\alpha}{\pi}\right) + 2 K_0^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + 3 K_0^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + \dots$$

where $K_0^{(6)}$, the whole sixth-order electron anomaly, is presently known only numerically. From the explicit knowledge of $K^{(2)}(b)$, $K^{(4)}(b)$, we find

$$K_1 = -\frac{5}{4} \left(\frac{\alpha}{\pi}\right)^2 + 2 K_1^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + \dots$$

with $K_1^{(4)}$ given in Eq. (4.8), and

$$K_2 = \left[\frac{13}{4} + 3 \zeta(2) \right] \left(\frac{\alpha}{\pi}\right) + \dots$$

The (missing) $(\alpha/\pi)^2$ term of K_2 seems to involve higher transcendental constants such as $\zeta(5)$ and the like, and we did not work it out. Collecting all partial results, one has, besides the fourth- and sixth-order terms already shown in Eqs. (2.13) and (2.25);

$$\begin{aligned} \left(\frac{\alpha}{\pi}\right)^4 A_1^{(8)}\left(\frac{m_e}{m_\mu}\right) = & \left(\frac{\alpha}{\pi}\right)^8 \left[\frac{4}{27} \log^3 \frac{m_\mu}{m_e} + \left(\frac{229}{324} + \frac{4}{9} \zeta(2) - \frac{8}{3} \zeta(2) \log 2 + \frac{2}{3} \zeta(3) \right) \log^2 \frac{m_\mu}{m_e} \right. \\ & + \left(\frac{56483}{7776} + 2 K_0^{(6)} - \frac{565}{54} \zeta(2) + \frac{31}{3} \zeta(2) \log 2 - \frac{35}{12} \zeta(3) + 4 C_4 \right) \log \frac{m_\mu}{m_e} \\ & \left. + (\text{unknown const.}) \right] . \end{aligned} \quad (4.21)$$

The above eighth-order term comes from the vac. pol. corrections of one photon line only. To this order, however, we have to consider also the vac. pol. corrections of two photon lines simultaneously. For this purpose, as a straightforward generalization of $K(a/M^2)$, Eq. (4.17), define $K_2(a_1/m_\mu^2, a_2/m_\mu^2)$ as the muon magnetic moment obtained by considering graphs with two "massive" photons of squared mass a_1, a_2 . It is a formal series in (α/π) , and we will be concerned only with the first term, $K_2^{(4)}(b, c)$ which is of fourth-order. It is unknown, except for the particular values

$$K_2(b, 0) = K_2(0, b) = K(b) . \quad (4.22)$$

By inserting an electron vacuum polarization into any two different photon lines in the muon vertex, we obtain the mass-dependent contribution to muon anomaly

$$A_2\left(\frac{m_e}{m_\mu}\right) = \int_{4m_e^2}^\infty \frac{da_1}{a_1} \varphi\left(\frac{a_1}{m_e^2}\right) \int_{4m_e^2}^\infty \frac{da_2}{a_2} \varphi\left(\frac{a_2}{m_e^2}\right) K_2\left(\frac{a_1}{m_\mu^2}, \frac{a_2}{m_\mu^2}\right) . \quad (4.23)$$

The two integrations can be worked out, one after the other, as the single integration of Eq. (4.18). To lowest order, on account of Eq. (4.22), one gets

$$\begin{aligned}
 A_2\left(\frac{m_e}{m_\mu}\right) &= \left(\frac{\alpha}{\pi}\right)^4 \left[\left(\frac{197}{324} + \frac{2}{9} \zeta(2) - \frac{4}{3} \zeta(2) \log 2 + \frac{1}{3} \zeta(3) \right) \log^2 \frac{m_\mu}{m_e} \right. \\
 &\quad + \left(\frac{2861}{972} - \frac{154}{27} \zeta(2) + \frac{20}{3} \zeta(2) \log 2 - \frac{7}{3} \zeta(3) + 2 C_4 \right) \log \frac{m_\mu}{m_e} \\
 &\quad \left. + (\text{unknown const.}) \right] + \dots
 \end{aligned} \tag{4.24}$$

By summing Eqs. (4.21) and (4.24), the eighth-order mass-dependent contribution given in Eq. (2.26) is obtained. It accounts for all eighth-order muon vertex graphs except for:

- i) diagrams with an electron-loop light-light subgraph (see, for example, Fig. 12a);
- ii) "mixed vac. pol." vertex diagrams, i.e. diagrams with an electron loop containing a muon loop. An example of a graph not considered is that of Fig. 12b. On the contrary, for example, the graphs of Fig. 12c are all included.

It would be straightforward to obtain the leading powers of $\log(m_\mu/m_e)$ for the higher order graphs corresponding to higher order vac. pol. insertions into the same second- and fourth-order vertex graphs, by using Eq. (4.19) and the renormalization group predictions for the leading logarithms of $\phi_\infty(z)$, $\Pi_\infty(z)$. Although perhaps appealing, this would be scarcely significant as the numerical value of the missing non-leading terms is likely to be of the same importance as the leading ones¹⁶⁾, which usually cancel each other out, and, further, there is no control on a host of other graphs (especially on those involving light-light scattering).

5. DESCRIPTION OF THE CALCULATION OF $K^{(4)}(b)$ AND $L^{(4)}(b)$

5.1 Generalities

We give in this section a brief description of the most technical ingredients of the calculation of $K^{(4)}(b)$ and $L^{(4)}(b)$. As in our previous works on similar subjects, the main tools to perform the calculations are: (i) the dispersive methods (the new approach presented in this paper does not alter substantially the technical problems one is faced with); (ii) Nielsen's function theory (see the Appendix of Ref. 6); (iii) the use of the Schoonschip program²³) by M. Veltman to do by computer all the straightforward, but otherwise impossible, algebraic manipulations of the analytic expressions in the intermediate steps of the calculations. The program was mainly used on the CERN CDC 7600 computer, typically in frequent runs of less than eight seconds -- the whole calculation takes a few minutes.

In working out the phase-space integrals encountered in applying the Cutkosky-Veltman cutting rules, one has often to deal with the characteristic square-root function

$$R(v, m^2, \mu^2) = (v^2 + m^4 + \mu^4 - 2vm^2 - 2v\mu^2 - 2m^2\mu^2)^{\frac{1}{2}} \quad (5.1)$$

For a two-body phase space, indeed, we have

$$\begin{aligned} \int d^4q \, \theta(-q_0) \delta(q^2 + m^2) \theta(p_0 + q_0) \delta((p+q)^2 + \mu^2) \Big|_{p^2 = -v} &= \\ &= \theta(v - (m+\mu)^2) \frac{1}{8v} R(v, m^2, \mu^2) \int d\Omega_q \end{aligned} \quad (5.2)$$

with the angular integration $d\Omega_q$ to be performed in the system where $p_\mu = (\sqrt{v}, \vec{0})$, and, for a three-body phase space

$$\begin{aligned} \int d^4k \, d^4q \, \theta(k_0) \delta(k^2 + m_1^2) \theta(q_0) \delta(q^2 + m_2^2) \theta(p_0 - q_0 - k_0) \delta((p-q-k)^2 + m_3^2) \Big|_{p^2 = -v} &= \\ &= \theta(v - (m_1 + m_2 + m_3)^2) \int \frac{db}{8v} R(v, b, m_1^2) \frac{1}{8b} R(b, m_1^2, m_3^2) \int d\Omega_q \int d\Omega_k \end{aligned} \quad (5.3)$$

where the angular integrations $d\Omega_k$ and $d\Omega_q$ are to be carried out in the frames $(p-q)_\mu = (\sqrt{b}, \vec{0})$ and $p_\mu = (\sqrt{v}, \vec{0})$, respectively.

The "natural variable"

$$z(v, m^2, \mu^2) = \frac{v - m^2 - \mu^2 - R(v, m^2, \mu^2)}{2 m \mu}, \quad (5.4)$$

$$v = \frac{m \mu}{z} + m^2 + \mu^2 + m \mu z$$

$$(m + \mu)^2 < v < \infty \quad \leftrightarrow \quad 1 > z > 0$$

rationalizes the square root $R(v, m^2, \mu^2)$,

$$R(v, m^2, \mu^2) = \frac{m \mu}{z} (1 - z^2)$$

and brings the integrands to a standard form as defined and dealt with in Ref. 6 (see in particular, the Appendix A there).

Of great use in simplifying the integrands were found to be the following recursion formulae, to be exploited, as a rule, before introducing the natural variable z ,

$$\int_{(\sqrt{b+1})^2}^u \frac{dv}{R(v, b, 1)} \frac{1}{v^n} f(v) = \frac{1}{(1-n)(b-1)^2} \left\{ \frac{R(u, b, 1)}{u^{n-1}} f(u) + \right. \\ \left. + \int_{(\sqrt{b+1})^2}^u \frac{dv}{v^{n-1}} \left[-R(v, b, 1) \frac{d}{dv} f(v) + \frac{(n-2)v + (3-2n)(b+1)}{R(v, b, 1)} f(v) \right] \right\} \quad (5.5)$$

$$\int_{(\sqrt{b+1})^2}^u \frac{dv v^n}{R(v, b, 1)} f(v) = \frac{1}{n} \left\{ R(u, b, 1) u^{n-1} f(u) + \right. \\ \left. + \int_{(\sqrt{b+1})^2}^u dv v^{n-2} \left[-v R(v, b, 1) \frac{d}{dv} f(v) + \frac{(2n-1)(b+1)v - (n-1)(b-1)^2}{R(v, b, 1)} f(v) \right] \right\} \quad (5.6)$$

$$\int_{(\sqrt{b+1})^2}^u \frac{dv}{R(v, b, 1)} \frac{1}{(v-1)^n} f(v) = \frac{1}{b(b-4)(1-n)} \left\{ \frac{R(u, b, 1)}{(u-1)^{n-1}} f(u) + \right. \\ \left. + \int_{(\sqrt{b+1})^2}^u \frac{dv}{(v-1)^{n-1}} \left[-R(v, b, 1) \frac{d}{dv} f(v) + \frac{(n-2)(v-1) + (3-2n)b}{R(v, b, 1)} f(v) \right] \right\} \quad (5.7)$$

Here, $f(v)$ is any function of v , including $f(v) \equiv 1$, regular and well-behaved enough to keep a meaning to these formulae. To prove them, one has simply to integrate back by parts in the left-hand sides the terms containing $f'(v)$.

Similarly, integrating by parts

$$\int_4^{(\sqrt{b}-1)^2} dv \, v^n \sqrt{v(v-4)} R(v, b, 1) \frac{d}{dv} f(v) \quad \text{for } f(v) = 1$$

one obtains

$$\int_4^{(\sqrt{b}-1)^2} \frac{dv}{\sqrt{v(v-4)}} \frac{v^n}{R(v, b, 1)} = \frac{1}{n-1} \int_4^{(\sqrt{b}-1)^2} \frac{dv}{\sqrt{v(v-4)}} \frac{v^{n-1}}{R(v, b, 1)} \cdot \left[(4n-10)(b-1)^2/v^2 - (n-2)(b+3)^2/v + (2n-3)(b+3) \right]. \quad (5.8)$$

Finally, from

$$\frac{\partial}{\partial v} \log \frac{v(v-b+3) + \sqrt{v(v-4)} R(v, b, 1)}{v(v-b+3) - \sqrt{v(v-4)} R(v, b, 1)} = \frac{b-3v+3}{\sqrt{v(v-4)} R(v, b, 1)}$$

one obtains

$$\int_4^{(\sqrt{b}-1)^2} \frac{dv}{\sqrt{v(v-4)}} \frac{v}{R(v, b, 1)} = \frac{b+3}{3} \int_4^{(\sqrt{b}-1)^2} \frac{dv}{\sqrt{v(v-4)} R(v, b, 1)}. \quad (5.9)$$

Note that Eqs. (5.8) and (5.9) are identities (possibly new in the literature) between integrals of elliptic type. All these identities are to be used recursively, both to decrease the order of transcendentality of $f(v)$ (usually a logarithm) and to reduce integrals with different powers of the factors v , $(v-1)$ to a standard type with one power only or no powers at all of the factors $1/v$, $1/(v-1)$. Only at this point the natural variable z can be introduced and the integrals actually performed (see below for elliptic integrals), avoiding an explosion of the number of terms and almost hopeless algebraic complications.

The preparatory algebra for constructing the function $G(u, \mu_1^2)$ with the requirements of Section 3a is a straightforward generalization of the procedure described in Section 3b.

At fourth order, every graph (better, every set of topologically equivalent graphs in the $\Delta \rightarrow 0$ limit) can, in general, be cut in various ways, so that its contribution to the anomaly can be written as

$$a = \int_0^\infty \frac{du}{u-M^2} \sum_i \theta(u-u_i) g_i(u)$$

where the index i runs over all the cuts and u_i is the threshold of the i^{th} cut. To handle the separate contributions we find it necessary to introduce a cut-off in the final dispersive integration and to write

$$a = \lim_{U \rightarrow \infty} \int_0^U \frac{du}{u-M^2} \sum_i \theta(u-u_i) g_i(u) = \lim_{U \rightarrow \infty} \sum_i \int_{u_i}^U \frac{du}{u-M^2} g_i(u).$$

As a matter of fact it turns out that, with the new dispersion relations we use, single cuts do not give separately finite contributions for $U \rightarrow \infty$, but develop in our case divergencies like $U \log U$, U , $\log U$. This is a different feature from the usual t -dispersion relations, in which all cuts give individually finite contributions to the anomaly.

5.2 A class of graphs

For the sake of brevity, we will comment only on the calculation of the contribution from the crossed type of diagram Fig. 10c -- by far the most difficult one. One has to deal with two three-body and two two-body cuts, as shown in Fig. 13. The inverse Feynman propagators D (the cut lines involve the corresponding δ -functions) are

$$\begin{aligned} D_1 &= (p-q)^2 + \mu_1^2, & D_2 &= (p-q-k)^2 + \mu_2^2, \\ D_3 &= (p-k)^2 + \mu_3^2, & D_a &= q^2 + a, & D_{\lambda^2} &= k^2 + \lambda^2. \end{aligned}$$

The masses μ_i , $i = 1, 2, 3$ are to be put equal to M after mass differentiations (if any) are performed. The mass λ is the usual "small photon mass", by which the infrared divergences, when present, are parametrized.

When dealing with the "three-electron" cut of Fig. 13d, after mass differentiations and putting $\mu_i^2 = M^2$ ($i = 1, 2, 3$), one is faced with integrals of the following kind

$$\int_{9M^2}^U \frac{du}{u-M^2} \int_{4M^2}^{(\sqrt{u}-M)^2} db \, f(b, u, a) \cdot \left[\frac{1}{\sqrt{b(b-4M^2)} R(u, b, M^2)} ; \right. \\ \left. \log \frac{b(u-b+3M^2) + \sqrt{b(b-4M^2)} R(u, b, M^2)}{b(u-b+3M^2) - \sqrt{b(b-4M^2)} R(u, b, M^2)} \right],$$

where $f(b, u, a)$ is a rational function. The logarithm is generated by the angular integration. As it stands, the b -integration cannot be performed, because of the two phase-space square roots. The basic idea to overcome this difficulty is to exchange the order of integrations obtaining

$$\int_{9M^2}^U du \int_{4M^2}^{(\sqrt{u}-M)^2} db = \int_{4M^2}^{(\sqrt{U}-M)^2} db \int_{(\sqrt{b}+M)^2}^U du.$$

The u -integration can then be performed exactly. Since this produces, in the place of $R(u, b, M^2)$, the new square root $R(U, b, M^2)$ the last b -integration, strictly speaking, is again elliptic; however one is interested in the $U \rightarrow \infty$ limit only,

and for large U simplifications occur which make possible the analytic integrations.

Because of the vanishing of one mass ($\lambda \rightarrow 0$), the other three-body cut (Fig. 13b) is somewhat simpler than the previous one, the only difficulty being represented in this case by a proper treatment of the infrared divergences. This task is accomplished in a way totally similar to that followed in Refs. 6 and 9, and described there in great detail. To the infrared integrands we add and subtract a suitable "infrared" function, depending on the mass λ , such that: (i) its integral is exactly feasible; (ii) the difference between the original integrand and its "infrared" approximation is infrared regular, in the sense that its integral can be performed after having put $\lambda = 0$.

As a consequence of the squared denominators in the starting expressions here, besides the usual $\lg \lambda$ terms, also factors $1/\lambda$ appear. All these divergences are cancelled by similar ones contained in the contribution of the two-body cut of Fig. 13d.

When considering the two-body cut diagrams of Figs. 13c and d, the one-loop subdiagrams are again computed using dispersion relations.

As a typical term consider

$$S = S(-p^2, -k^2, -(p-k)^2) \equiv i \int \frac{d^4 q}{D_1^2 D_2^2 D_a} (pq)^4,$$

to be evaluated at $-p^2 = u$, $-k^2 = 0$, $-(p-k)^2 = M^2$. It could be evaluated by means of a dispersion relation in $(-p^2)$. It is not vanishing, however, for $-p^2 \rightarrow \infty$. The standard way of mastering the subtractions consists in writing

$$\begin{aligned} S &= p_\mu p_\nu p_\rho p_\sigma i \int \frac{d^4 q}{D_1^2 D_2^2 D_a} q_\mu q_\nu q_\rho q_\sigma = \\ &= p_\mu p_\nu p_\rho p_\sigma \left[s_1 p_\mu p_\nu p_\rho p_\sigma + s_2 (p_\mu p_\nu p_\rho k_\sigma + \text{symm.}) + \dots \right] \end{aligned} \quad (5.10)$$

The invariant amplitudes $s_i \equiv s_i(-p^2, -k^2, -(p-k)^2)$ do satisfy unsubtracted dispersion relations (squared denominators can be treated as in Sections 3.1 and 3.2). However, as there are four tensorial indices to be carried by the two four-vectors p, k , explicit working out of the s_i , although completely straightforward, is exceedingly long and unfeasible in practice.

The actual procedure we follow consists in observing that

$$pq = \frac{1}{2} (p^2 + \mu_1^2 - a - D_1 + D_a) \quad (5.11)$$

and then substituting in Eq. (5.10):

$$S = \frac{1}{2} \left[(p^2 + \mu_1^2 - a) i \int \frac{d^4 q}{D_1^2 D_2^2 D_a} (pq)^3 - i \int \frac{d^4 q}{D_1 D_2^2 D_a} (pq)^3 + i \int \frac{d^4 q}{D_1^2 D_2^2} (pq)^3 \right], \quad (5.12)$$

The three terms on the right-hand side of Eq. (5.12) are simpler than the original one, as they involve less denominators and/or less powers of (pq) . In the terms with three denominators, the identity (5.11) is to be substituted repeatedly, until no powers of (pq) are left. In the other terms with two denominators generated in this procedure, the q -integral depends on a single vector only [k in the case of the third term in Eq. (5.12)]. They can be evaluated by unsubtracted dispersion relations in that squared vector by following the standard way outlined above. Indeed, as a single vector is present, going through the algebra of the invariant amplitudes is much simpler than in the previous case.

Equation (5.11) expresses (pq) , linear in q , as the difference of terms quadratic in q , so that, even starting from convergent integrals, separately divergent terms can appear. Divergent terms, on the other hand, can also be present from the beginning in the two-body cuts involving a vertex subgraph, such as that in Fig. 13c. It is therefore convenient to regularize, by means of an over-all factor $A/(q^2 + A)$, so that all the various terms become separately convergent. As always when dealing with integrals, linearly divergent in the absence of a regularizer, attention is to be made in shifting the loop momentum. For example, in a term like

$$X \equiv i \int \frac{d^4 q}{D_1 D_2} (pq) \frac{A}{q^2 + A},$$

(whose q -integral depends on both p and k) the shift $q \rightarrow q - p$ gives, for large A ,

$$X \simeq i \int \frac{d^4 q}{(q^2 + \mu_1^2) [(k+q)^2 + \mu_2^2]} \frac{A}{q^2 + A} - \frac{1}{2} p^2 i \int \frac{d^4 q}{q^2 + \mu_2^2} \frac{A}{(q^2 + A)^2},$$

to be regarded as a q -integral depending on k only, plus a simple contact term.

The limit $A \rightarrow \infty$ is to be taken at the end of the calculation. When needed, one has first to introduce vertex renormalization counter terms $(-e)(\alpha/\pi)L^{(2)}\gamma_\mu$, with $(M = 1$ for simplicity)

$$L^{(2)}(A, a) = \frac{1}{4} \log A + \frac{9}{8} + \frac{3}{4} a + \left(\frac{1}{2} - \frac{3}{8} a^2 \right) \log a - \\ - \sqrt{a(a-4)} \left(\frac{3}{4} + \frac{3}{8} a + \frac{3}{2(a-4)} \right) \log \frac{\sqrt{a} - \sqrt{a-4}}{\sqrt{a} + \sqrt{a-4}}$$

and

$$L^{(2)}(A, \lambda^2) \simeq \frac{1}{4} \log A + \frac{9}{8} + \log \lambda$$

for graphs 11 and 13 of Fig. 4, respectively.

In the case of these cut diagrams, besides infrared divergencies (to be treated as discussed above) another technical difficulty arises, already mentioned in Section 3: for large values of the squared mass a ($a > \mu_1^2 + \mu_2^2 + 2M^2$), external to the uncut loop, an anomalous threshold appears. To overcome this difficulty without computing any anomalous discontinuity, it is sufficient to do all the calculations, here as well as in all the other cut graphs, for $0 < a < 2M^2$, and to sum up at the end the partial results so obtained. Analytic continuation of the sum to $a \geq 2M^2$ is then straightforward. To exemplify, in terms of the usual variable $y = (\sqrt{a} - \sqrt{a - 4M^2})/(\sqrt{a} + \sqrt{a - 4M^2})$, one encounters the integral

$$y \int_0^1 dz \left[\frac{1}{z^2 + y^2} + \frac{1}{1 + y^2 z^2} \right] = \frac{\pi}{2} \frac{\operatorname{Re} y}{|\operatorname{Re} y|} = -\frac{\pi}{2}$$

because, for $0 < a < 2M^2$, it is $\operatorname{Re} y < 0$ (see Fig. 11).

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Figure captions

- Fig. 1 : a) The only fourth-order vertex graph with vac. pol. insertion;
b) second-order vac. pol.; c) second-order vertex; d) second-order
vertex with "massive photon".
- Fig. 2 : Fourth-order vertex graphs with vac. pol. insertion.
- Fig. 3 : Fourth-order vac. pol. graphs.
- Fig. 4 : Fourth-order vertex graphs without vac. pol. insertion with one
"massive photon".
- Fig. 5 : Fourth-order vertex graphs with vac. pol. insertion and one "massive
photon".
- Fig. 6 : Sixth-order vertex graphs with fourth-order vac. pol. insertion.
- Fig. 7 : Sixth-order vertex graph with a single second-order vac. pol. inser-
tion.
- Fig. 8 : Sixth-order vertex graphs with mixed fourth-order vac. pol. insertion.
- Fig. 9 : a) Second-order vertex amplitude $M_{\mu}^{(2)}(P, \Delta)$; b) its $\Delta = 0$ value;
c) its Δ -derivative at $\Delta = 0$.
- Fig. 10 : Topology of the self-mass-like graphs obtained from the vertex graphs
of Fig. 4, in the $\Delta \rightarrow 0$ limit.
- Fig. 11 : Path of the variable y as $0 < b < \infty$.
- Fig. 12 : Examples of eighth-order diagrams. The contributions from (a) and
(b) are not contained in Eq. (2.26); those from (c) are.
- Fig. 13 : Cuts of the diagram (c) of Fig. 10.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

$$M_{\mu}^{(2)}(p, \Delta) =$$

a)

$$V_{\mu}^{(2)}(p) =$$

b)

$$T_{\nu\mu}^{(2)}(p) =$$

c)

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13