CONVERGENT PERTURBATION EXPANSIONS AND ANALYTICITY PROPERTIES
IN STATIC FIELD - THEORETIC MODELS

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A B S T R A C T

Simplified field-theoretic models, consisting of a charged pion field coupled linearly and quadratically to a fixed extended nucleon with spin and isospin, are considered. For small couplings, it is shown that perturbation theory converges and yields the dressed one-nucleon states and the resolvent of the total Hamiltonian for complex energy. The elastic scattering and one-pion production amplitudes are shown : i) to be analytic functions of the respective complex energies, and ii) to be given by convergent perturbation expansions in certain regions of the associated complex energy planes.

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1. INTRODUCTION

The approach to low energy strong interaction dynamics by means of explicit and simplified field theoretic models has attracted and continues to attract a persistent attention ¹⁾. Historically, non-trivial static models, like the well-known one for P wave pion-nucleon dynamics ²⁾ [and other related theories for S wave interactions ³⁾], have played a prominent role, in spite of the fact that they are not explicitly soluble, and that many problems about their structure and properties from a field theory point of view remain unsolved. In fact, static models allowed to account for certain quantitative features of low energy pion-nucleon scattering, through certain dispersion relation arguments ^{4),2)}, and, moreover, they served as helpful sources of inspiration for other approaches towards strong interactions, like dispersion theory. Given the persistent interest that field theory, in its various aspects, attracts generally, one is naturally led to push further the study of static models and to try to understand better their properties, as they constitute a class of non-trivial theories which are mathematically tractable, and, as said above, they provide a reasonable basis for low energy pion-nucleon dynamics.

This paper is devoted to a mathematical study of certain convergence and analyticity properties in those models. We shall consider a fixed extended nucleon with spin and isospin and a charged pion field coupled to the nucleon internal degrees of freedom both linearly and quadratically, corresponding, say, to P and S wave couplings, respectively. The main results presented here are :

- a) it is shown, for small couplings, that the perturbation expansion for the dressed nucleon states converges ;
- b) it is shown that the elastic scattering and one-pion production amplitudes are analytic functions of the various complex energies (for fixed scattering angles) and that they are given by convergent perturbation series in certain regions of the complex energy planes, for small couplings.

In Section 2, we introduce the class of static models to be treated and present a useful bound for the interaction Hamiltonian. Section 3 contains a study of the convergence of the perturbation series for the resolvent and its subsequent analytic continuation in the energy. Section 4 deals with the corresponding perturbation theory approach to the dressed one-nucleon states, and, in so doing, some mathematical results are obtained about the convergence of Wigner-Brillouin perturbation theory. Section 5 presents, in rather condensed form, representations for the on-shell elastic scattering and one-pion

production amplitudes, in terms of the resolvent and the dressed nucleon states, which generalize other related representations previously given in the literature. Sections 6 and 7 are concerned with deriving analyticity properties for those amplitudes and with showing that the latter exist and are given by perturbation series in some complex energy regions, for small couplings. Section 8 summarizes some features of the two-pion-nucleon elastic scattering amplitude. In Section 9, some estimates of the convergence conditions for the P wave and S wave models are given.

For other mathematical studies of simplified field theoretic models, more or less related to the ones treated here, we refer to Refs. 6) and 7).

2. CHARACTERIZATION OF THE MODELS

2.A Definition of a class of static models :

We shall consider simplified field theoretic models for the low energy pion-nucleon interactions, in which antinucleons, kaons, hyperons as well as direct pion-pion interactions are absent, by assumption. Let the mass and the third isospin component of our (pseudoscalar) pions be denoted by $\mu(>0)$ and $j(=1,2,3)$, respectively. The bare pion states with definite three-momenta $\vec{k}_1 \dots \vec{k}_n$, energies $\omega(\vec{k}_1) \dots \omega(\vec{k}_n)$ [$\omega(\vec{k}) = (\mu^2 + \vec{k}^2)^{\frac{1}{2}}$] and isospin projections $j_1 \dots j_n$ are the fully symmetric kets $\psi(\vec{k}_1 j_1 \dots \vec{k}_n j_n)$, which are normalized as

$$\begin{aligned} (\Psi(\vec{k}'_1 j'_1 \dots \vec{k}'_n j'_n), \Psi(\vec{k}_1 j_1 \dots \vec{k}_n j_n)) = & \frac{\delta_{n'n}}{n!} \left[\delta^{(3)}(\vec{k}'_1 - \vec{k}_1) \delta_{j'_1 j_1} \dots \delta^{(3)}(\vec{k}'_n - \vec{k}_n) \delta_{j'_n j_n} + \right. \\ & \left. + \text{permutations} \right] \end{aligned} \quad (2.A.1)$$

Specifically, the models under consideration will be characterized by the following assumptions :

- a) there is just one recoil-less nucleon which absorbs and emits pions, and it can be regarded as a fixed source, described by a real density $\rho(|\vec{x}|)$. The third component of the spin and isospin degrees of freedom of the nucleon will be denoted, respectively, by $\sigma(=\pm\frac{1}{2})$ and $\tau(=\pm\frac{1}{2})$, and the bare nucleon states will be represented by the four-dimensional spinors $\psi(\sigma, \tau)$. It is always possible, and convenient in our case, to assume that the bare nucleon energy vanishes ;
- b) the total Hamiltonian H reads, in terms of the standard pion creation and destruction operators [$a(\vec{k}j)^+$ and $a(\vec{k}j)$]

$$H = H_\pi + H_1 + H_2 \quad (2.A.2)$$

$$H_{\pi} = \sum_{j=1}^3 \int d^3 \bar{k} \cdot \omega a(\bar{k}j)^{\dagger} a(\bar{k}j) \quad , \quad \omega = \omega(\bar{k}) \quad (2.A.3)$$

$$H_1 = \sum_{j=1}^3 \int d^3 \bar{k} v(k) \frac{k}{\omega^{1/2}} \left[u(\hat{k}j) a(\bar{k}j) + u(\hat{k}j)^{\dagger} a(\bar{k}j)^{\dagger} \right] \quad (2.A.4)$$

$k = |\bar{k}| \quad , \quad \hat{k} = \bar{k}/k$

$$H_2 = \sum_{j_1, j_2=1}^3 \int d^3 \bar{k}_1 d^3 \bar{k}_2 v(k_1) v(k_2) \left\{ w_1(\bar{k}_1 j_1 \bar{k}_2 j_2) a(\bar{k}_1 j_1) a(\bar{k}_2 j_2) + \right. \\ \left. + w_1(\bar{k}_1 j_1 \bar{k}_2 j_2)^{\dagger} a(\bar{k}_2 j_2)^{\dagger} a(\bar{k}_1 j_1)^{\dagger} + [w_2(\bar{k}_1 j_1 \bar{k}_2 j_2) + w_2(\bar{k}_2 j_2 \bar{k}_1 j_1)^{\dagger}] a(\bar{k}_1 j_1)^{\dagger} \right. \\ \left. \cdot a(\bar{k}_2 j_2) \right\} + H'_2 \quad (2.A.5)$$

where $v(k) = \int d^3 \bar{x} \rho(|\bar{x}|) \exp i \bar{k} \bar{x}$, with $v(k)^* = v(k)$, $v(0) = 1$, and it is assumed that $v(k) \rightarrow 0$ as $k \rightarrow +\infty$ in a way to be specified later. We assume that H_1 corresponds to the well-known P wave Chew-Low Hamiltonian ^{4),5),2)}, that is, $u(\hat{k}j) = i/[2(2\pi)^3]^{1/2} \cdot f_0/u \cdot (\bar{\sigma} \hat{k}) \otimes \tau_j$, where f_0 is the dimensionless (unrenormalized) pion-nucleon coupling constant and the Pauli spin and isospin matrices $\bar{\sigma}$, $\bar{\tau}$ represent the dynamical variables of the nucleon. The w 's and 4×4 matrices, independent on a , $a^{\dagger}$ but dependent on $\bar{k}$'s, j 's and $\bar{\sigma}$, $\bar{\tau}$, and H'_2 is a real constant times the unit 4×4 matrix. The term H_1 is invariant under space and time inversions and under rotations in ordinary and isotopic spin spaces, and we shall assume that the same is true for H_2 . Later, we shall impose further restrictions upon the w 's (see Section 5). In this class of models, the bare and the physical pion states coincide as there is no vacuum polarization, but the interaction dresses the nucleon, so that the physical nucleon states are different from the bare ones.

Notice that in principle $\rho(|\bar{x}|)$ does not vanish necessarily beyond a certain radius, but it does as $|\bar{x}| \rightarrow +\infty$. Consequently, if one considers, for instance, the pion current operator, one sees easily that, in principle, the commutator of two such current operators vanishes only at infinite distances from the source.

For later convenience, we shall introduce the pion number operator

$$N_{\pi} = \sum_{j=1}^3 \int d^3 \bar{k} a(\bar{k}j)^{\dagger} a(\bar{k}j) \quad (2.A.6)$$

as well as the complex constants $\beta_h(\sigma\tau\sigma'\tau' \hat{k}j)$, $h = 1, 2$ $\gamma_r(\sigma\tau\sigma'\tau' \bar{k}_1 j_1 \bar{k}_2 j_2)$, $r = 1, 2, 3$ which express how u 's and w 's act upon the bare nucleon states :

$$u(\hat{k}j)\Psi(\sigma\tau) = \sum_{\sigma'\tau'=-1/2}^{+1/2} \beta_1(\sigma\tau\sigma'\tau' \hat{k}j) \Psi(\sigma'\tau')$$

$$w_1(\bar{k}_1 j_1 \bar{k}_2 j_2)\Psi(\sigma\tau) = \sum_{\sigma'\tau'=-1/2}^{+1/2} \gamma_1(\sigma\tau\sigma'\tau' \bar{k}_1 j_1 \bar{k}_2 j_2) \Psi(\sigma'\tau')$$

(2.A.7)

and so on for $u(\hat{k}j)^+$, $w_1(\bar{k}_1 j_1 \bar{k}_2 j_2)^+$ and $[w_2(\bar{k}_1 j_1 \bar{k}_2 j_2) + w_2(\bar{k}_2 j_2 \bar{k}_1 j_1)^+]$ associated with the three constants β_2 , γ_2 and γ_3 , respectively.

The term H_2 embodies, in particular, the S wave interaction Hamiltonian used by Drell, Friedman and Zachariasen which, in terms of the pion field operator $\bar{\varphi}(\bar{x})$ and the conjugate momentum $\bar{\pi}(\bar{x})$ reads

$$\frac{f_1}{\mu} \int d^3\bar{x}_1 d^3\bar{x}_2 \varphi(\bar{x}_1) \varphi(\bar{x}_2) [\bar{\varphi}(\bar{x}_1) \bar{\varphi}(\bar{x}_2)] + \frac{f_2}{\mu^2} \int d^3\bar{x}_1 d^3\bar{x}_2 \varphi(\bar{x}_1) \varphi(\bar{x}_2) \left\{ \bar{\pi}(\bar{x}_1) \wedge \bar{\pi}(\bar{x}_2) \right\}$$

where f_1 and f_2 are dimensionless coupling constants. Notice that, in this case, $H_2' = f_1/\mu \int d^3\bar{k} [v(k)]^2 / (2\pi)^3 2\omega$.

It must be remarked that, in general, we shall be dealing with cut-off versions of rather singular field theoretic models. In fact, in the limit of a point source, and at least for the Chew-Low Hamiltonian H_1 , the coupling of the pion field to the nucleon spin gives rise, in principle, to an infinite number of primitively divergent contributions to the nucleon self-energy. Such an unpleasant feature is absent, in general, in the models without internal degrees of freedom for the nucleon ^{6), 7)}.

2.B Some mathematical notions and a basic inequality :

The Hilbert space $\mathcal{H}$ is formed by normalizable states ψ describing situations with one nucleon always present and any number of mesons, for arbitrary values of σ , τ , $\bar{k}$'s and j 's. More precisely, let us consider all kets ψ which be linear superpositions of the tensor products of $\psi(\sigma\tau)$ and $\psi(\bar{k}_1 j_1 \dots \bar{k}_n j_n)$, namely

$$\Psi = \sum_{n=0}^{\infty} \sum_{\sigma, \tau = -1/2}^{+1/2} \sum_{j_1, \dots, j_n=1}^3 \int d^3 \bar{k}_1 \dots d^3 \bar{k}_n \alpha^{(n)}(\sigma \tau, \bar{k}_1 j_1, \dots, \bar{k}_n j_n) \cdot \left[\Psi(\sigma \tau) \otimes \Psi(\bar{k}_1 j_1, \dots, \bar{k}_n j_n) \right] \quad (2.B.1)$$

where the complex amplitude $\alpha^{(n)}(\sigma \tau, \bar{k}_1 j_1, \dots, \bar{k}_n j_n)$ is symmetric under the exchange of $(\bar{k}_r j_r)$ and $(\bar{k}_s j_s)$, $r, s = 1 \dots n$, and let us define the scalar product of Ψ_1, Ψ_2 as

$$(\Psi_1, \Psi_2) = \sum_{n=0}^{\infty} \sum_{\sigma, \tau = -1/2}^{+1/2} \sum_{j_1, \dots, j_n=1}^3 \int d^3 \bar{k}_1 \dots d^3 \bar{k}_n \alpha^{(n)}(\sigma \tau, \bar{k}_1 j_1, \dots, \bar{k}_n j_n)^* \cdot \alpha^{(n)}(\sigma \tau, \bar{k}_1 j_1, \dots, \bar{k}_n j_n) \quad (2.B.2)$$

[Equation (2.B.2) is, of course, consistent with Eq. (2.A.1) which refers to unnormalizable kets lying outside $\mathcal{H}$]. Then, $\mathcal{H}$ is formed only by those kets Ψ such that $\|\Psi\| = (\Psi, \Psi)^{1/2} < +\infty$. As usually, the norm $\|A\|$ of an operator A is the least upper bound (l.u.b.) of $\|A\Psi\|/\|\Psi\|$ as Ψ varies throughout the domain of A . As it is well known, operators like H_π and N_π are not defined throughout all $\mathcal{H}$, but only in certain dense domains of it such that, say, $H_\pi \Psi$ also belongs to $\mathcal{H}$. Throughout this paper, we shall not treat those subtle aspects related to the domains of definition of the operators involved and we shall simply accept that H_π and N_π are well-defined self-adjoint operators. The spectrum of H_π in $\mathcal{H}$ consists of a discrete eigenvalue, $E = 0$, with a fourfold degeneracy, corresponding to the four bare nucleon states $(\Psi(\sigma \tau))$ and a continuum from $E = \mu > 0$ up to $E = +\infty$, corresponding to the bare nucleon plus $n \geq 1$ pion states. We shall admit without further discussion that when $v(k)$ and the w 's are smooth and vanish suitably at infinity, then both H_1 and H_2 can be defined as symmetric operators in suitable domains.

At this point, one can show that, for any normalizable state Ψ belonging to the domains of definition of H_π, H_1 and H_2 , the following important inequality holds :

$$\|(H_1 + H_2)\Psi\| \leq e_1 \|H_\pi \Psi\| + e_2 \|\Psi\| \quad (2.B.3)$$

where e_1 and e_2 are positive constants. Their expressions in terms of the cut-off function, β 's and γ 's, as well as a sketch of the proof of the inequality (2.B.3) are given in Appendix A. It is assumed that the cut-off function, β 's and γ 's are such that $e_1, e_2 < +\infty$. Moreover, by playing properly with the domains of definition of H_1, H_2 and H_π and imposing that $e_1 < 1$, one can invoke the Kato-Rellich theorem^{8),9)} and conclude that: H is a self-adjoint operator, whose spectrum is real and bounded from below. Notice that if $H_2 = 0$, so that only the Chew-Low Hamiltonian remains, by playing with the bounds given just at the end of Appendix A, one can always make e_1 smaller than 1 (provided that $a'_s < +\infty$, $s = 1, 2$, of course), and, hence, arrive at the self-adjointness of $H_\pi + H_1$ under rather general conditions. This remark is due to Professor V. Glaser. It is pointed out that our results in Appendix A and Section 3 constitute a generalization to (non-soluble) static models with spin, isospin and linear and quadratic couplings to the pion field, of some results obtained earlier by Y. Kato⁶⁾ for the simpler (soluble) model without internal degrees of freedom and linear coupling. Unfortunately, in our case we cannot invoke the solvability of the models in order to construct the eigenkets of the total Hamiltonian, whereas Y. Kato was able to do so in his case. Consequently, we shall be forced to undertake a direct mathematical study of the convergence properties of the dressed nucleon states in Section 4.

3. THE RESOLVENT AND ITS DETERMINATION

We shall construct the resolvent of the total Hamiltonian H through a convergent perturbation series with the aid of (2.B.3).

Let $G(z) = (z-H)^{-1}$ and $G_\pi(z) = (z-H_\pi)^{-1}$ be the resolvents of H and H_π , z being an arbitrary real or complex energy. Let us generate formally the perturbation series for $G(z)$ in terms of $G_\pi(z)$:

$$G(z) = G_\pi(z) + G_\pi(z)(H_1 + H_2)G(z) = G_\pi(z) \sum_{n=0}^{\infty} \left[(H_1 + H_2)G_\pi(z) \right]^n \quad (3.1)$$

Let $\rho(z)$ be the smallest distance from the point z to the whole spectrum of H_π (including the origin), and let $\rho(z) > 0$, which holds if $\text{Im } z \neq 0$ and also if $\text{Im } z = 0$ but either $-\infty < \text{Re } z < 0$ or $0 < \text{Re } z < \mu$. One has the typical resolvent estimates $\|G_\pi(z)\| = 1/\rho(z)$, $\|H_\pi G_\pi(z)\| \leq 1 + |z|/\rho(z)$ [see 9) for further details], which, through (2.B.3), imply:

$$\|(H_1 + H_2)G_\pi(z)\| \leq e_1 \left[1 + \frac{|z|}{\rho(z)} \right] + \frac{e_2}{\rho(z)} \equiv e(z) \quad (3.2)$$

$e(z)$ being dimensionless. Coming to the perturbation series (3.1), majorizing by means of the above inequalities and summing the resulting geometrical series one bounds the norm of the resolvent :

$$\|G(z)\| \leq \frac{1}{\rho(z)[1-e(z)]} \quad (3.3)$$

Once e_1 and e_2 are given (see Appendix A), let us assume $e_1 < \frac{1}{2}$, whose necessity will be shortly realized, and let us find out the regions in the z plane for which $e(z) < 1$, so that $\|G(z)\| < +\infty$ and, hence, the perturbation series (3.1) converges in norm. From the definition of $e(z)$ and after a glance at the spectrum of H_π we conclude that :

- a) in the half-plane $\operatorname{Re} z \leq \mu/2$, the convergence region is $|z| > e_2/1-2e_1$;
- b) in the strip $\mu/2 < \operatorname{Re} z < \mu$, the series (3.1) converges if

$$e_1 \left\{ 1 + \left[\frac{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}{(\operatorname{Re} z - \mu)^2 + (\operatorname{Im} z)^2} \right]^{1/2} \right\} + \frac{e_2}{[(\operatorname{Re} z - \mu)^2 + (\operatorname{Im} z)^2]^{1/2}} < 1$$

- c) in the half-plane $\mu \leq \operatorname{Re} z$, the convergence region is

$$e_1 \left\{ 1 + \left[1 + \left(\frac{\operatorname{Re} z}{\operatorname{Im} z} \right)^2 \right]^{1/2} \right\} + \frac{e_2}{|\operatorname{Im} z|} < 1$$

It is remarked that : i) the condition $e_1 < \frac{1}{2}$ implies that the three regions are non-empty, ii) $e(z) < 1$ holds in cases a), b) for $|\operatorname{Im} z|$ large enough and in case b) when, in addition, $|\operatorname{Re} z|$ is sufficiently small, iii) the interval $-\infty < \operatorname{Re} z < e_2/1-2e_1$, $\operatorname{Im} z = 0$, always belongs to region a), iv) if $e_1 + e_2/\mu < \frac{1}{2}$ holds, there exists a complex domain of convergence about $\operatorname{Re} z = \mu/2$, $\operatorname{Im} z = 0$, which belongs to regions a) and b), v) the interval $\mu \leq \operatorname{Re} z < +\infty$, $\operatorname{Im} z = 0$ is completely outside the convergence regions.

As a matter of fact, for the given e_1, e_2 with either $e_1 < \frac{1}{2}$ or $e_1 + e_2/\mu < \frac{1}{2}$, $G(z)$ can be calculated, at least in principle, for any z not belonging to the spectrum of H (which was seen to be real and lower bounded, by virtue of the Kato-Rellich theorem). In fact, one has $\|G(z)\| = 1/\tilde{\rho}(z)$ where $\tilde{\rho}(z)$ is the smallest distance from z to the full real spectrum of H , so that for any complex z_0 lying inside the above convergence regions a), b), c) and sufficiently close to their boundaries $\|G(z_0)\|$ remains finite since

$\tilde{p}(z_0) > p(z_0)[1 - e(z_0)]$ necessarily. Then, by using $G(z) = G(z_0) + (z_0 - z)G(z_0)G(z) = G(z_0) \sum_{n=0}^{\infty} (z_0 - z)^n G(z_0)^n$ and performing a direct majorization, one gets $\|G(z)\| \leq 1/(\tilde{p}(z_0) - |z - z_0|) < +\infty$ for some z outside the above regions a), b), c). Thus, the analytic continuation in z of the resolvent (and its matrix elements) outside the regions a), b), c) is necessarily possible.

4. THE PHYSICAL ONE-NUCLEON STATES AND THEIR DETERMINATION

Let $\psi(\sigma\tau)_+$ be the dressed one-nucleon states with physical energy M , so that $H\psi(\sigma\tau)_+ = M\psi(\sigma\tau)_+$ and $(\psi(\sigma'\tau')_+, \psi(\sigma\tau)_+) = \delta_{\sigma'\sigma}\delta_{\tau'\tau}$. Here, σ and τ denote the spin and isospin projections of that bare nucleon state $\psi(\sigma\tau)$ such that $\psi(\sigma\tau)_+ \rightarrow \psi(\sigma\tau)$ as the interaction $H_1 + H_2$ is turned off. There are several ways of determining the dressed states $\psi(\sigma\tau)_+$.

One of them proceeds through the introduction of the projector P_+ on the four-dimensional space spanned by all $\psi(\sigma\tau)_+$, inside the Hilbert space $\mathcal{H}$, and the related operator HP_+ . Let us give a sketch of this method. Upon applying well-known recipes from operator theory ^{8),9)}, one gets the following representations for P_+ and HP_+ in terms of $G(z)$

$$P_+ = \frac{1}{2\pi i} \oint_{\Gamma} G(z) dz, \quad HP_+ = \frac{1}{2\pi i} \oint_{\Gamma} z G(z) dz \quad (4.1)$$

Here, Γ denotes a circle in the complex z plane of radius $\mu/2$ and whose centre is the origin. Then, by noting that for any z on Γ one has $\|G_{\pi}(z)\| \leq 2/\mu$, $\|(H_1 + H_2)G_{\pi}(z)\| \leq 2(e_1 + e_2/\mu)$, inserting the perturbation series (3.1) into the right-hand sides of Eqs. (4.1) and majorizing, one obtains

$$\|P_+\| \leq \frac{1}{1 - 2(e_1 + e_2/\mu)}, \quad \|HP_+\| \leq \frac{\mu}{2[1 - 2(e_1 + e_2/\mu)]}$$

Thus, one concludes that convergent perturbation theory yields both P_+ and HP_+ when $e_1 + e_2/\mu < \frac{1}{2}$. Then, in order to obtain $\psi(\sigma\tau)_+$ and the physical mass, the only remaining step consists in diagonalizing the 4×4 matrix HP_+ in the four-dimensional subspace of $\mathcal{H}$ determined by P_+ . We shall not discuss this, as one can appeal to standard methods ¹⁰⁾ and we shall turn to another more transparent way of finding out the physical nucleon states, which consists in using Wigner-Brillouin perturbation theory ¹¹⁾. In so doing, we shall present a mathematical treatment which is well suited for convergence studies.

Let P denote the projector on the four-dimensional subspace spanned by all the bare states $\psi(\sigma\tau)$ and let us introduce temporarily the new dressed state $\psi_1(\sigma\tau)_+$ such that $(\psi(\sigma\tau), \psi_1(\sigma\tau)_+) = 1$. One has $\psi(\sigma\tau)_+ = z_2^{-\frac{1}{2}} \psi_1(\sigma\tau)_+$, where z_2 is a normalization constant which can always be determined trivially once $\psi_1(\sigma\tau)_+$ has been found. One can easily derive the following integral equation à la Wigner-Brillouin for $\psi_1(\sigma\tau)_+$:

$$\psi_1(\sigma\tau)_+ = \psi(\sigma\tau) + (1-P)(G_\pi(M)(H_1+H_2)\psi_1(\sigma\tau)_+ \quad (4.2)$$

while the physical energy, M , is given in our case by

$$M = \left(\psi(\sigma\tau), (H_1+H_2)\psi_1(\sigma\tau)_+ \right) \quad (4.3)$$

It will be convenient to allow M be complex, for a while. Let $\rho_2(M)$ be the smallest distance from M to the continuous spectrum of H_π , and let us assume $|M| < \mu/2$, so that $\rho_2(M) \geq \mu/2 > 0$. Then, since $(1-P)G_\pi(M)$ is the resolvent of H_π in the infinite dimensional subspace orthogonal to the four-dimensional one spanned by P , one derives $\|(1-P)G_\pi(M)\| \leq 1/\rho_2(M)$, $\|H_\pi(1-P)G_\pi(M)\| \leq 1+|M|/\rho_2(M)$ and, using (2.B.3):

$$\|(H_1+H_2)(1-P)G_\pi(M)\| \leq e_1 \left[1 + \frac{|M|}{\rho_2(M)} \right] + \frac{e_2}{\rho_2(M)} = e_4(M) \quad (4.4)$$

Upon iterating Eq. (4.2), majorizing the resulting series with the aid of the above inequalities and using (2.B.3) as well as $H_\pi\psi(\sigma\tau) = 0$ and $\|\psi(\sigma\tau)\|=1$, one gets finally:

$$\|\psi_1(\sigma\tau)_+\| \leq 1 + \frac{e_4}{\rho_2(M)[1-e_4(M)]}$$

For fixed e_1, e_2 such that $e_1 + e_2/\mu < \frac{1}{2}$ and for complex M such that $|M| < \mu/2$, we have $e_4(M) < 1$, so that the series obtained by iterating (4.2) converges in norm and yields $\psi_1(\sigma\tau)_+ = \psi_1(\sigma\tau, M)_+$ as a function of the complex parameter M . The real physical nucleon energy has to be determined from the equation $M = O(M)$, where $O(M) \equiv (\psi(\sigma\tau), (H_1+H_2)\psi_1(\sigma\tau, M)_+)$. We can show that for e_1, e_2 small enough, then the real physical energy can be found by applying the contraction mapping principle ¹²⁾ to the function $M \rightarrow M' = O(M)$. Let us sketch this construction. First, we notice that if the additional condition $\|H_2'\| + [2(e_1+e_2/\mu)/(1-2(e_1+e_2)/\mu)] \cdot e_2 < \mu/2$ holds, then the complex domain $|M| < \mu/2$ maps into itself by means of the function $O(M)$. In fact, upon inserting the series for $\psi_1(\sigma\tau, M)_+$ into the definition of $O(M)$ and majorizing, one derives $|O(M)| \leq \|H_2'\| + e_4(M)e_2/(1-e_4(M))$, which is bounded by $\mu/2$ by virtue of the above condition, whose fulfilment is assumed in what follows. Second, if we consider two complex values M_i and their associates $M_i' = O(M_i)$, $i = 1, 2$, then one can show that

$$|M'_1 - M'_2| \leq e_2 \eta |M_1 - M_2| \quad (4.5)$$

where η is real and positive and the product $e_2 \eta$ can be made smaller than 1 for sufficiently small e_1, e_2 . The proof of (4.5) is skipped to Appendix B. By invoking the contraction mapping principle¹²⁾ one concludes that there exists a value M such that: i) $M = O(M)$, ii) it is locally unique. Since this fixed point, M , is the eigenvalue of a self-adjoint operator, H , it is necessarily real.

5. THE ELASTIC SCATTERING AND THE ONE-PION PRODUCTION AMPLITUDES

We shall treat the scattering and production processes of the pions by the fixed nucleon which are not trivial due to the nucleon degrees of freedom and to the quadratic coupling. For that purpose and in what follows, we shall assume that $w_h(\bar{k}_1 j_1 \bar{k}_2 j_2)$, $h = 1, 2$, bears the form

$$\sum_{n=1}^N g_{1hn}(\omega_1) g_{2hn}(\omega_2) \tilde{w}_{hn}(\hat{k}_1 j_1 \hat{k}_2 j_2)$$

where the $\tilde{w}$'s are 4×4 matrices independent on ω_1, ω_2 , the g 's are arbitrary (kinematical) real functions of the pion energies and $N < +\infty$ for simplicity. The interest of this assumption is twofold: i) the physically interesting S wave Drell, Friedman and Zachariasen Hamiltonian fulfils it, ii) it will enable one to derive analyticity properties for elastic scattering and production amplitudes. For convenience, we shall define for $l, h = 1, 2$, $g_{lho}(\omega) = (k/\omega^{1/2}) \delta_{l1} \delta_{h1}$, $R_{lho}(\hat{k}j) = u(\hat{k}j) \delta_{l1} \delta_{h1}$ and for $n = 1 \dots N$

$$\begin{aligned} R_{11n}(\hat{k}j) &= \sum_{j_1=1}^3 \int d^3 \bar{k}_1 v(k_1) g_{21n}(\omega_1) \tilde{w}_{1n}(\hat{k}j \hat{k}_1 j_1) a(\bar{k}_1 j_1) \\ R_{12n}(\hat{k}j) &= \sum_{j_1=1}^3 \int d^3 \bar{k}_1 v(k_1) g_{22n}(\omega_1) \tilde{w}_{2n}(\hat{k}j \hat{k}_1 j_1) a(\bar{k}_1 j_1)^+ \end{aligned} \quad (5.1)$$

and so on for $R_{21n}(\hat{k}j)$, $R_{22n}(\hat{k}j)$, corresponding respectively to

$$g_{11n} \tilde{w}_{1n}(\hat{k}_1 j_1 \hat{k}j)^+ a(\bar{k}_1 j_1) \quad g_{12n} \tilde{w}_{2n}(\hat{k}_1 j_1 \hat{k}j) a(\bar{k}_1 j_1)^+$$

Let us consider an incoming free pion, $\psi(\vec{k}j)$, which in the remote past is far from the physical nucleon, and the corresponding elastic scattering amplitude $T(\sigma'\tau'\vec{k}'j'; \sigma\tau\vec{k}j)$ and the one-pion production amplitude $T(\sigma'\tau'\vec{k}'j'; \vec{k}_2j_2; \sigma\tau\vec{k}j)$. In general, the amplitude $T(f,i)$ for the process $i \rightarrow f$ is related to the s matrix element $s(f,i)$ by $s(f,i) = \delta_{fi} + 2\pi i \delta(E_f - E_i) T(f,i)$ where δ_{fi} is given by the right-hand side of Eq. (2.A.1) times $\delta_{\sigma'\tau', \sigma\tau}$. Upon generalizing the reduction technique à la Chew-Low-Wick 4), 5), 2) and performing some calculations, which we omit for brevity, one obtains explicit representations for those amplitudes, in terms of the resolvent, g 's, w 's, R 's and the dressed nucleon states. Those representations, which constitute a generalization of related ones given in Refs. 4), 5) and 13), read :

$$\frac{T(\sigma'\tau'\vec{k}'j'; \sigma\tau\vec{k}j)}{v(\kappa')v(\kappa)} = (\Psi(\sigma'\tau')_+, [w_2(\vec{k}'j'\vec{k}j) + w_2(\vec{k}j\vec{k}'j')^\dagger] \Psi(\sigma\tau)_+ + \sum_{n,n'=0}^N \sum_{\ell h \ell' h'=1}^2 g_{\ell h n}(\omega) g_{\ell' h' n'}(\omega) t(\ell' h' n' \ell h n, \omega), \quad \kappa' = \kappa \quad (5.2)$$

$$\frac{T(\sigma'\tau'\vec{k}'_1j'_1\vec{k}'_2j'_2; \sigma\tau\vec{k}j)}{v(\kappa'_1)v(\kappa'_2)v(\kappa)} = t^{(0)}(\omega'_1\omega'_2) + \sum_{n,n_1,n_2=0}^N \sum_{\ell h \ell_1 h_1 \ell_2 h_2=1}^2 g_{\ell h n}(\omega) \cdot g_{\ell_1 h_1 n_1}(\omega_1) g_{\ell_2 h_2 n_2}(\omega_2) t(\ell_1 h_1 n_1 \ell_2 h_2 n_2 \ell h n, \omega'_1 \omega'_2), \quad \omega'_1 + \omega'_2 = \omega \quad (5.3)$$

where the auxiliary amplitudes, the t 's, are given by

$$t(\ell' h' n' \ell h n, z) = (\Psi(\sigma'\tau')_+, [R_{\ell' h' n'}(\vec{k}'j')^\dagger G(M+z) R_{\ell h n}(\vec{k}j) + R_{\ell h n}(\vec{k}j) G(M-z) R_{\ell' h' n'}(\vec{k}'j')^\dagger] \Psi(\sigma\tau)_+ \quad (5.4)$$

$$t^{(0)}(z_1, z_2) = (\Psi(\sigma'\tau')_+, \left\{ \left([w_1(\vec{k}'_1j'_1\vec{k}'_2j'_2) + w_1(\vec{k}'_2j'_2\vec{k}'_1j'_1)^\dagger] G(M+z_1+z_2) \cdot Q(\omega, \vec{k}j) + Q(\omega, \vec{k}j) G(M-z_1-z_2) [w_1(\vec{k}'_1j'_1\vec{k}'_2j'_2) + w_1(\vec{k}'_2j'_2\vec{k}'_1j'_1)^\dagger]^\dagger \right) + \right. \\ \left. + \left(Q(\omega'_2 \vec{k}'_2j'_2)^\dagger G(M+z_2) [w_2(\vec{k}'_1j'_1\vec{k}j) + w_2(\vec{k}j\vec{k}'_1j'_1)^\dagger] + [w_2(\vec{k}'_1j'_1\vec{k}j) + w_2(\vec{k}j\vec{k}'_1j'_1)^\dagger] G(M-z_2) Q(\omega'_2, \vec{k}'_2j'_2) \right) + (1 \leftrightarrow 2) \right\} \Psi(\sigma\tau)_+ \quad (5.5)$$

$$\begin{aligned}
 t(\ell_1 h_1 n_1 \ell_2 h_2 n_2, z_1 z_2) = & \left(\Psi(\sigma' \tau')_+, \left\{ R_{\ell_2 h_2 n_2}(\hat{k}'_2 j'_2)^+ G(M+z_2) R_{\ell_1 h_1 n_1}(\hat{k}'_1 j'_1)^+ \right. \right. \\
 & \cdot G(M+z_1+z_2) R_{\ell h n}(\hat{k} j) + (1 \rightleftharpoons 2) + R_{\ell_1 h_1 n_1}(\hat{k}'_1 j'_1)^+ G(M+z_2) R_{\ell h n}(\hat{k} j) \cdot \\
 & \cdot G(M-z_1) R_{\ell_1 h_1 n_1}(\hat{k}'_1 j'_1)^+ + (1 \rightleftharpoons 2) + R_{\ell h n}(\hat{k} j) G(M-z_1-z_2) R_{\ell_1 h_1 n_1}(\hat{k}'_1 j'_1)^+ \\
 & \left. \left. \cdot G(M-z_2) R_{\ell_2 h_2 n_2}(\hat{k}'_2 j'_2)^+ + (1 \rightleftharpoons 2) \right\} \Psi(\sigma \tau)_+ \right) \quad (5.6)
 \end{aligned}$$

The symbol $(1 \rightleftharpoons 2)$ denotes in each case the contribution which is obtained from the one preceding it, just by interchanging $(z_1 \omega_1 \hat{k}'_1 j'_1)$ by $(z_2 \omega_2 \hat{k}'_2 j'_2)$, we have defined $Q(\omega \hat{k} j) = \sum_{n=0}^N \sum_{h=1}^2 g_{\ell h n}(\omega) R_{\ell h n}(\hat{k} j)$, and as in what follows $\hat{k} j \hat{k}' j' \hat{k}_1 j_1 \hat{k}_2 j_2$ will be held fixed (at their physical values, of course), the dependence of the t 's on the latter has not been explicit. Notice that the first term on the right-hand side of (5.2) and $t^{(0)}$ are due to the Hamiltonian H_2 , and hence vanish in the Chew-Low model.

6. PROPERTIES OF THE ELASTIC SCATTERING AMPLITUDE

As in the simpler Chew-Low model, one can derive formally analyticity properties for $t(\ell' h' n' \ell h n, z)$ in the complex z plane. In fact, upon inserting a suitable complete set of eigenstates of H for each resolvent on the right-hand side of Eq. (5.4), separating the one physical nucleon contribution and manipulating, one gets :

$$\begin{aligned}
 t(\ell' h' n' \ell h n, z) = & \frac{\Phi_0(\ell' h' n' \ell h n)}{z} + \int_{-\infty}^{+\infty} dE' \left[\frac{\Phi_1(\ell' h' n' \ell h n, E')}{z - E'} + \right. \\
 & \left. + \frac{\Phi_2(\ell' h' n' \ell h n, E')}{z + E'} \right] \quad (6.1)
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi_0(\ell' h' n' \ell h n) = & \sum_{\sigma'' \tau'' = -1/2}^{+1/2} \left[\left(\Psi(\sigma' \tau')_+, R_{\ell' h' n'}(\hat{k}' j')^+ \Psi(\sigma'' \tau'')_+ \right) \cdot \right. \\
 & \cdot \left(\Psi(\sigma'' \tau'')_+, R_{\ell h n}(\hat{k} j) \Psi(\sigma \tau)_+ \right) - \left(\Psi(\sigma' \tau')_+, R_{\ell h n}(\hat{k} j) \Psi(\sigma'' \tau'')_+ \right) \cdot \\
 & \left. \cdot \left(\Psi(\sigma'' \tau'')_+, R_{\ell' h' n'}(\hat{k}' j')^+ \Psi(\sigma \tau)_+ \right) \right] \quad (6.2)
 \end{aligned}$$

and so on for φ_1, φ_2 , corresponding to intermediate states with $n \geq 1$ pions. Equation (6.1) implies that $t(\ell'h'n'\ell hn z)$ is an analytic function of z in the whole z plane, except for a pole at $z = 0$ and two real non-overlapping cuts, so that for $\omega \geq \mu$, $\lim_{\epsilon \rightarrow 0^+} t(\ell'h'n'\ell hn, \omega + i\epsilon)$ gives rise, through Eq. (5.2) to the physical elastic amplitude $T(\sigma'\tau'\bar{k}'j'; \sigma\tau\bar{k}j)$ and so on for $\omega \leq -\mu$ and the crossed process. Since these properties have been derived only formally, as we know nothing about φ_1, φ_2 and the eigenstates of H corresponding to $n \geq 1$ pions, let us turn to more rigorous derivations. When e_1, e_2 are small enough for Wigner-Brillouin perturbation theory to give the dressed states $\psi_1(\sigma\tau)_+$, then one can show that $t(\ell'h'n'\ell hn z)$ exists and can be obtained by perturbation theory and further analytic continuation in z , except perhaps at the real singularities. This result can be proved by :

i) majorizing Eq. (5.4) :

$$|t(\ell'h'n'\ell hn, z)| \leq |z_2| \left\{ \|R_{\ell'h'n'}(\hat{k}'j')\psi_1(\sigma\tau)_+\| \cdot \|G(M+z)\| \cdot \|R_{\ell hn}(\bar{k}j)\psi_1(\sigma\tau)_+\| + \right. \\ \left. + \|R_{\ell hn}(\hat{k}j)^+\psi_1(\sigma\tau')_+\| \cdot \|G(M-z)\| \cdot \|R_{\ell'h'n'}(\hat{k}'j')^+\psi_1(\sigma\tau)_+\| \right\}$$

ii) using $\|R_{\ell hn}(\hat{k}j)\psi_1(\sigma\tau)_+\| < +\infty$ whose proof is sketched in Appendix C and so on for the others ; iii) recalling the properties of the resolvent obtained in Section 3, which imply that both $G(M+z)$ and $G(M-z)$ can be determined by perturbation series in certain non-empty domains (for instance, for small $|\operatorname{Re} z|$ and large $|\operatorname{Im} z|$, and in a small domain about $\operatorname{Re} z = \mu/2, \operatorname{Im} z = 0$), and by analytic continuation in z outside those domains.

We remark that in physically interesting cases, as the Chew-Low and Drell-Friedman-Zachariasen models, the dependence on $\hat{k}, \hat{k}'$ or on the scattering angles, which is hidden inside the R 's, factorizes so that at the end $T(\sigma'\tau'\bar{k}'j'; \sigma\tau\bar{k}j)$ has a polynomial (rotational invariant) dependence on $\hat{k}, \hat{k}'$, as expected physically.

7. PROPERTIES OF THE ONE-PION PRODUCTION AMPLITUDE

One can obtain analyticity properties for $t(\ell, h, n, \ell_2 h_2 n_2, z_1 z_2)$ in the direct product $\{z_1\} \otimes \{z_2\}$ of the complex z_1 and z_2 planes. In fact, by inserting one complete set of states for each resolvent on the right-hand side of (5.6), separating the one-physical nucleon states and manipulating, one obtains :

$$\begin{aligned}
 t(l_1 h_1 n_1 l_2 h_2 n_2 l h n, z_1 z_2) = & \frac{A_0}{z_1 z_2} + \frac{B_0}{z_1(z_1+z_2)} + \frac{C_0}{z_2(z_1+z_2)} + \\
 & + \frac{1}{z_1+z_2} \int_{-\infty}^{+\infty} dE' \left[\frac{A_1(E')}{z_1-E'} + \frac{A_2(E')}{z_1+E'} + \frac{A_3(E')}{z_2-E'} + \frac{A_4(E')}{z_2+E'} \right] + \\
 & + \frac{1}{z_2} \int_{-\infty}^{+\infty} dE' \left[\frac{B_1(E')}{z_1+z_2-E'} + \frac{B_2(E')}{z_1+z_2+E'} + \frac{B_3(E')}{z_1-E'} + \frac{B_4(E')}{z_1+E'} \right] + \\
 & + \frac{1}{z_1} \int_{-\infty}^{+\infty} dE' \left[\frac{C_1(E')}{z_1+z_2-E'} + \frac{C_2(E')}{z_1+z_2+E'} + \frac{C_3(E')}{z_2-E'} + \frac{C_4(E')}{z_2+E'} \right] + \\
 & + \int_{-\infty}^{+\infty} \frac{dE' dE''}{z_1+z_2-E''} \left[\frac{D_1(E'E'')}{z_2-E'} + \frac{D_2(E'E'')}{z_1+E'} \right] + \\
 & + \int_{-\infty}^{+\infty} dE' dE'' \left[\frac{D_3(E'E'')}{(z_1+z_2+E')(z_2+E'')} + \frac{D_4(E'E'')}{(z_1+z_2+E')(z_1+E'')} \right] + \\
 & + \int_{-\infty}^{+\infty} dE' dE'' \left[\frac{D_5(E'E'')}{(z_2-E')(z_1+E'')} + \frac{D_6(E'E'')}{(z_1-E')(z_2+E'')} \right]
 \end{aligned} \tag{7.1}$$

where the dependence of A's, B's and C's on $l_1 h_1 n_1$, etc., has not been explicated. Representation (7.1) shows that $t(l_1 h_1 n_1 l_2 h_2 n_2 l h n, z_1 z_2)$ is an analytic function of both z_1 and z_2 simultaneously in a non-empty domain $\mathcal{D}$ contained inside $\{z_1\} \otimes \{z_2\}$. The domain $\mathcal{D}$, which is defined by (7.1), is complementary inside $\{z_1\} \otimes \{z_2\}$, of the domain formed by the union of the following sets of pairs (z_1, z_2) :

- a) $z_1 + z_2 = 0$;
- b) $z_1 = 0$;
- c) $z_2 = 0$;
- d) $\text{Im}(z_1+z_2) = 0$ with either $\text{Re}(z_1+z_2) \geq \mu$ or $\text{Re}(z_1+z_2) \leq -\mu$;
- e) $\text{Im} z_2 = 0$ with either $\text{Re} z_2 \geq \mu$ or $\text{Re} z_2 \leq -\mu$;
- f) $\text{Im} z_1 = 0$ with either $\text{Re} z_1 \geq \mu$ or $\text{Re} z_1 \leq -\mu$.

In particular, the following pairs (z_1, z_2) belong to $\mathcal{D}$: i) $\text{Im} z_1 > 0$, $\text{Im} z_2 > 0$, ii) $\text{Im} z_1 < 0$, $\text{Im} z_2 < 0$, iii) $\text{Im} z_1 = \text{Im} z_2 = 0$ with either $0 < \text{Re} z_1$, $\text{Re} z_2 < \mu$ or $-\mu/2 < \text{Re} z_2 < 0$.

By using the decomposition of Q into R 's given after Eq. (5.6) in order to decompose $t^{(0)}(z_1, z_2)$ into pieces, and manipulating with those pieces in a similar way, one can obtain formally representations for them analogous to (but simpler than) (7.1), and one can show that they are analytic in both z_1 and z_2 in the domain $\mathcal{D}$. By taking the limit of those representations and (7.1) when $z_s = \omega'_s + i\epsilon$, $s=1,2$, $\omega'_s \geq \mu$, $\epsilon \rightarrow 0^+$, and using Eq. (5.3) one gets the physical one-pion production amplitude. One can also use those representations and (7.1) in order to derive a variety of crossing relations for the six allowed channels.

A rigorous treatment of $t^{(0)}(z_1, z_2)$ and $t(\ell_1 h_1 n_1 \ell_2 h_2 n_2 \ell h n z_1 z_2)$ can be done following our earlier study of $t(\ell' h' n' \ell h n z)$ in Section 6 and Appendix C, without new essential difficulties. The only new feature appears when majorizing the right-hand side of (5.6), as one faces the problem of showing, for instance, that $\|R_{\ell_1 h_1 n_1}(\vec{k}_1 j_1)^+ G(M+z_1+z_2)\| < +\infty$ for complex $z_1 + z_2$. Fortunately, by using the iterations of Eq. (3.1), this reduces to proving that $\|R_{\ell_1 h_1 n_1}(\vec{k}_1 j_1)^+ G_\pi(M+z_1+z_2)\| < +\infty$ which, in turn, can be done by extending to it the methods presented in Appendix C for $\|R_{\ell h n}(\vec{k} j)(1-P)G_\pi(M)\|$.

8. REMARKS ON THE CONNECTED ELASTIC SCATTERING AMPLITUDE OF TWO PIONS BY THE SOURCE

We shall extend our previous analysis to the connected elastic scattering amplitude of two pions by the dressed source, $T^C(\sigma' \tau' \vec{k}_1 j_1 \vec{k}_2 j_2 ; \sigma \tau \vec{k}_1 j_1 \vec{k}_2 j_2)$, which is related to the s matrix by

$$S(f, i) = \delta_{fi} - 2\pi i \delta[E_f - E_i] \left\{ \delta^{(3)}(\vec{k}_1 - \vec{k}_i) \delta_{j_1 j_i} T(\sigma' \tau' \vec{k}_2 j_2 ; \sigma \tau \vec{k}_2 j_2) + \right. \\ \left. + (\text{other 3 combinations}) + T^C \right\}$$

One can obtain a representation for T^C analogue to Eqs. (5.2-6), but, due to its length, we shall simply write and discuss the most singular terms of it, which contain three resolvents. The omitted contributions, containing one and two resolvents, look like Eqs. (5.4) and (5.6), respectively, (with R 's replaced suitably by Q 's and sums of w 's), and they vanish if $H_2 = 0$. The sum, T_{3R}^C , of all contributions to T^C containing three resolvents, can be shown to be :

$$T_{3R}^C = v(k'_1) v(k'_2) v(k_1) v(k_2) \left\{ t_1(\omega'_1, \omega_1 + \omega_2, \omega_1) + t_1(\omega'_2, \omega_1 + \omega_2, \omega_1) + \right. \\ \left. + t_1(\omega'_1, \omega_1 + \omega_2, \omega_2) + t_1(\omega'_2, \omega_1 + \omega_2, \omega_2) + t_2(\omega'_1, \omega_1 - \omega'_2, \omega_1) + \right. \\ \left. + t_2(\omega'_2, \omega_1 - \omega'_1, \omega_1) + t_2(\omega'_1, \omega_2 - \omega'_2, \omega_2) + t_2(\omega'_2, \omega_2 - \omega'_1, \omega_2) \right\} \quad (8.1)$$

$$\begin{aligned}
 t_1(\omega'_1, \omega_1 + \omega_2, \omega_1) = & \left(\Psi(\sigma' \tau)_+ \right) \left\{ Q(\omega'_1 \hat{k}'_1 j'_1)^+ G(M + \omega'_1 + i\epsilon) Q(\omega'_2 \hat{k}'_2 j'_2)^+ \right. \\
 & \cdot G(M + \omega_1 + \omega_2 + i\epsilon) \cdot Q(\omega_2 \hat{k}_2 j_2) G(M + \omega_1 + i\epsilon) Q(\omega_1 \hat{k}_1 j_1) + \\
 & + Q(\omega_1 \hat{k}_1 j_1) G(M - \omega_1) Q(\omega_2 \hat{k}_2 j_2) G(M - \omega_1 - \omega_2) Q(\omega'_2 \hat{k}'_2 j'_2)^+ G(M - \omega'_1) \cdot \\
 & \left. \cdot Q(\omega'_1 \hat{k}'_1 j'_1)^+ \right\} \Psi(\sigma \tau)_+ \quad (8.2)
 \end{aligned}$$

$$\begin{aligned}
 t_2(\omega'_1, \omega_1 - \omega'_2, \omega_1) = & \left(\Psi(\sigma' \tau)_+ \right) \left\{ \left[Q(\omega'_1 \hat{k}'_1 j'_1)^+ G(M + \omega'_1 + i\epsilon) Q(\omega_2 \hat{k}_2 j_2) + \right. \right. \\
 & + Q(\omega_2 \hat{k}_2 j_2) G(M - \omega'_1 + (\omega_1 - \omega'_2)) Q(\omega'_1 \hat{k}'_1 j'_1)^+ \left. \right] G(M + \omega_1 - \omega'_2 + i\epsilon) \cdot \\
 & \cdot \left[Q(\omega'_2 \hat{k}'_2 j'_2)^+ G(M + \omega_2 + i\epsilon) Q(\omega_1 \hat{k}_1 j_1) + Q(\omega_1 \hat{k}_1 j_1) G(M - \omega_1 + (\omega_1 - \omega'_2)) \cdot \right. \\
 & \left. \left. \cdot Q(\omega'_2 \hat{k}'_2 j'_2)^+ \right] \right\} \Psi(\sigma \tau)_+ \quad (8.3)
 \end{aligned}$$

and so on for the remaining t_1 's and t_2 's, with the appropriate changes for $\hat{k}$'s and j 's. Using the expressions for Q 's in terms of R 's, one can decompose all the t_1 's and t_2 's into sums of products of g 's times some "elementary" pieces $\tilde{t}_{1s}(z_1 z_2 z_3)$ and $\tilde{t}_{2s}(z'_1 z'_2 z'_3)$, respectively, where s is a discrete index representing the various possible orderings of $\hat{k}$'s, j 's and values of l 's, h 's and n 's, and z_1, z_2, z'_2 , etc., are the complexifications of $\omega'_1, \omega_1 + \omega_2, \omega_1 - \omega'_2$, etc. Both $\tilde{t}_{1s}$ and $\tilde{t}_{2s}$ are given by representations similar to Eqs. (8.2-3), with Q 's replaced by R 's. Upon introducing three suitable complete sets of states and playing the usual game, one obtains

$$\begin{aligned}
 \tilde{t}_{1s}(z_1 z_2 z_3) = & \frac{A_s}{z_1 z_2 z_3} + \left\{ \frac{1}{z_1 z_2} \int_{-\infty}^{+\infty} dE' \left[\frac{B_{1s}^{(1)}(E')}{z_3 - E'} + \frac{B_{1s}^{(2)}(E')}{z_3 + E'} \right] + \text{permutations} \right\} + \\
 & + \left\{ \frac{1}{z_1} \int_{-\infty}^{+\infty} dE' dE'' \left[\frac{C_{1s}^{(1)}(E' E'')}{(z_2 - E')(z_3 - E'')} + \frac{C_{1s}^{(2)}(E' E'')}{(z_2 + E')(z_3 + E'')} \right] + \text{permutations} \right\} + \\
 & + \int_{-\infty}^{+\infty} dE' dE'' dE''' \left\{ \frac{D_{1s}^{(1)}(E' E'' E''')}{(z_1 - E')(z_2 - E'')(z_3 - E''')} + \frac{D_{1s}^{(2)}(E' E'' E''')}{(z_1 + E')(z_2 + E'')(z_3 + E''')} \right\} \quad (8.4)
 \end{aligned}$$

$$\begin{aligned} \tilde{t}_{2s}(z'_1 z'_2 z'_3) = & \frac{A_{2s}^{(1)}(z'_1 z'_2) A_{2s}^{(2)}(z'_3 z'_2)}{z'_2} + \int_{-\infty}^{+\infty} \frac{dE'}{z'_2 - E'} \left\{ \frac{q_1(E')}{z'_1} + \frac{q_2(E')}{-z'_1 + z'_2} + \right. \\ & + \int_{-\infty}^{+\infty} dE'' \left[\frac{q_1(E' E'')}{z'_1 - E''} + \frac{q_2(E' E'')}{-z'_1 + z'_2 - E''} \right] \left. \right\} \cdot \left\{ \frac{q_3(E')}{z'_3} + \frac{q_4(E')}{-z'_3 + z'_2} + \right. \\ & + \left. \int_{-\infty}^{+\infty} dE''' \left[\frac{q_3(E' E''')}{z'_3 - E'''} + \frac{q_4(E' E''')}{-z'_3 + z'_2 - E'''} \right] \right\} \end{aligned} \quad (8.5)$$

where $A_{2s}^{(1)}(z'_1 z'_2)$, $A_{2s}^{(2)}(z'_3 z'_2)$ possess representations, respectively analogous to the ones inside the two curly brackets on the right-hand side of (8.5). Clearly, each $\tilde{t}_{1s}(z_1 z_2 z_3)$ is analytic in all z_1, z_2, z_3 in a domain contained in $\{z_1\} \otimes \{z_2\} \otimes \{z_3\}$ which is defined by the representation (8.4). The developments of Sections 6 and 7 can be generalized to show that all $\tilde{t}_{1s}(z_1, z_2, z_3)$, for all possible values of s , exist and are given by perturbation theory for small e_1, e_2 when all $|\operatorname{Re} z_i|$ are small and all $|\operatorname{Im} z_i|$ are large. Such "simultaneous calculation" of all $\tilde{t}_{1s}$ is not spoiled by the restrictions among the various z 's appearing in different $\tilde{t}_{1s}$'s due to energy conservation. On the other hand, each $\tilde{t}_{2s}(z'_1 z'_2 z'_3)$ is also analytic in $\{z'_1\} \otimes \{z'_2\} \otimes \{z'_3\}$, and it is also given by perturbation theory in some parts of such a domain. Unfortunately, the variables z' appearing in different $\tilde{t}_{2s}$'s "conspire against one another" due to energy conservation. In fact, as a glance at the right-hand sides of Eqs. (8.1) and (8.3) reveals, if one of the $\tilde{t}_{2s}$'s is analytic for certain values of its arguments, in principle there will be another amplitude $\tilde{t}_{2s}$ which may be singular for the corresponding values of its own variables. Moreover, the $\tilde{t}_{2s}$'s are rather singular objects in the physical region, as the associated z'_2 always corresponds to the difference of two energies. For instance, Eq. (8.3) indicates the existence of a pole singularity at $\omega_1 = \omega'_2$. These singularities are nothing but the typical ones of any connected three-body amplitude for "forward" configurations.

A physically interesting feature of the t_2 's is worth noticing. Let us set $H_2 = 0$ for simplicity and let us consider all the processes $(\sigma \tau \bar{k} j) \rightarrow (\sigma' \tau' \bar{k}' j' \bar{k}_1 j_1' \dots \bar{k}_n j_n')$, and the one-pion inclusive distribution $F(\bar{k}' j'; \bar{k} j \sigma \tau)$, that is, the total cross-section for all possible values of $\sigma' \tau' n (\geq 0)$, $\bar{k}_1 j_1' \dots \bar{k}_n j_n'$, for fixed $\bar{k} j \sigma \tau \bar{k}' j'$. Then, by using completeness and the reduction technique à la Chew-Low-Wick, one can develop here the well-known Mueller's reasoning ¹⁴⁾ and arrive at the following formal representation :

$$\begin{aligned}
 F(\vec{k}'j'; \vec{k}j \sigma \tau) = & (\Psi(\sigma \tau)_+ , [Q(\omega \vec{k}j)^+ G(M + \omega - i\epsilon) Q(\omega' \vec{k}'j') + \\
 & + Q(\omega' \vec{k}'j') G(M - \omega') Q(\omega \vec{k}j)^+] \delta(M + \omega - \omega' - H) . [Q(\omega' \vec{k}'j')^+ . \\
 & . G(M + \omega + i\epsilon) Q(\omega \vec{k}j) + Q(\omega \vec{k}j) G(M - \omega') Q(\omega' \vec{k}'j')^+] \Psi(\sigma \tau)_+)
 \end{aligned}
 \tag{8.6}$$

which looks like a sort of analytic continuation of one of the "discontinuities" of t_2 , say, the "discontinuity" of the resolvent standing in the middle, for "forward" configurations.

9. ESTIMATES OF CONVERGENCE CONDITIONS IN CHEW-LOW AND DRELL-FRIEDMAN-ZACHARIASEN MODELS

We shall present some estimates of the validity of the condition $e_1 + e_2/\mu < \frac{1}{2}$, which ensures the reliability of perturbation theory for $G(z)$, P_+ and HP_+ , in the cases of the Chew-Low Hamiltonian and the two pieces of the Drell-Friedman-Zachariasen one separately. Upon evaluating in each case the corresponding constants β 's and γ 's, using Eqs. (A.1-3) as well as Eqs. (A.4-5) for the most favourable values of the d 's, and performing some calculations, one gets, respectively, the following conditions for the coupling constants in terms of the dimensionless cut-off function $V(\mu y)$ ($\mu y = k$) :

$$1.1 |f_0| \left\{ 2 \left(\int_0^\infty dy \frac{[V(\mu y)]^2 y^4}{(1+y^2)^{3/2}} \right)^{1/2} + \left(\int_0^\infty dy \frac{[V(\mu y)]^2 y^4}{(1+y^2)^{1/2}} \right)^{1/2} \right\} < 1
 \tag{9.1}$$

$$|f_1| \int_0^\infty dy \frac{[V(\mu y)]^2 y^2}{(1+y^2)^{1/2}} < 1
 \tag{9.2}$$

$$1.3 |f_2| \left\{ \int_0^\infty dy \frac{[V(\mu y)]^2 y^2}{(1+y^2)^{1/2}} \right\}^{1/2} \cdot \left\{ \int_0^\infty dy [V(\mu y)]^2 y^2 (1+y^2)^{1/2} \right\}^{1/2} < 1
 \tag{9.3}$$

For simplicity, let us choose a cut-off function such that $V(k)=1$, $k \leq k_m$, $V(k)=0$, $k > k_m$ with $5 \leq k_m/\mu \leq 6$, which corresponds to reasonable estimates for the source radius ²⁾. Then, upon performing the integrations, the convergence conditions (9.1-3) become approximately $|f_0| < 0.04$, $|f_1| < 0.03$, $|f_2| < 0.01$ which, roughly speaking, are one or two orders of magnitude smaller than the (unrenormalized) coupling constants corresponding to the real low energy pion-nucleon interaction ^{2),3)}.

APPENDIX A

SOME USEFUL BOUNDS AND DERIVATION OF THE INEQUALITY (2.B.3)

Here we shall present some bounds for operators appearing in the Hamiltonian H , which will be useful in order to construct convergent perturbation series. They can be proved by judicious use of Eqs. (2.A.1), (2.A.3-7), (2.B.1-2), the relation $a(\bar{k}j)^+ \psi(\bar{k}_1 j_1 \dots \bar{k}_n j_n) = (n+1) \psi(\bar{k} j \bar{k}_1 j_1 \dots \bar{k}_n j_n)$ and its analogue for $a(\bar{k}j)$, as well as the Schwartz inequality. Their derivations will not be presented, as they follow similar lines as the ones given in Refs. 7) and 6) for other more or less related models. For any normalizable ket ψ belonging to the common domain of definition of the operators involved, the bounds read :

$$a) \left\| \sum_{j=1}^3 \int d^3 \bar{k} v(k) \frac{k}{\omega^{1/2}} U(\bar{k}j) a(\bar{k}j) \psi \right\| \leq a_1 \| H_{\pi} \psi \|$$

$$b) \left\| \sum_{j=1}^3 \int d^3 \bar{k} v(k) \frac{k}{\omega^{1/2}} U(\bar{k}j)^+ a(\bar{k}j)^+ \psi \right\| \leq a_2 \| H_{\pi} \psi \| + a_3 \| \psi \|^2$$

$$c) \left\| \sum_{j_1, j_2=1}^3 \int d^3 \bar{k}_1 d^3 \bar{k}_2 v(k_1) v(k_2) w_1(\bar{k}_1 j_1 \bar{k}_2 j_2) a(\bar{k}_1 j_1) a(\bar{k}_2 j_2) \psi \right\| \leq$$

$$\leq b_1 \| [N_{\pi}(N_{\pi}-1)]^{1/2} \psi \|^2$$

$$d) \left\| \sum_{j_1, j_2=1}^3 \int d^3 \bar{k}_1 d^3 \bar{k}_2 v(k_1) v(k_2) w_2(\bar{k}_1 j_1 \bar{k}_2 j_2)^+ a(\bar{k}_2 j_2)^+ a(\bar{k}_1 j_1)^+ \psi \right\| \leq$$

$$\leq b_2 \| [(N_{\pi}+2)(N_{\pi}+1)]^{1/2} \psi \|^2$$

$$e) \left\| \sum_{j_1, j_2=1}^3 \int d^3 \bar{k}_1 d^3 \bar{k}_2 v(k_1) v(k_2) [w_2(\bar{k}_1 j_1 \bar{k}_2 j_2) + w_2(\bar{k}_2 j_2 \bar{k}_1 j_1)^+] a(\bar{k}_1 j_1)^+ a(\bar{k}_2 j_2) \right.$$

$$\left. \cdot \psi \right\| \leq b_3 \| N_{\pi} \psi \|^2$$

The positive constants a 's and b 's are given by

$$a_s = \left\{ \sum_{\sigma \tau \sigma' \tau' = -1/2}^{+1/2} \sum_{j=1}^3 \left| d^3 \bar{k} \left| \frac{v(k)}{\omega} \cdot \frac{k}{\omega^{1/2}} \beta_s (\sigma \tau \sigma' \tau' \hat{k}_j) \right| \right|^2 \right\}^{1/2}, \quad s = 1, 2 \quad (A.1)$$

$$a_3 = \left\{ \sum_{\sigma \tau \sigma' \tau' = -1/2}^{+1/2} \sum_{j=1}^3 \left| d^3 \bar{k} \left| v(k) \frac{k}{\omega^{3/2}} \beta_2 (\sigma \tau \sigma' \tau' \hat{k}_j) \right| \right|^2 \right\}^{1/2} \quad (A.2)$$

$$b_r = \left\{ \sum_{\sigma \tau \sigma' \tau' = -1/2}^{+1/2} \sum_{j_1, j_2=1}^3 \left| d^3 \bar{k}_1 d^3 \bar{k}_2 \left| v(k_1) v(k_2) \gamma_r (\sigma \tau \sigma' \tau' \bar{k}_1 j_1 \bar{k}_2 j_2) \right| \right|^2 \right\}^{1/2} \quad (A.3)$$

$r = 1, 2, 3$

We shall also require the following bound for N_π . If $l_1 \geq l_2 \geq 0$ and also in the case $l_1 = 0, l_2 = -1$;

$$f) \left\| [N_\pi + l_1] (N_\pi + l_2) \right\|^{1/2} \Psi \leq d_1(l_1, l_2) \|N_\pi \Psi\| + d_2(l_1, l_2) \|\Psi\|$$

where $d_1(l_1, l_2) \geq 1$, $2d_1(l_1, l_2) \cdot d_2(l_1, l_2) \geq l_1 + l_2$, $d_2(l_1, l_2) \geq (l_1, l_2)^{\frac{1}{2}}$ [in practice $d_1(l_1, l_2) = 1$, $d_2(l_1, l_2) = l_1 + l_2/2$]. It can be proved upon squaring the right-hand side of the inequality f), using $\|N_\pi \Psi\| \cdot \|\Psi\| \geq (\Psi, N_\pi \Psi)$ and manipulating a little.

Finally, we shall always assume that

$$g) \|H'_2\| < +\infty$$

From Eqs. (2.A.4-5) and the above inequalities a)-g), and by using the trivial relation $\|H_\pi \Psi\| \geq \mu \|N_\pi \Psi\|$, one derives easily the important inequality (2.B.3), with the following expressions for e_1 and e_2 :

$$e_1 = a_1 + a_2 + \frac{1}{\mu} [b_1 \cdot d_1(0, -1) + b_2 \cdot d_2(2, 1) + b_3] \quad (A.4)$$

$$e_2 = a_3 + b_1 \cdot d_2(0, -1) + b_2 \cdot d_2(2, 1) + \|H'_2\| \quad (A.5)$$

Notice that e_1 is dimensionless, while e_2 has the dimension of energy. One can obtain other bounds, just by changing the majorization techniques. For instance, one can replace the right-hand sides of a) and b) by $a'_1 \|N_{\pi}^{\frac{1}{2}} \psi\|$ and $a'_2 \|(N_{\pi}+1)^{\frac{1}{2}} \psi\|$, respectively, where a'_s , $s = 1, 2$, are given precisely by the same expressions as a_s , but replacing $v(k)/\omega$ simply by $v(k)$ inside the integrals. Then, by using the inequality ($l \geq 0$) $\|(N_{\pi}+l)^{\frac{1}{2}} \psi\| \leq \lambda \|N_{\pi} \psi\| + c(\lambda) \|\psi\|$, which holds for any $\lambda > 0$ and a suitable $c(\lambda) > 0$ (for instance, $\lambda \geq \frac{1}{2}$, $c(\lambda) \geq 1+l$, or else, if $l=0$, then $c(\lambda)=1/4\lambda$), one can also arrive at the inequality (2.B.3) with other constants e'_1, e'_2 .

APPENDIX B

PROOF OF THE INEQUALITY (4.5)

Let us introduce the ket $\Phi = \psi_1(\sigma \tau M_1)_+ - \psi_1(\sigma \tau M_2)_+$, which, by using Eq. (4.3), leads easily to $|M_1 - M_2| \leq e_2 \|\Phi\|$. Thus, the only remaining problem is to show the existence of some η such that

$$\|\Phi\| \leq \eta \cdot |M_1 - M_2| \quad (\text{B.1})$$

By using Eq. (4.2) one derives easily the following integral equation for Φ :

$$\Phi = \Phi_c + (1-P) G_{\pi}(M_2) (H_1 + H_2) \Phi \quad (\text{B.2})$$

where

$$\begin{aligned} \Phi_c &= (1-P) [G_{\pi}(M_1) - G_{\pi}(M_2)] (H_1 + H_2) \psi_1(\sigma \tau M_1)_+ = \\ &= (M_2 - M_1) (1-P) G_{\pi}(M_2) G_{\pi}(M_1) (H_1 + H_2) \psi_1(\sigma \tau M_1)_+ \end{aligned} \quad (\text{B.3})$$

Upon iterating Eq. (B.2) and majorizing, one obtains

$$\|\Phi\| \leq \left\{ 1 + \frac{e_2}{\mathcal{E}_2(M_2) [1 - e_2(M_2)]} \right\} \|\Phi_0\| + \frac{e_1}{\mathcal{E}_2(M_2) [1 - e_2(M_2)]} \|H_{\pi} \Phi_0\| \quad (\text{B.4})$$

On the other hand, by majorizing directly on the right-hand side of (B.3), one arrives at

$$\|\Phi_0\| \leq |M_1 - M_2| \cdot \|(1-P)G_\pi(M_2)G_\pi(M_1)\| \cdot \|(H_1 + H_2)\Psi_1(\sigma\tau M_1)_+\| \quad (\text{B.5})$$

and similarly

$$\|H_\pi \Phi_0\| \leq |M_1 - M_2| \cdot \|H_\pi(1-P)G_\pi(M_2)G_\pi(M_1)\| \cdot \|(H_1 + H_2)\Psi_1(\sigma\tau M_1)_+\| \quad (\text{B.6})$$

Finally, by using Eq. (4.2), one obtains

$$\|(H_1 + H_2)\Psi_1(\sigma\tau M_1)_+\| \leq \left[1 + \frac{e_1(M_1)}{1 - e_1(M_1)}\right] e_2 \quad (\text{B.7})$$

Due to the projector $1-P$, the operators $(1-P)G_\pi(M_2)G_\pi(M_1)$ and $H_\pi(1-P)G_\pi(M_2)G_\pi(M_1)$ are restricted to the infinite dimensional subspace of $\mathcal{H}$ orthogonal to the four-dimensional one spanned by P , so that for $|M_1| < u/2$, both $\|(1-P)G_\pi(M_2)G_\pi(M_1)\|$ and $\|H_\pi(1-P)G_\pi(M_2)G_\pi(M_1)\|$ are finite and, of course, independent on e_1, e_2 . Consequently, the combination of the inequalities (B.4-7) leads immediately to (4.5) as well as to an expression for the real positive constant η . Clearly, one can always fulfil $e_2\eta < 1$, provided that both e_1 and e_2 be small enough.

APPENDIX C

FINITENESS OF $\|R_{\ell hn}(\hat{k}j)\Psi_1(\sigma\tau)\|$

By using the iterations of Eq. (4.2) and majorizing, one obtains :

$$\begin{aligned} \|R_{\ell hn}(\hat{k}j)\Psi_1(\sigma\tau)_+\| &\leq \|R_{\ell hn}(\hat{k}j)\Psi(\sigma\tau)\| + \\ &+ \frac{e_2}{1 - e_1(M)} \cdot \|R_{\ell hn}(\hat{k}j)(1-P)G_\pi(M)\| \end{aligned} \quad (\text{C.1})$$

Let us show that $\|R_{\ell hn}(\hat{k}j)\Psi(\sigma\tau)\|$ and $\|R_{\ell hn}(\hat{k}j)(1-P)G_\pi(M)\|$ are both finite.

1. $\|R_{\ell hn}(\hat{k}j)\Psi(\sigma\tau)\|$:

The cases $(n=0, \ell, h=1, 2)$ and $(h=1, \ell=1, 2, n=1 \dots N)$ are trivial, while the cases $(h=2, \ell=1, 2, n=1 \dots N)$ can be treated by adapting the bound b) of Appendix A (or its modification given at the end of Appendix A) to the present situation. This leads to $\|R_{\ell 2n}(\hat{k}j)\Psi(\sigma\tau)\| \leq a_{\ell 2n}$, where $a_{\ell 2n}$ is a positive constant, similar to the ones given in Eqs. (A.1-2) and finite for the cut-off functions under consideration.

2. $\|R_{\ell n n}(\hat{k}_j)(1-P)G_{\pi}(M)\| :$

The case $(n=0, \ell, h=1, 2)$ is trivial, as $\|(1-P)G_{\pi}(M)\| < +\infty$, so that we come directly to the case $(\ell=h=1, n=1\dots N)$. For any normalizable ψ in the domain of $R_{11n}(\hat{k}_j)(1-P)G_{\pi}(M)$, the Schwartz inequality yields :

$$\left(\|R_{11n}(\hat{k}_j)(1-P)G_{\pi}(M)\psi\| / \|\psi\| \right)^2 \leq \left(\|(1-P)G_{\pi}(M)\psi\| / \|\psi\| \right) \cdot \left(\|R_{11n}(\hat{k}_j)^+ R_{11n}(\hat{k}_j)(1-P)G_{\pi}(M)\psi\| / \|\psi\| \right) \quad (C.2)$$

On the other hand, since $R_{11n}(\hat{k}_j)^+ R_{11n}(\hat{k}_j)$ is in normal form, and having in mind the inequality e) in Appendix A, one shows that :

$$\|R_{11n}(\hat{k}_j)^+ R_{11n}(\hat{k}_j)(1-P)G_{\pi}(M)\psi\| / \|\psi\| \leq \frac{b_{11n}}{\mu} \left(\|H_{\pi}(1-P)G_{\pi}(M)\psi\| / \|\psi\| \right) \quad (C.3)$$

where b_{11n} is a positive and finite constant, similar to the ones given in Eq. (A.3). By using $\|H_{\pi}(1-P)G_{\pi}(M)\| < +\infty$, the inequalities (C.2-3) and the very definition of norm, we conclude that $\|R_{11n}(\hat{k}_j)(1-P)G_{\pi}(M)\| < +\infty$. Concerning the case $(\ell=1, h=2, n=1\dots N)$, by using $R_{12n}^+ R_{12n} = R_{12n} R_{12n}^+ + [R_{12n}^+, R_{12n}]$, we reduce it to the ones previously treated since $R_{12n} R_{12n}^+$ is in normal form and the commutator is a c number, independent of creation and annihilation operators. And so on for the cases $(\ell=h=2, n=1\dots N)$, $(\ell=2, h=1, n=1\dots N)$.

The results obtained in paragraphs 1. and 2., together with the inequality (C.1) lead directly to the finiteness of $\|R_{\ell n n}(\hat{k}_j)\psi_1(\sigma\tau)_+\|$.

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