NP Internal Report 74-9
21 October 1974

ON THE PRECISION WITH WHICH THE PHASE OF THE FORWARD ELASTIC AMPLITUDE
CAN BE DETERMINED IN COULOMB INTERFERENCE MEASUREMENTS

T. Ekelöf

G E N E V A

1974

1. INTRODUCTION

In high-energy elastic scattering between charged hadrons the hadronic force dominates the interaction, and it is only at very small momentum transfers that the much weaker Coulomb force becomes important because of its long range. The two forces are of equal magnitude at t -values around -0.003 (GeV/c)^2 [at $t \sim -0.0018 \text{ (GeV/c)}^2$ for pp scattering, at $t \sim -0.0029 \text{ (GeV/c)}^2$ for πp scattering, and at $t \sim -0.0037 \text{ (GeV/c)}^2$ for Kp scattering fairly independently of energy above 10 GeV]. From data on the differential cross-section in the interference region the phase between the two amplitudes can be deduced; and since the phase of the Coulomb amplitude is known to good accuracy -- there is only some uncertainty in a small model-dependent correction (the Bethe phase) -- the phase of the hadronic amplitude can thus be obtained from such data.

The interest in studying this hadronic phase has been greatly stimulated by the recent discovery of rising total cross-sections at the CERN ISR and at FNAL. Various theories have been developed to describe this rise. Since the s -dependence of the total cross-section is related to the phase of the elastic amplitude at zero momentum transfer through dispersion relations, one can use data on this phase to test the validity of these various theories at very high energies¹⁾. Another object of interest is to test the predictions of the Regge theory about the behaviour of the elastic scattering phase for different particles as the secondary trajectories die out with energy²⁾.

Elastic pp scattering in the Coulomb interference region has been measured up to 500 GeV/c³⁾. For πp scattering, measurements up to 60 GeV/c are reported⁴⁾ and for Kp scattering up to 14 GeV/c⁵⁾. Further such measurements are planned at Serpukhov⁶⁾, at FNAL⁷⁾, and at the ISR⁸⁾ and the SPS⁹⁾ at CERN. In this paper we report a study of the precision with which the phase of the elastic hadronic amplitude in the Coulomb interference region can be determined from such measurements.

2. THE COULOMB INTERFERENCE FORMULA

The differential cross-section for elastic scattering of p 's, π 's, or K 's from protons in the Coulomb interference region can be well described by the formula¹⁰⁾:

$$\frac{\partial \sigma}{\partial t} = \underbrace{\frac{4\alpha^2\pi}{\beta^2|t|^2} \cdot e^{bt}}_{f_{\text{Coulomb}}^2} + \underbrace{\frac{\sigma_{\text{tot}}^2}{16\pi} (1 + \rho^2) \cdot e^{bt}}_{f_{\text{strong}}^2} + \underbrace{\frac{\alpha\sigma_{\text{tot}}}{\beta|t|} (\sin 2\delta + \rho \cos 2\delta) \cdot e^{bt}}_{f_{\text{Coulomb}} \cdot f_{\text{strong}}}$$

where

α is the fine structure constant ($=1/137$)

β is the velocity of the incident particle

t is the four-momentum transfer squared

σ_{tot} is the total cross-section
 ρ is the ratio $\text{Re } f_s(0)/\text{Im } f_s(0)$
 b is the form factor slope
 δ is the Bethe phase = $-(\ln |t| + \ln b + 0.577) \cdot \frac{\alpha}{\beta}$.

In this parametrization the following approximations have been made:

1. All spin-dependent effects have been neglected. From general considerations these effects are predicted to be small and to decrease with energy. A recent analysis¹¹⁾ of experimental data indicates that the spin-dependent amplitudes are of the order of 10% and that they do not decrease with energy. In the same analysis it is shown, however, that the influence of spin effects on the determination of the value of ρ is very small ($\lesssim 0.01$).
2. The hadronic and electromagnetic form factors have been set equal. The values of these two form factors in the Coulomb interference region are both close to unity and, as is shown below, the evaluation of ρ is very insensitive to the value of b .
3. The greatest uncertainty in the determination of ρ is probably due to the model-dependent expression for the Bethe phase. The expression used here¹⁰⁾ gives a value of 2δ ranging from 0.06 to 0.02 in the region $0.001 \lesssim |t| \lesssim 0.02$. Note that 2δ is an approximately additive correction to ρ in the interference term.

In Fig. 1 the three terms of this formula and their sum are plotted separately for the case of π^+p scattering [$\sigma_{\text{tot}} = 25 \text{ mb}$, $b = 8 (\text{GeV}/c)^{-2}$] and for a few values of ρ around zero. Apparently the variation of $d\sigma/dt$ with ρ is dominated by the variation of the interference term at small t -values. (The variation of the f_{strong}^2 -term does not exceed the width of the line-dotted curve.) The deviation from zero of the interference term corresponding to $\rho = 0$ shows the importance of the Bethe phase.

In Fig. 2 are shown $d\sigma/dt$ curves, normalized to the $d\sigma/dt$ curve with $\rho = 0$, for a series of ρ -values in the case of pp , π^+p , and K^+p scattering. When going toward small t -values the ρ -independent Coulomb term ($\sim |t|^{-2}$) becomes dominant over the interference term ($\sim |t|^{-1}$) and the relative ρ -dependence of $d\sigma/dt$ goes to zero as is seen in the figure. At the t -value where the interference term has a relative maximum [i.e. when $f_{\text{Coulomb}}^2 = f_{\text{strong}}^2$ at $-t \sim 0.003 (\text{GeV}/c)^2$] the relative variation of $d\sigma/dt$ with ρ is also at maximum. As the interference term decreases with growing t , the relative variation of $d\sigma/dt$ with ρ also decreases but, as this happens, the ρ -variation of the strong term [$\sim (1 + \rho^2)$] also becomes important. The contribution from this term is symmetric around $\rho = 0$ and therefore the dependence of $d\sigma/dt$ on ρ is enforced for positive ρ and weakened for negative ρ . The

sensitivity of $d\sigma/dt$ to ρ is therefore very different for positive and negative ρ -values in this t -region. At the t -value of strongest ρ -dependence, a shift of 1% in the $d\sigma/dt$ curve can be seen to correspond to a shift of 0.01 in ρ .

When comparing the experimental data with the theoretical expression with the aim of determining ρ , available experimental data on the parameters σ_{tot} and b should be used. In Fig. 3 is shown the variation of $d\sigma/dt$ with σ_{tot} and b around the mean values 25 mb and 8 (GeV/c)⁻², respectively, and for a ρ -value of zero. The variation of $d\sigma/dt$ with σ_{tot} and b is seen to be different in type from that with ρ (see Fig. 1), and the shift in $d\sigma/dt$ corresponding to a typical error, say 0.5 (GeV/c)⁻², in b , is very small. In the case of σ_{tot} a typical error, say 0.5 mb, gives a somewhat greater shift.

Looking at Fig. 2 again, we find that the normalized $d\sigma/dt$ curves are fairly parallel to the normalization line over a wide t -range. A shift in the absolute normalization of the cross-section and a shift in ρ thus have very similar effects on the differential cross-section.

3. THE CALCULATION OF THE ERROR IN ρ

The parametrization of $d\sigma/dt$ given in the previous section can be fitted to experimental data using the χ^2 method, and the error in the value of ρ obtained from such a fit is given in the error matrix of the χ^2 fit. In the work reported here, this error matrix has been calculated using a standard χ^2 computer code¹²⁾. The formula for $d\sigma/dt$ was evaluated for a series of t -values within a certain t -range [100 equidistant points in the range $0.002 < |t| < 0.02$ (GeV/c)²], and then the χ^2 code was used to fit these points to the same formula, allocating to each point a "statistical" error equal to a normalization constant times the square root of the $d\sigma/dt$ -value at that point. The normalization constant was chosen such that the integral of $d\sigma/dt$ over the chosen t -range corresponded to a given number of events.

In the fit the various parameters of the $d\sigma/dt$ formula were used as constraint parameters. The start values of these parameters in the fit were set equal to the values used in the calculation of the $d\sigma/dt$ points, and the strengths of the constraints were expressed through the constraint standard errors used in the fit. In varying these standard errors, one at a time, the influence of the corresponding constraints was studied.

The only parameter left free was ρ . The other parameters used -- σ_{tot} , b , and an over-all normalization factor K (multiplying the cross-section) -- were constraint parameters. (These could, however, also be made free parameters by setting the corresponding errors to a very large number " ∞ ", or made fixed parameters by setting the corresponding error to zero.)

The χ^2 "fit" performed under these conditions is degenerate ($\chi^2 = 0$) and the calculus is exactly resolved after one matrix inversion. The resulting error matrix contains the standard errors of the fit parameters. The error of interest in this report is the error in ρ . In Figs. 4 to 7 this error has been plotted as function of the total number of events for a series of different constraint situations. In these figures the t -range is $0.002 \leq |t| \leq 0.02$, and the values of the physical parameters have been chosen to be $\rho = 0$, $\sigma_{\text{tot}} = 25 \text{ mb}$, and $b = 8 \text{ (GeV/c)}^{-2}$, to approximately correspond to πp scattering at high energies. The numbers attached to each curve are the errors of the various constraint parameters. The "standard constraint error cases" around which each error is varied is $\Delta\sigma_{\text{tot}} = 0.5 \text{ mb}$, $\Delta K = 1\%$, and $\Delta b = 0.5 \text{ (GeV/c)}^{-2}$ in Figs. 5, 6a, and 7a, and $\sigma_{\text{tot}} = 0.5 \text{ mb}$, $\Delta K = \infty$ (no normalization), and $\Delta b = 0.5 \text{ (GeV/c)}^{-2}$ in Figs. 6b and 7b.

The same two "standard cases" are represented in Tables 1a and 1b, respectively, where the error in ρ is given as function of the upper and the lower limits of the t -interval over which $d\sigma/dt$ is measured. The number of events for each table element is proportional to the $d\sigma/dt$ integral over the corresponding t -interval and is normalized to 10^5 events in the interval $0.002 < |t| < 0.02 \text{ (GeV/c)}^2$.

4. DISCUSSION OF THE RESULTS

Figure 4 shows $\Delta\rho$ as function of the number of events for a series of χ^2 fit cases, each with one or several free fit parameters (ρ , σ_{tot} , K , and/or b) but no constraint parameters. The fit corresponding to the bottom curve (labelled 0 0 0) represents a lower limit for $\Delta\rho$ since all parameters except ρ are assumed to be exactly known in the fit. From the other curves it can be seen that the constraint parameter that has the strongest influence on the error in ρ is the absolute normalization K , and that the parameter of least importance is the form factor parameter b . To give an example of the influence of K on the fit, it can be seen that with 10^4 events the error in ρ can never be smaller than 4.2×10^{-2} if the normalization K is free, whereas the same limit is 1.5×10^{-2} if the normalization is known.

Also from Figs. 5, 6, and 7 one can draw the conclusion that the normalization constraint is the most important one. Provided that the normalization is known to within 1% then the constraints in σ_{tot} and b are only of minor importance, as is seen in Figs. 6a and 7a. In Figs. 6b and 7b it is shown that if the normalization is not known, the σ_{tot} and b constraints play a greater role, and also that all the curves are shifted upwards considerably, compared to the case where the normalization is known.

From Tables 1a and 1b the significance of the $d\sigma/dt$ data at low t -values can be seen. Choosing as an example the table element corresponding to the interval

$0.002 < |t| < 0.02 \text{ (GeV/c)}^2$ ($\Delta\rho = 1.13 \times 10^{-2}$) and going to the right in the table, the error in ρ is seen to grow rapidly when the lower limit of the t -range is increased. On the other hand, going to the left in the table, i.e. extending the measurements below $|t| = 0.002 \text{ (GeV/c)}^2$, does not make the $\Delta\rho$ fall off dramatically. The same has been verified to be true also if σ_{tot} and b are left as free parameters. If instead the normalization is left free (Table 1b) the decrease in $\Delta\rho$ is considerable when the lower limit is extended to very low t -values. This is so because at low t -values the Coulomb term dominates completely and can serve to "renormalize" the data. It should be noted, however, that to have the ρ -dependent interference term to be less than 1% of the pure Coulomb term, $|t|$ must be less than $[1.5 \times 10^{-5} / (2\delta + \rho)] \text{ (GeV/c)}^2$.

To demonstrate the influence on ρ from a systematic error in the t -measurement, Table 2 has been calculated. For each t -range is given the shift in ρ obtained from the fit when the t -scale is multiplied by a factor of 1.01 between the generation and the fit of the $d\sigma/dt$ points. The shift can be seen to be great when in the measurement is included the very low t -region [$t \leq 0.002 \text{ (GeV/c)}^2$].

Any statistical error in the t -measurement can be compensated for, provided it is known, by folding the t -resolution curve into the $d\sigma/dt$ distribution. If the statistical error is big, the effect of the folding on the $d\sigma/dt$ curve at low t -values is considerable, and an error in the resolution curve will introduce an error in ρ . From Fig. 2 it can be seen that if the error in the folding correction is of the order of 1% in the differential cross-section, then this will introduce an error of 0.01 in ρ .

The major part of this work was carried out during the author's stay at the Leningrad Nuclear Physics Institute, USSR. Many discussions with A.A. Vorobiev on topics related to the subject of the paper are gratefully acknowledged.

REFERENCES

- 1) W. Bartl and A.N. Diddens, CERN NP Internal Report 73-4 (1973).
P. Kroll, Nuovo Cimento Letters 7, 745 (1973).
C. Bourrely and J. Fischer, Nuclear Phys. B61, 513 (1973).
H. Cheng, J.K. Walker and T.T. Wu, Phys. Letters 44 B, 283 (1973).
D.W. Joynson and W. von Schlippe, Nuclear Phys. B65, 333 (1973).
R.W. Meyer and H. Winzeler; Paper 351, 2nd Int. Conf. on Elementary Particles, Aix-en-Provence, 1973.
J.B. Bronzan, G.L. Kane and U.P. Sukhatme, Phys. Letters 49 B, 272 (1974).
G. Höhler and H.P. Jacob, Univ. Karlsruhe preprint TKP9/74, 1974.
- 2) Ya.I. Azimov, E.M. Levin, M.G. Riskin and V.A. Khoze, Proc. 9th Winter School on Nuclear and Elementary Particle Physics, Leningrad, 1974. (Leningrad Nuclear Physics Institute, 1974) Vol. II, p. 5 [English translation: CERN Trans. 74-8 (1974)].
- 3) G.C. Beznogikh, A. Bujak, L.F. Kirillova, B.A. Morozov, V.A. Nikitin, P.V. Nomokonov, A. Sandacz, M.G. Shafranov, V.A. Sviridov, Truong Bien, V.I. Zayachki, N.K. Zhidkov and L.S. Zolin, Phys. Letters 39 B, 411, (1972).
U. Amaldi, R. Biancastelli, C. Bosio, G. Matthiae, J.V. Allaby, W. Bartel, M.M. Block, G. Cocconi, A.N. Diddens, R.W. Dobinson, J. Litt and A.M. Wetherell, Phys. Letters 43 B, 231 (1973).
V. Bartenev, R.A. Carrigan Jr., I-Hung Chiang, R.L. Cool, K. Goulianos, D. Gross, A. Kuznetsov, E. Malmud, A.C. Melissinos, B. Morozov, V. Nikitin, S.L. Olsen, Y. Pilipenko, V. Popov, R. Yamada and L. Zolin, Phys. Rev. Letters 31, 1367 (1973).
- 4) V.D. Apokin et al., papers Nos. 786 and 789 submitted to the 17th Int. Conf. on High-Energy Physics, London, 1974.
- 5) J.R. Campbell et al. Nuclear Phys. B64, 1 (1973).
R.W. Meijer, Ph.D. Thesis (Amsterdam, 1973).
- 6) Proposal at Serpukhov by A. Vorobiev et al.
- 7) Proposal No. 69 at FNAL by M. Atac et al.
- 8) Proposal No. ISRC/74-17 at CERN by the CERN-Rome Collaboration.
- 9) Proposal No. SPSC/74-94/P26 at CERN by J. Berthot et al.
- 10) H.A. Bethe, Ann. Phys. (USA) 3, 190 (1958).
M.P. Locher, Nuclear Phys. B2, 525 (1967).
G.B. West and D.R. Yennie, Phys. Rev. 172, 1413 (1968).
- 11) C. Bourrely, J. Soffer and D. Wray, CNRS Marseilles preprint 74/P.597.
- 12) I.N. Silin, FORTRAN program FUMILI, JINR Program Library, Dubna.

Table captions

The tables are matrices in which each element corresponds to an interval on the $|t|$ scale, the upper limit of which is given by the vertical scale and the lower limit by the horizontal scale.

Table 1

The element of these tables represents the error with which $\rho = \text{Re } f_s(0)/\text{Im } f_s(0)$ for elastic πp scattering [$\rho = 0$, $\sigma_{\text{tot}} = 25 \text{ mb}$, $b = 8 \text{ (GeV/c)}^{-2}$] can be determined from a χ^2 fit of elastic $d\sigma/dt$ data in the corresponding t -interval. The number of events for each table element is proportional to the integral of the $d\sigma/dt$ curve over the corresponding t -interval, and the scale is set so that there are 10^5 events in the interval $0.002 < |t| < 0.02 \text{ (GeV/c)}^2$. The errors used for the constraint parameters σ_{tot} and b are 0.5 mb and 0.5 (GeV/c)^{-2} , respectively. In Table 1a the value of ΔK is 0.01 and in Table 1b K has been used as a free parameter (no absolute normalization of the experimental data).

Table 2

The element of this table is the shift obtained in $\rho = \text{Re } f_s(0)/\text{Im } f_s(0)$ if the t -scale of the experimental data has a systematic error of 1% . The parameters used in the fit are the same as for Table 1a.

Error in $\rho = \text{Re } f(0)/\text{Im } f(0)$ when the normalization in $d\sigma/dt$ is known to 1%

TMIN	1.5	1.0	2.0	3.0	4.0	5.0	7.0	10.0	14.0	20.0	30.0	50.0
TMAX												
100.0	.88	.89	.91	1.03	1.22	1.43	1.94	2.87	4.37	7.18	12.82	23.63
50.0	.99	1.02	1.07	1.25	1.51	1.82	2.53	3.76	5.53	8.40	15.33	
30.0	1.01	1.05	1.11	1.30	1.57	1.88	2.61	3.92	6.36	12.81		
20.0	1.03	1.07	1.13	1.34	1.66	2.05	3.06	5.37	10.23			
14.0	1.11	1.16	1.25	1.54	2.00	2.60	4.31	7.83				
10.0	1.27	1.31	1.46	1.92	2.63	3.56	5.73					
7.0	1.49	1.53	1.80	2.50	3.44	4.36						
5.0	1.71	1.75	2.14	2.91	3.65							
4.0	1.82	1.86	2.28	2.94								
3.0	1.88	1.92	2.31									
2.0	1.90	1.94										
1.0	2.42											

Each table entry should be multiplied by 10^{-2}
t-scale unit: $10^{-3} (\text{GeV}/c)^2$

Table 1a

Error in $\rho = \text{Re } f(0)/\text{Im } f(0)$ when the normalization in $d\sigma/dt$ is not known

TMAX	TMIN	.5	1.0	2.0	3.0	4.0	5.0	7.0	10.0	14.0	20.0	30.0	50.0
100.0	1.14	1.46	1.87	1.95	1.95	1.95	2.02	2.36	3.22	4.84	8.22	16.79	43.06
50.0	1.23	1.72	2.38	2.48	2.51	2.51	2.65	3.23	4.56	6.79	10.84	21.89	
30.0	1.24	1.77	2.48	2.59	2.62	2.62	2.76	3.36	4.78	7.86	19.66		
20.0	1.25	1.84	2.64	2.75	2.80	2.80	3.02	4.00	7.07	19.38			
14.0	1.39	2.14	3.24	3.36	3.51	4.04	6.48	18.83					
10.0	1.64	2.75	4.53	4.66	5.24	6.86	16.83						
7.0	2.05	3.68	7.91	8.32	11.28	20.95							
5.0	2.49	5.25	19.57	22.87	53.89								
4.0	2.70	5.75	42.62	81.47									
3.0	2.78	5.82	50.92										
2.0	2.89	7.53											
1.0	6.27												

Each table entry should be multiplied by 10^{-2}
t-scale unit: $10^{-3} \text{ (GeV/c)}^2$

Table 1b

Shift in $\rho = \text{Re } f(0)/\text{Im } f(0)$ when the t-scale is shifted by 1%

	TMIN	.5	1.0	2.0	3.0	4.0	5.0	7.0	10.0	14.0	20.0	30.0	50.0
TMAX													
100.0	1.04	1.17	.98	.79	.65	.56	.42	.28	.12	-.14	-.67	-1.69	
50.0	1.06	1.31	1.08	.83	.66	.52	.31	.05	-.23	-.50	-.42		
30.0	1.01	1.31	1.06	.80	.61	.47	.27	.09	-.02	.24			
20.0	1.04	1.37	1.11	.86	.68	.56	.40	.29	.42				
14.0	1.18	1.54	1.24	.97	.79	.67	.51	.46					
10.0	1.42	1.78	1.39	1.07	.86	.71	.53						
7.0	1.71	2.04	1.50	1.09	.81	.64							
5.0	1.94	2.17	1.45	.97	.75								
4.0	1.98	2.14	1.35	.94									
3.0	1.92	2.05	1.30										
2.0	1.99	2.11											
1.0	3.56												

Each table entry should be multiplied by 10^{-2}
t-scale unit: $10^{-2} (\text{GeV}/c)^2$

Table 2

Figure captions

- Fig. 1 : Plot showing the Coulomb amplitude squared, the strong amplitude squared, the interference term, and the differential cross-section for elastic π^+p scattering in the Coulomb interference region for a series of values of $\rho = \text{Re } f_s(0)/\text{Im } f_s(0)$ around zero.
- Fig. 2 : Plot showing the differential elastic cross-section in the Coulomb interference region for a series of values of $\rho = \text{Re } f_s(0)/\text{Im } f_s(0)$ around zero and in which each curve has been normalized to the $d\sigma/dt$ curve with $\rho = 0$. The figure shows plots for pp ($\sigma_{\text{tot}} = 40$ mb), π^+p ($\sigma_{\text{tot}} = 25$ mb) and K^+p ($\sigma_{\text{tot}} = 20$ mb) scattering.
- Fig. 3 : Plots showing the differential cross-section for elastic π^+p scattering for a series of values of the form factor parameter b (Fig. 3a) and the total cross-section σ_{tot} (Fig. 3b).
- Fig. 4 : Plot showing the error with which $\rho = \text{Re } f_s(0)/\text{Im } f_s(0)$ can be determined from a χ^2 fit of elastic πp scattering data [parameters values used are $\rho = 0$, $\sigma_{\text{tot}} = 25$ mb, and $b = 8$ (GeV/c) $^{-2}$] in the region $0.002 < |t| < 0.02$ (GeV/c) 2 as function of the total number of events. For a given curve, each of the parameters σ_{tot} (the total cross-section), b (the form factor parameter), and K (the absolute normalization factor) have either been used as a free parameter (error = ∞) or as a fit parameter (error = 0) in the fit.
- Fig. 5 : This plot is of the same type as in Fig. 4, but in the fit the parameters σ_{tot} , b , and K have been used as constraint parameters with the errors $\Delta\sigma_{\text{tot}} = 0.5$ mb, $\Delta b = 0.5$ (GeV/c) $^{-2}$, and ΔK equal to the parameter indicated in the figure.
- Fig. 6 : These plots are the same as in Fig. 5 but for the case $\Delta b = 0.5$ (GeV/c) $^{-2}$ and $\Delta\sigma_{\text{tot}}$ equal to the parameter indicated in the figure. In Fig. 6a ΔK is equal to 0.01 and in Fig. 6b K has been used as a free parameter (no absolute normalization of the experimental data).
- Fig. 7 : These plots are the same as in Fig. 5 but for the case $\Delta\sigma_{\text{tot}} = 0.5$ mb and Δb equal to the parameter indicated in the figure. In Fig. 7a ΔK is equal to 0.01 and in Fig. 7b K has been used as a free parameter (no absolute normalization of the experimental data).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6a

Fig. 6b

Fig. 7a

Fig. 7b