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STUDY OF NON-LINEAR EFFECTS OF A RESONANT CAVITY ON
THE LONGITUDINAL DYNAMICS OF A COASTING PARTICLE BEAM

by

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Abstract

A computational method has been developed to simulate the longitudinal motion of a relativistic beam of charged particles circulating in a strong focusing synchrotron or storage ring. Unlike most analytical approaches the simulation is not restricted to small perturbations and thus the growth of instabilities can be followed into the non-linear region. The investigations show that, as a result of an instability, the final energy spread of an initially unbunched beam interacting with a resonant cavity is larger than the value at threshold and tends to a limit which depends on both the initial energy spread and on the spread at threshold. Although the cavity has a high impedance only at the frequency of oscillation of the beam, the numerical analysis of particle distributions contains higher beam modes. Apart from these typical non-linear results for which experimental evidence by ISR experiments with a special cavity already exists, the computational method demonstrates that linear theories provide a good approximation to the physical behaviour of the system.

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1. INTRODUCTION

The high density of particles in circular accelerators and storage rings causes various self-field effects associated with the longitudinal motion of particles. In particular, longitudinal density fluctuations are formed spontaneously due to the electromagnetic interaction of a coasting beam with the self-excited fields of its surroundings. Depending on the beam intensity and the nature of the surroundings such as unexcited cavities, smooth or corrugated resistive vacuum chambers etc., well known phenomena as the negative-mass or the resistive-wall instability may occur. These instabilities result in an increase in the energy spread of the circulating beam. Both growth rates in the linear regime and thresholds were theoretically derived ^{1,2,3)} prior to the phenomenon being observed, but recent experiments ⁴⁾ with the experimental cavity at the ISR indicate that the final energy spread exceeds the threshold value which can not be explained yet by analytic theories. Since the increase of the energy spread reduces the phase space density in proton storage rings or might lead to substantial losses of particles due to the aperture limitations of accelerators, it is evident that a deeper theoretical understanding of the instabilities is required.

Linear theories by their nature cannot predict the large amplitude or asymptotic behaviour of an instability. Their analytical extension to the non-linear region has proved to be very difficult, as it involves approximations whose validity are not established on purely theoretical grounds ^{5,6)}. This also suggests the use of numerical techniques and a high speed computer to allow non-linear effects to be studied.

A possible method, used by N.J.D. Jacobs ⁷⁾, is to integrate the single particle equations of motion numerically with a step size corresponding to many revolutions. Another possibility is directly to simulate particles travelling round the accelerator taking into account

momentum changes resulting from the electromagnetic interaction of the particles with cavity-like objects. Our studies are based on the second method as it does not depend on the implicit assumption of the first one, namely that all particles have nearly the same revolution frequency. As resonant structure the closed cylindrical cavity shown in Figure 1 has been chosen.

Figure 1 - Passive cavity (external RF system not in operation) traversed near the axis by superparticles of charge Ze .

In order to give an adequate description of the continuous beam, we consider a number N of superparticles, i.e. particles of charge Ze , where e is the elementary charge and Z is to be chosen so that the quantity

$$I_0 = \frac{N Ze \Omega}{2\pi}$$

corresponds to a realistic current circulating in an accelerator with angular frequency Ω . N should be as large as possible but obviously an upper limit is given by the computing time. The superparticles pass through a cylindrical cavity in which essentially two effects arise. Firstly, the particles excite electromagnetic fields and consequently lose energy and, secondly, each of the particles is accelerated or de-

celerated by the electromagnetic fields which are already in the cavity due to the passages of earlier particles and of earlier passages of the same particle. Either effect changes the energy of the particle and therefore its subsequent revolution frequency.

Our theory includes only the fundamental mode of the cavity and it neglects all interaction of the azimuthal motion with vertical and radial motions. Furthermore, we assume that outside the cavity the particles do not interact with their surroundings. Physical structures, of course, have a beam pipe connecting the exit of the cavity with the entrance of another or the same cavity. But as has been verified ⁸⁾ already, the beam pipe does not make a large change in the energy loss for the first few cavity modes. Hence, the closed cavity model is an adequate representation for cylindrical cavities in which all modes except the fundamental one are suppressed.

A summary of the mathematical approach to calculate the energy loss of a single particle is followed by a more quantitative description of the simulation programme. After a review of the main results of the linear theory, numerical results are given for a variety of parameters specifying particle beam, cavity and accelerator.

2. MATHEMATICAL FORMALISM

2.1. Calculation of the energy loss

To calculate the energy loss which is associated with the radiation of a particle, we shall use the mathematical formalism which has been developed by W. Heitler ⁹⁾ and E.U. Condon ¹⁰⁾ and which already has been applied to a series of problems ^{11,12,13,14)}.

The wave equation for the field created by the moving charged particle *)

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \quad (1)$$

connects the current density $\vec{j}$ related to the moving charge to the vector potential $\vec{A}(\vec{r}, t)$. The solution of the differential equation (1) may be found in the form of a series

$$\vec{A}(\vec{r}, t) = \sum_{\lambda} q_{\lambda}(t) \vec{A}_{\lambda}(\vec{r}) \quad (2)$$

where the $\vec{A}_{\lambda}(\vec{r})$ are orthogonal functions depending only on the space coordinates and satisfying the boundary conditions of the problem at hand. They are normalised such that

$$\int_V \vec{A}_{\lambda}(\vec{r}) \cdot \vec{A}_{\lambda'}^*(\vec{r}) dV = \frac{\delta_{\lambda\lambda'}}{\epsilon_0} . \quad (3)$$

From the periodicity of the $q_{\lambda}(t)$ follows from the homogenous equation (1) that the $\vec{A}_{\lambda}(\vec{r})$ obey the equation

$$\nabla^2 \vec{A}_{\lambda}(\vec{r}) + \frac{\omega_{\lambda}^2}{c^2} \vec{A}_{\lambda}(\vec{r}) = 0 .$$

By substituting series (2) into equation (1) and multiplying with $\vec{A}_{\lambda'}^*(\vec{r})$, the integration with respect to the volume V of the resonator yields the equations for the time dependent field coordinates $q_{\lambda}(t)$:

$$\ddot{q}_{\lambda}(t) + \omega_{\lambda}^2 q_{\lambda}(t) = \int_V \vec{j} \cdot \vec{A}_{\lambda}^*(\vec{r}) dV . \quad (4)$$

*) Throughout the report mksA units will be used.

If there are no sources in the enclosed volume, the electric and magnetic fields in the resonator are related to the vector potential $\vec{A}_\lambda(\vec{r}, t)$ by

$$\vec{E} = - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad (5)$$

$$\vec{B} = \text{curl } \vec{A}(\vec{r}, t) .$$

Using (5) and the defining equations for $\vec{A}_\lambda(\vec{r})$, the energy stored in the volume V is found from

$$W = \frac{\epsilon_0}{2} \int_V \left(|\vec{E}|^2 + c^2 |\vec{B}|^2 \right) dV,$$

hence

$$W = \frac{1}{2} \sum_\lambda \left(|\dot{q}_\lambda|^2 + \omega_\lambda^2 |q_\lambda|^2 \right) . \quad (6)$$

For the geometry of the closed cylindrical cavity shown in Figure 1 we find the vector potential ²²⁾

$$A_{\lambda z} = C_\lambda \cos \frac{p \pi z}{h} J_n(v_{ns} \frac{r}{d}) e^{in \theta}$$

where the mode λ is characterized by the mode numbers s , p , and n . C_λ is a normalization constant and v_{ns} is determined by the zeros of the Bessel function $J_n(v_{ns})$. For a cavity with all modes damped except the fundamental one, i.e. the TM_{010} -mode, we find with the normalization (3)

$$A_{oz} = \frac{1}{d \sqrt{\pi \epsilon_0 h}} \frac{J_0(v_{01} \frac{r}{d})}{J_1(v_{01})}$$

with resonance frequency of

$$\omega_0 = \frac{v_{01} c}{d} , \quad c \text{ velocity of light in vacuum. } (7)$$

In the following the suffices indicating the mode number will be omitted as only the TM_{010} -mode is considered. For the charge density which is related to a superparticle of charge Ze traversing the cavity along the z -axis with velocity v ,

$$j_z = \frac{Ze v}{2\pi r} \delta(r) \delta(z - vt) , \quad (8)$$

the integral of equation (4) is

$$f(t) = \int_V j_z A_z dV = \frac{Ze v}{d \sqrt{\pi \epsilon_0} h J_1(v_{01})} H\left(\frac{h}{2} - |z - vt|\right) . \quad (9)$$

$H(x)$ is the Heaviside step function, $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$. With (9) a solution of equation (4) for $t > h/2v$ is

$$\begin{aligned} q(t) &= \frac{1}{\omega} \int_{-h/2v}^{h/2v} f(\tau) \sin[\omega(t - \tau)] d\tau \\ &= \frac{Ze h}{\omega d \sqrt{\pi \epsilon_0} h J_1(v_{01})} \frac{\sin(\frac{\omega h}{2v})}{(\frac{\omega h}{2v})} \sin(\omega t) . \end{aligned} \quad (10)$$

The same form of the solution and its time derivative is obtained if at $t = 0$, when the particle is in the middle of the cavity, $\dot{q}$ is abruptly changed by the amount

$$\Delta \dot{q} = \frac{Ze h}{d \sqrt{\pi \epsilon_0} h J_1(v_{01})} \frac{\sin(\frac{\Omega h}{2v})}{(\frac{\Omega h}{2v})} \quad (11)$$

If the complete solution of equation (4) for $t > h/2v$ is written in the form

$$q(t) = q_0 \cos(\omega t) + \frac{1}{\omega} \dot{q}_0 \sin(\omega t) + D \sin(\omega t) ,$$

with

$$D = \frac{Ze h}{\omega d \sqrt{\pi \epsilon_0} h J_1(v_{01})} \frac{\sin(\frac{\omega h}{2v})}{(\frac{\omega h}{2v})} ,$$

then the stored energy in the cavity after the passage of a particle may be calculated using equation (6)

$$W = W_0 + \dot{q}_0 \omega D + \frac{1}{2} \omega^2 D^2 ,$$

where

$$W_0 = \frac{1}{2} (\dot{q}_0^2 + \omega^2 q_0^2) \quad (12)$$

is the stored energy before the particle entered the cavity. Energy conservation demands that the difference between the stored energy before and after the passage must be equal and of opposite sign to the energy change ΔU of the particle, hence

$$\Delta U = - \Delta W = - \omega D (\dot{q}_0 + \frac{1}{2} \omega D) . \quad (13)$$

Another way of calculating the energy change ΔU of a particle begins with the complete solution of equation (4) for $|t| \leq h/2v$, which is

$$\tilde{q}(t) = q_0 \cos(\omega t) + \frac{1}{\omega} \dot{q}_0 \sin(\omega t) + \frac{Ze v}{\omega d \sqrt{\pi \epsilon_0} h J_1(v_{01})} \left\{ 1 - \cos[\omega(t + \frac{h}{2v})] \right\} .$$

The electric field on the z-axis excited by the particle due to equation (5) may be written

$$E_z(t) = - \ddot{q}(t) A_z(0) .$$

Integrating the force on the particle over the time of the cavity traversal yields the same result as above:

$$\Delta U = Ze v \int_{-h/2v}^{h/2v} E_z(t) dt = -\omega D(\dot{q}_0 + \frac{1}{2}\omega D) .$$

In the preceeding analysis it is implicitly assumed that the effect of damping is negligible during the short time a particle takes to cross the gap. However, in the time between the successive passage of two particles there is the free oscillation of a damped cavity obeying the equation

$$\ddot{q} + \frac{\omega}{Q} \dot{q} + \omega^2 q = 0 \quad (14)$$

with the solution

$$q(t) = \exp(-\frac{\omega t}{2Q}) [q_0 \cos(\omega' t) + \frac{1}{\omega'} (\dot{q}_0 + \frac{\omega}{2Q} q_0) \sin(\omega' t)] \quad (15)$$

where

$$q_0 = q(0), \quad \dot{q}_0 = \dot{q}(0)$$

and

$$\omega'^2 = \omega^2 (1 - \frac{1}{4Q^2}) .$$

As is shown in the Appendix, the factor Q introduced in equation (14) corresponds to the usual quality factor of a cavity.

Now, the interaction between many particles and the cavity can be visualised as follows: Before a particle passes the cavity, there is the free oscillation (14) of a damped cavity. The passage of a particle

then causes an abrupt change of $\dot{q}$ by the amount given in equation (11). This is followed by another free oscillation (14) of the damped cavity, then by another passage of a particle and so on.

2.2. Preexcitation of the cavity

If the longitudinal distribution of particles is uniform or only varies slightly and if the cavity is initially not excited, then it may take a long time until the stored energy in the cavity and thus the beam-cavity interaction is large enough to provoke a longitudinal instability. Therefore, in order to reduce the corresponding computer time of a simulation programme, it is advantageous to start the simulation with the cavity already excited. Assuming the particle distribution to be constant up to the time we want to begin with the simulation, the Fourier analysis of the longitudinal distribution yields the coordinates of the cavity oscillation, q_0 and $\dot{q}_0$, and thus the energy stored in the cavity at that time.

If damping is small and is measured by the quantity Q in the usual way, then equation (4) can be replaced by

$$\ddot{q} + \frac{\omega}{Q} \dot{q} + \omega^2 q = \tilde{f}(t) \quad (16)$$

which differs from equation (14) in that the rhs is different from zero. The driving time $\tilde{f}(t)$ is given by equation (9) but for a current density defined by

$$j_z = \frac{\lambda v}{2\pi r} \delta(r) \cos(\Omega t - kz) \quad (17)$$

where λ is the amplitude of the line density and $k = \Omega/v$, v being the velocity of the particles. We find

$$\tilde{f}(t) = \frac{2\lambda v^2 \sin(\frac{\Omega h}{2v})}{\Omega d \sqrt{\pi \epsilon_0} h J_1(v_{01})} \cos(\Omega t) ,$$

and by integrating equation (16) over the time interval $(-\infty, t)$:

$$q = \frac{\lambda v h \alpha}{d \sqrt{\pi \epsilon_0} h' J_1(v_{01})} \frac{\sin(\frac{\Omega h}{2v})}{(\frac{\Omega h}{2v})} \cos(\Omega t - \epsilon) \quad (18)$$

with

$$\alpha = \left[(\omega^2 - \Omega^2)^2 + \frac{\omega^2 \Omega^2}{Q^2} \right]^{-\frac{1}{2}}$$

and

$$\epsilon = \arctan \left[\frac{\omega \Omega / Q}{\omega^2 - \Omega^2} \right] .$$

In accordance with the preceeding analysis where the first mode of excitation has been considered only, the yet unknown quantity λ must be related to the first Fourier component of the particle distribution. For $i = 1, 2, \dots, N$ superparticles distributed around the orbit and passing through the cavity at times $0 \leq t_i < T$, the dc-current density is

$$\lambda_0 = \frac{\sum_{i=1}^N (Ze)}{2\pi R} , \quad \text{R circumference of the accelerator}$$

hence

$$\lambda_0 v = \frac{NZe}{T} .$$

The latter notation suggests the ac-components of equation (18), $\lambda_{1v} \sin(\Omega t)$ and $\lambda_{1v} \cos(\Omega t)$, for N particles to become

$$\lambda_{sv} = \frac{2Ze}{T} \sum_{i=1}^N \sin(2\pi t_i/T) \quad \text{and} \quad \lambda_{cv} = \frac{2Ze}{T} \sum_{i=1}^N \cos(2\pi t_i/T) ,$$

therefore

$$q = \frac{h\alpha}{d \sqrt{\pi\epsilon_0} h' J_1(v_{01})} \frac{\sin(\frac{\Omega h}{2v})}{(\frac{\Omega h}{2v})} \left\{ \lambda_c v \cos \epsilon - \lambda_s v \sin \epsilon \right\} \quad (19)$$

$$\dot{q} = \frac{-h\alpha \Omega}{d \sqrt{\pi\epsilon_0} h' J_1(v_{01})} \frac{\sin(\frac{\Omega h}{2v})}{(\frac{\Omega h}{2v})} \left\{ \lambda_s v \cos \epsilon + \lambda_c v \sin \epsilon \right\} .$$

Combining equations (12) and (19) we may compute the energy initially stored in the cavity for a particle beam circulating with fixed configuration up to the time we want to start with the simulation and the values of q and $\dot{q}$ at that moment.

3. DESCRIPTION OF THE SIMULATION PROGRAMME

3.1. Initial conditions

The distribution of the $i = 1, 2, \dots, N$ particles is described by pairs of coordinates (E_i, t_i) where the energy scale is chosen so that $E_i = 0$ for particles with revolution time

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega' \xi} \quad (20)$$

and $0 < t_i < T$ is the next time the particles arrive at the cavity. The parameter ξ is near 1.0 and provides the possibility to shift the revolution frequency around the resonance frequency ω' of the damped cavity.

The initial values of (E_i, t_i) are chosen at random according to the following procedures. The energies are selected from a Gaussian distribution of rms energy spread $\langle \Delta E_{in} \rangle$. As has been found in the calculation of thresholds ¹⁵⁾, a Gaussian distribution function leads to

approximately the same results as a squared cosine distribution or parabolas of power 2 to 5 except for very reactive impedances; thus realistic distributions may be represented well enough. In order to shorten the time to excite the cavity it may be expedient to introduce an initial modulation of the particle beam. This is done by a premodulation factor ρ in selecting the times t_i as

$$t_i = T[x_i + \rho \sin(2\pi x_i)]$$

where $x_i = (i - 1 + r)/N$, and r is a random number uniformly distributed in the interval $[0, 1]$.

The particular case in which the energies E_i were selected from a squared cosine distribution is described in more detail in section 5.6.

3.2. Equations for synchrotron oscillations including beam-cavity interaction

After each passage the effect of the beam-cavity interaction modifies for each particle the pair (E_i, t_i) , and the cavity coordinates q and $\dot{q}$, too. The energy error E_{ik} of the i 'th superparticle after the k 'th passage through the cavity at time t is given by equation (13). For a particle of unit charge it follows:

$$E_{ik} = E_{i,k-1} - \frac{\omega D}{Z}(\dot{q}_k + \frac{\omega D}{2}) , \quad (21)$$

where $E_{i,k-1}$ is the energy error after the preceeding passage. For any particle the change of energy will alter the time of its next arrival. This effect depends on the characteristics of the focusing magnetic field of an AG-synchrotron which may be summarized in defining a parameter η by

$$\left(\frac{d\Omega}{\Omega}\right)_{ik} = -\frac{\eta}{\beta^2} \frac{E_{ik}}{E_0\gamma} ; \quad E_0\gamma = \text{total particle energy.} \quad (22)$$

η is related to the momentum compaction factor α_p and the particle energy via (see for example Ref. 16):

$$\eta = \alpha_p - \gamma^{-2} \quad \text{or} \quad \eta = \gamma_T^{-2} - \gamma^{-2} ;$$

β and γ have the usual relativistic meaning and γ_T equals γ at which η vanishes. Thus the time of the next arrival for a particle of energy error E_{ik} follows from the preceeding arrival time $t_{i,k-1}$ by

$$t_{ik} = t_{i,k-1} + T \left(1 + \frac{\eta E_{ik}}{\beta^2 \gamma E_0} \right) . \quad (23)$$

The quantity $\dot{q}_k$ in equation (21) can be computed from the coordinates q_k , and $\dot{q}_k$, calculated at the passage of the particle which traversed the cavity just before at the time $t - \delta t$:

$$\dot{q}_k = -\frac{\omega}{2Q} q_k + e^{-\frac{\omega}{2Q} \delta t} \left\{ -\omega' q_k \sin(\omega' \delta t) + \left[\dot{q}_k + \frac{\omega}{2Q} q_k \right] \cos(\omega' \delta t) \right\} + D\omega$$

with

$$q_k = e^{-\frac{\omega}{2Q} \delta t} \left\{ q_k \cos(\omega' \delta t) + \frac{1}{\omega'} \left[\dot{q}_k + \frac{\omega}{2Q} q_k \right] \sin(\omega' \delta t) \right\} .$$

At the beginning of the simulation q_k , and $\dot{q}_k$, are determined by equations (19).

The transformations (21) and (23) of the pairs (E_{ik}, t_{ik}) for N particles, in the following referred to as a "turn", describe the synchrotron motion due to the interaction of the particles with the

cavity. Clearly it may happen that in a turn a particle overtakes one or several others. This can be treated very quickly by assigning two pointers to each particle, the first pointer giving the address of the next particle in time and the second the address of the previous particle in time. This arrangement, named a bidirectional string¹⁷⁾, is completely cyclic and computing time is required only when one particle overtakes another. In that case only six pointers belonging to five particles have to be changed. It appears that this is by far the fastest method of rearrangement; the average computer time required - including the transformations above and input and output jobs - on the CDC 7600 at CERN is 23 μ sec per particle and turn.

3.3. Compilation of input and output data for the simulation programme

a) Input data

Describing the beam:

- N - number of superparticles
- Z - multiplicity of unit charge e per superparticle
- ρ - premodulation factor
- $\langle \Delta E_{in} \rangle$ - initial rms energy spread of the beam
- m_0 - mass of a particle at rest
- γ - energy of a particle in units of $m_0 c^2$

Describing the accelerator:

- η - variation of T with ΔE
- ξ - ratio of angular frequency Ω to resonance frequency ω'

Describing the cavity:

- h - length
- d - radius
- Q - quality factor

All other quantities quoted in previous sections may be computed using this list of input parameters.

b) Output data

After each $k_m/100$ turns, where k_m is a suitably chosen total number of turns, a list of data is printed out:

W	- energy stored in the cavity	
E	- mean energy	} of the beam
$\langle \Delta E^2 \rangle$	- energy variance	
F_1	- first Fourier component in units of the dc-component	
ϕ	- phase of cavity oscillation with respect to the beam modulation	

The last two quantities are found from

$$F_n = \frac{2}{N} \left\{ \left[\sum_{i=1}^N \cos\left(\frac{2\pi n}{T} (t_i - t_N)\right) \right]^2 + \left[\sum_{i=1}^N \sin\left(\frac{2\pi n}{T} (t_i - t_N)\right) \right]^2 \right\}^{\frac{1}{2}}$$

and

$$\phi = - \arctan\left(\frac{\dot{q}}{\omega' q}\right) + \arctan \left[\frac{\sum_{i=1}^N \sin\left[\frac{2\pi}{T}(t_i - t_N)\right]}{\sum_{i=1}^N \cos\left[\frac{2\pi}{T}(t_i - t_N)\right]} \right] .$$

There also is an option to print the list of pairs (E_i, t_i) after any desired turn.

4. REVIEW OF THE LINEAR THEORY

For comparison with computed results, we summarize below the results of the linear theory concerning threshold and growth-time.

The linearized Vlasov equation leads to the well known dispersion equation 3)

$$1 = \frac{U - iV}{n|k_0|\Delta^2} \text{sign}(k_0) I$$

relating the coefficients U and V to the integral I which is a complex function depending on the energy distribution of the particles. n denotes the mode number and

$$\Delta = \pi \frac{\Delta E}{\Omega} = \frac{\pi \gamma E_0}{\Omega} \left(\frac{\Delta E}{E} \right)$$

is a measure of the energy spread of the particles, with $\Delta E/E$ the total energy spread at half height in units of the total energy of the particles $E = E_0 \gamma$. In evaluating the integral I in the dispersion relation above for realistic distribution functions ¹⁵⁾, a condition for stability is found for not too reactive impedances:

$$\frac{|U - iV|}{n|k_0|\Delta^2} \lesssim 0.7 \quad . \quad (24)$$

The growth-time for the instability pertaining to any distribution tends towards the growth-time of a δ -function distribution if the spread Δ is much smaller than the threshold value given by (24). The latter growth-time can be calculated from the dispersion equation of a δ -function,

$$1 = (U - iV) \frac{n k_0}{(\Omega - n \Omega_0)^2} \quad ,$$

with the solution

$$\Omega = n \Omega_0 \pm \sqrt{\frac{n k_0 V}{2}} \left\{ \sqrt{1 + \left(\frac{U}{V}\right)^2} + \frac{U}{V} - i \sqrt{1 + \left(\frac{U}{V}\right)^2} - \frac{U}{V} \right\}$$

where Ω_0 is the unperturbed angular frequency of the particles. Near or on resonance $U \ll V$, and the e-folding growth-time of the perturbation amplitude is

$$\tau = \sqrt{\frac{2}{n|k_0|V}} \quad . \quad (25)$$

The complex quantity $U - iV$ depends on the geometry and the electromagnetic properties of the beam surroundings and the current I_0 of the beam; it is ¹⁸⁾

$$U - iV = -i e I_0 Z_c ,$$

where Z_c is the sum of the impedances seen by the beam. For the case of the closed cylindrical cavity the impedance Z_c is given in the Appendix. The quantity k_0 appearing in the equations above is a constant of the accelerator; it is found to be

$$k_0 = \frac{\Omega}{2\pi} \frac{d\Omega}{dE} = - \frac{\Omega^2 \eta}{2\pi \beta^2 \gamma E_0} .$$

With the defining equations for the quantities k_0 , U and V follows the Keil - Schnell criterion ¹⁸⁾ for stability, which in terms of the rms energy spread $\langle \Delta E/E \rangle$ for Gaussian distributed particles is

$$\left| \frac{Z_c}{n} \right| \lesssim 2.8\pi \ln 2 \frac{E_0}{e} \frac{|\eta| \gamma}{\beta^2 I_0} \left(\langle \frac{\Delta E}{E} \rangle \right)^2 .$$

For the growth-time we find

$$\tau = \frac{2\beta}{n \Omega_0} \sqrt{\frac{\pi E_0 \gamma}{e |\eta| I_0 (Z_c/n)}} .$$

This result corresponds to Hübner's formula quoted in Ref. 19. With equations (A 3) and (A 4) we may express both the energy threshold and the growth-time of the linear theory for the closed cylindrical cavity as function of our input variables, and obtain:

$$\left\langle \frac{\Delta E_{th}}{E} \right\rangle_\ell = c_1 \sqrt{\frac{NZ Q}{|\eta| \gamma h}} \beta^2 \sin\left(\frac{1.202 h}{\beta d}\right) \quad (26)$$

$$\tau_{\ell} = c_2 \sqrt{\frac{\gamma h}{|\eta| N Z Q}} \frac{d}{\sin\left(\frac{1.202 h}{\beta d}\right)} \quad (27)$$

with

$$c_1 = \left(\frac{e^2 Z_0 c}{1.4 \pi^3 \nu_{01}^2 J_1^2(\nu_{01}) E_0 \ln 2} \right)^{\frac{1}{2}} = 6.413 \cdot 10^{-10} \text{ m}^{\frac{1}{2}}$$

and

$$c_2 = \left(\frac{2 \pi^3 E_0 J_1^2(\nu_{01})}{e^2 c^3 Z_0} \right)^{\frac{1}{2}} = 3.105 \text{ m}^{-3/2} \text{ s} .$$

5. NUMERICAL RESULTS

5.1. Resonance line shape and beam analysis

The resonance curve has been investigated in the neighbourhood of the resonance frequency by varying the frequency ratio $\xi = \Omega/\omega'$. Input data have been taken for which the linear theory predicts instability. They are listed in the caption of Figure 2 which shows the results for the energy W stored in the cavity for two different quality factors as functions of ξ . Though the resonance shape depends on the number of turns executed in the simulation programme, the relationship between the quality factor Q of the cavity and the full width at half height $\Delta\xi$ of the function W^2 is approximately

$$Q = \frac{1}{\Delta\xi} = \frac{\omega'}{\Delta\Omega} . \quad (28)$$

Another expected result is that the closer ξ is to unity the earlier the energy spread crosses its threshold value. On the basis of these results we have decided to use $\xi = 1.0$ for all subsequent runs.

With a few exceptions the computer model was run with data yielding a working point in the unstable region. A standard set of input parameters has been chosen such that realistic conditions are obtained. This set of data is compiled in Table 1. The quantities listed there result in a current of 91.9 mA and a coupling impedance at resonance of $Z_c/Q = 68.7 \text{ Ohm}$; these were taken as input parameters unless otherwise specified or chosen as independent variables. All numerical results have been obtained for superparticles with a charge to mass ratio corresponding to that of a proton.

In order to demonstrate the longitudinal density fluctuations resulting from unstable conditions, Figure 3 shows typical distributions of particles in the longitudinal phase-space after various numbers of turns. Obviously the fastest particles form a cluster in which the initial density is maintained for a long time. The formation and the shapes of clusters resemble those obtained by M.S. Grewal and J.A. Byers ²¹⁾ who investigated the negative mass instability by numerical computations for the case of a cylindrical relativistic beam gyrating within two perfectly conducting cylinders concentric with the beam. For the same parameters, the diagrams of Figures 4a - 4d show the energy stored in the cavity, the first, second and third Fourier component of the particle distribution, and the mean energy of the particle beam. The diagrams 4e and 4f show plots of the phase ϕ of the cavity oscillation relative to the first ac-component F_1 , and the energy variance, where all plots are given with respect to the number of turns. A horizontal scale marked with the time in units of the linear e-folding time (27) has been added to Figure 4b.

As expected, the relative maxima of the first Fourier component coincide with the maxima of the stored energy, whereas at the minima the mean beam energy does not change significantly. Depending on the number N of particles, the Fourier amplitudes as well as the stored energy dis-

appear in the noise at about ten to thirty times the difference between the start of the simulation programme and the time of maximum excitation.

Figure 4e illustrates that at resonance ($\xi = 1.0$) the phase of the cavity oscillation at the beginning lags behind the phase of the beam modulation by an amount of $-\pi/2$. This result is in agreement with the phase difference expected from equation (18). When the beam modulation has reached its maximum, the phase is about the same as at the beginning. It then decreases slightly as time and thus the mean angular frequency of the beam increases. Each relative minimum of the excitation is accompanied by a dip in the phase plot. If there is no prebunching, $\rho = 0$, the results for the phase are the same with the only exception that it takes a longer time until the threshold is reached and therefore the initial phase is maintained for a longer time.

In a number of standard runs different sets of random numbers were used to generate the pairs (E_i, t_i) . It turned out that the choice of any of them does not significantly influence the results as differences up to only a few percent have been found. Therefore the results, such as growth-time, threshold, etc. referred to below, have been obtained always using the same set of random numbers.

5.2. Growth rate of the instability

The e-folding growth-time has been calculated from fits of an exponential function to the first ac-component in the range of time between two subsequent outputs covering at least 100 turns. Generally, the growth-time behaved qualitatively in the following way. In the course of developing instability it increases steadily, reaching a relative maximum and then decreases slightly to a relative minimum. The fluctuations which subsequently appear occur in the interval between these two extrema as long as the amplitude F_1 of the perturbed current has not yet reached about 40% to 60% of its dc-value. In the remaining time

until the threshold of the instability is reached, the growth-time increases rapidly. Since it turned out that any prebunching extended the transient time, i.e. the time until the first relative maximum is reached, into a region in which the Fourier component is no longer small (50% - 70% in units of its dc-value), we decided to compute the non-linear growth-time without premodulation of the beam ($\rho = 0$).

The range of growth-times between the two relative extrema mentioned above is plotted in Figure 5 together with the growth-time calculated from equation (27), both as function of the cavity dimensions d and h , the quality factor Q , and the quantities γ and η , respectively. To distinguish the results of our non-linear method from the corresponding predictions of the linear theory of suffix " ℓ ", we have indexed them by " $n\ell$ ". Since the growth-time τ_ℓ for an unstable relativistic beam depends linearly on the geometrical quantity d/κ where $\kappa = [\sin(1.202 h/d)]/\sqrt{h}$, we have plotted the results obtained from different cavity dimensions as function of this quantity. There is a good agreement between the computed $\tau_{n\ell}$ and the expected τ_ℓ . The same has been found from runs in which the current I_0 was varied by varying the quantities N or Z . A discrepancy between them obviously arises only for high Q factors if the damping time constant $2Q/\omega$ becomes larger than the linear growth-time τ_ℓ .

When the transient time is over, the Fourier amplitude has reached 5 to 10% of its maximum value, and the phase ϕ is still about $-\pi/2$. The e-folding time of the increase of stored energy is half the value $\tau_{n\ell}$ calculated from the increase of the amplitude of perturbation F_1 . This is expected as the stored energy is a function of the square of the perturbation amplitude.

5.3. Threshold and overshoot

The small-amplitude threshold that determines whether a beam is stable or unstable initially, is the same in our simulation as in the linear theory. If the beam is initially unstable, the first growth of the first beam harmonic stops, and it begins to decrease at the moment when the beam width crosses that threshold. This can be verified by Figure 6 which shows the results of runs with $N = 50$ and $Z = 10^8$ as function of the initial spread $\langle \Delta E_{in}/E \rangle$. The rms energy spread at the first maximum of the fundamental beam harmonic being denoted by $\langle \Delta E_{in}/E \rangle_{n\ell}$ agrees well with the threshold width $\langle \Delta E_{th}/E \rangle_{\ell}$ given by the stability criterion (24). If the initial energy width is larger than this value, the energy variance will increase only a little for ratios of $\langle \Delta E_{in}/E \rangle / \langle \Delta E_{th}/E \rangle_{\ell}$ slightly larger than one and it will be unchanged for ratios larger than two. These results also have been obtained from runs with different sets of parameters.

Figures 7 and 8 represent how the spread $\langle \Delta E_{th}/E \rangle_{n\ell}$ depends on the current $I_0 \sim NZ$, and on the quantities such as d , h , Q , γ , and η , respectively. The results of our non-linear theory agree in most of the runs within a few percent with the small-amplitude threshold derived from linear theory and marked by a straight line in the diagrams. There are noticeable deviations only for values of κ larger than one, i.e. for very short cavities and for cavities with a length to diameter ratio of $\pi\beta/v_{01}$ or a multiple of this.

As has already been seen in Figure 4 the increase of the energy spread does not stop when the perturbation amplitude and therefore the cavity excitation have reached their first maxima. It seems to converge at $t \rightarrow \infty$ to a final value which we estimate by fitting the function

$$\left\langle \frac{\Delta E}{E} \right\rangle = r_1 - r_2 e^{-r_3 t} \quad , \quad r_i > 0$$

where the constants r_1 , r_2 and r_3 after each run are determined by least square fits. The final spread $\langle \Delta E_f / E \rangle = r_1$ is plotted in Figure 6 and 7 by error bars resulting from different fits. It is seen that the final energy spread is proportional to the square of the threshold width for constant initial spread $\langle \Delta E_{in} \rangle$. If the latter is varied (Figure 6), the product $\langle \Delta E_{in} \rangle \langle \Delta E_f \rangle$ is almost constant. These results suggest that the quantity

$$\epsilon = \frac{\langle \Delta E_{in} \rangle \langle \Delta E_f \rangle}{\langle \Delta E_{th} \rangle^2 n \ell}$$

be defined as a measure of the overshoot of $\langle \Delta E_f \rangle$. Figures 6 and 7 indicate that the overshoot is approximately given by $\epsilon = 1$ if the ratio $\Delta E_{in} / \Delta E_{th}$ is not too small. This result has already been obtained by R.A. Dory²⁰⁾ who studied the broad-band longitudinal space-charge interaction, i.e. for a coupling impedance Z_c/n independent of n up to high values of n . There it was found that the final energy spread attained in an unstable configuration never exceeds three times the stabilizing value. By our method, however, ratios of $\Delta E_f / \Delta E_{th}$ of ten and larger are predicted for $\Delta E_{th} / \Delta E_{in}$ of the same order of magnitude.

A few runs with the standard set of parameters but with $\eta < 0$ (below transition energy) yielded the same results on threshold and overshoot as for a corresponding value of $\eta > 0$. This can be explained by the symmetry of both the cavity frequency response and the particle spectrum. A further consequence of that symmetry is the behaviour of the phase ϕ ; thus for the resonance case we found that $\phi(\eta < 0) = -\pi - \phi(\eta > 0)$.

5.4. The dependence of the relative amount of beam harmonics on the initial energy spread

The Fourier components plotted in Figure 4 as function of the number of turns represent well the results generally obtained for an initially unstable beam. The first peaks of the higher beam harmonics

appear at about the time at which the beam variance passes its threshold value. For each beam harmonic studied *) the first peak is bigger than the subsequent ones and, in agreement with the findings quoted in ref. 7, the first Fourier component always dominates the higher components. The amplitude F_1 of the perturbed current which is measured in units of the de-current attained peak values in the range of 1.0 to 1.6.

By several simulations we studied how the amount of the harmonics developed in the beam depends on the initial energy spread. Since the bunches resulting from a cavity oscillation are presumably shorter and thus richer in harmonics when the phase-space density is high, the relative amount of higher beam harmonics should be higher if the initial energy spread is less. This is fully confirmed by the results obtained and listed in Table 2. For decreasing initial beam spread the peak values of the 2nd, 3rd, and 4th beam harmonic increase monotonously with respect to the maximum of the first harmonic which is increasing too. However, for ratios of the initial width to the threshold width decreasing from 10^{-2} to 10^{-3} the amount of the first Fourier component and the peak ratios remain about constant.

5.5. The effect of a high Q factor

In all the runs of which results have been discussed so far the energy spreads were far below the resonance width of the cavity. The latter is, from equations (22) and (28), measured at full width at half maximum

$$\frac{\Delta E_c}{E} = \frac{\beta^2}{|n| Q}$$

*) Up to the 4th order.

With this relation and equations (26) and (27) it is readily seen that the ratio $\Delta E_{th}/\Delta E_c$ as well as the ratio of the damping constant $2Q/\omega$ to the linear growth-time τ_ℓ is proportional to $Q^{3/2}$. Hence for sufficient high Q factor several effects may arise. One of them has been mentioned above in the discussion of Figure 5b. For values of the damping constant of the order of the e-folding growth-time τ_ℓ or larger it was found that the instability develops more slowly than predicted by the linear theory. By comparing the resonance width ΔE_c with the full widths at half height which correspond to the initial spreads of the runs leading to the results plotted in Figure 8b, we see that the instability sets in for beam spreads smaller than the resonance width ΔE_c of the cavity.

Another effect, due to a beam spread of the order of ΔE_c , might change the threshold. It has been studied in further detail by varying the initial beam spread. The parameters were chosen so that the quantity $2Q/(\omega\tau_\ell)$ was slightly larger than one ($N = 50$, $Z = 10^8$, $Q = 4300$). The results are plotted in Figure 9 from which one deduces that the non-linear threshold width amounts to about 90% of the linear threshold width, independently of initial spreads $\langle\Delta E_{in}\rangle$ below threshold. It is also noticeable that the overshoot is much less than for the case $\Delta E_c/\Delta E_{th} \gg 1$. It seems to converge to a limit for decreasing initial spreads, which is about $\langle\Delta E_f\rangle/\langle\Delta E_{th}\rangle_\ell \approx 1.3$.

For Q -values increasing up to 30 000 the ratio $\langle\Delta E_{th}\rangle_{n\ell}/\langle\Delta E_{th}\rangle_\ell$ decreases to about 87% and the final beam spread becomes less than $\langle\Delta E_{th}\rangle_{n\ell}$. However, strong fluctuations of the energy variance render it difficult to derive exact quantitative results. These fluctuations of the energy variance can be correlated with the phase oscillation, the frequency of which becomes comparable to the linear growth-time of the instability for high values of Q .

5.6. Simulations with another distribution function

In order to test the dependence of the results on the distribution function, random numbers for the energy distribution of the particles were selected from a squared cosine distribution instead of a Gaussian distribution. This distribution can be obtained from a uniform distribution of random numbers y in the interval $(0, 1)$ by solving the equation

$$Q(z) = \int_{-2}^z P(x) dx = y$$

where

$$P(x) = \frac{1}{2} \cos^2\left(\frac{\pi x}{4}\right), \quad x \leq 2$$

Thus the initial energies of the particles are found by the relation

$$E_i = s \cdot z_i \left\langle \frac{\Delta E_{in}}{E} \right\rangle$$

with $\langle \Delta E_{in}/E \rangle$ the rms energy spread of the squared cosine distribution and $s = \left(\frac{4}{3} - \frac{8}{\pi^2}\right)^{-\frac{1}{2}}$.

To have a measure of the shape of the distribution function we define a parameter μ giving the ratio of the number of particles within the rms energy spread with respect to the total number of particles in the beam. For the squared cosine function this parameter is $\mu_{sc} = 0.650$ whereas the integration of a Gaussian distribution yields $\mu_G = 0.683$. In our runs we obtained for the initial distribution function $\mu_{sc} = 0.660$ and $\mu_G = 0.690$.

Table 3 shows the results obtained with $N = 800$ and $Z = 5 \cdot 10^7$ for the squared cosine function and the Gaussian shape for different initial rms spreads. No systematic difference for the non-linear threshold is found though for $U \ll V$, the threshold of a particle beam with

an energy distribution similar to a squared cosine function is expected to be about 4% above the threshold of the beam with Gaussian energy shape 15). Supposedly, the shape of the particle layer at threshold is independent of the initial shape. This supposition is favoured by inspection of Figure 3 and by the values of the parameter μ at threshold (Table 3) which obviously are not correlated anymore to their initial values.

5.7. Stored energy and mean beam energy

It may be of interest to quote the energy balance at threshold as function of the various parameters involved in the physics of longitudinal instabilities. From fits to the results we found that the decrease of the mean beam energy behaves as

$$\frac{\langle U \rangle}{E_0 \gamma} = - 7.6 \cdot 10^{-10} e^{a(\kappa-1)} \left[\frac{NZ Q}{\eta \gamma} \right]^{\frac{1}{2}}$$

and the energy stored in the cavity as

$$\frac{W}{E_0 \gamma} = 5.6 \cdot 10^{-29} e^{b(\kappa-1)} \left[\frac{NZ Q}{\eta \gamma} \right]^2 ;$$

all quantities in mksA units. The values for a and b are 3.2 and 5.0 for $\kappa < 1$, or 0.65 and 1.3 for $\kappa \geq 1$, respectively. These relations are good approximations for $NZ \geq 10^9$, $200 \leq Q \leq 2000$, $\gamma \leq 100$, and $10^{-2} \leq \eta \leq 0.5$. In cases where W is larger than $|\langle U \rangle|$, the difference is due to the energy W_0 already stored in the cavity when we start with the computer simulation, i.e. where $\langle U \rangle = 0$.

6. CONCLUSIONS

The longitudinal instability of a coasting particle beam has been investigated in the non-linear region by computer simulation. It is found that the perturbation amplitude of the beam is maximum when the

energy width of the developing instability corresponds to the threshold. For initial spreads above this value no amplification of the perturbation nor a noticeable increase of the energy variance occurs.

The stability limit obtained by our non-linear method agrees very well with the threshold derived from linear theory. There is also a good agreement between our growth-time and the growth-time predicted by the linear theory for δ -function distributions; discrepancies for high values of Q can be explained by a damping time constant larger than the linear growth-time.

In the case of instability the increase of energy variance of the particle beam does not stop when the threshold is reached. The further the initial energy spread is below the stability limit, the larger the final spread ΔE_f will be. For initial energy spreads ΔE_{in} not too small with respect to the threshold ΔE_{th} , we find in agreement with Dory ²⁰⁾

$$\langle \Delta E_{in} \rangle \langle \Delta E_f \rangle \approx \langle \Delta E_{th} \rangle^2 .$$

In spite of the absence of coupling impedances at higher harmonics of the fundamental frequency, modulations of higher ac-current components of an unstable beam occur. This is a typical non-linear effect of the beam-cavity interaction which has already been found experimentally ⁴⁾. The relative amount of harmonics developed in the beam is more pronounced for a beam of high density and small initial energy spread.

For a damping rate of the cavity oscillation superior to the amplification rate of the instability derived from linear theory for high quality factors of the cavity, the growth of an instability sets in for beam spreads smaller than the resonance width of the cavity. However, the growth of the perturbation takes a longer time and the energy spread at threshold seems to be reduced by a few per cents.

The rms energy spreads at threshold obtained for an initial energy distribution of the beam corresponding to a squared cosine distribution are not as large with respect to the threshold widths for beams of initial Gaussian shape as one might expect from ref. 15. This is a consequence resulting from the shape of the particle beam in longitudinal phase-space which is independent of the initial distribution of the particles.

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TABLE 1

Input parameters (in mksA units) of the standard run

$N = 100$	$\gamma = 15$	$d = 0.5$
$Z = 2.5 \times 10^7$	$\eta = 0.1$	$h = 0.2$
$\langle \Delta E_{in}/E \rangle = 10^{-4}$	$\xi = 1.0$	$Q = 200$

TABLE 2

Peak values of higher beam harmonics with respect to the first one as function of the initial energy spread
($N = 200$, $Z = 2.5 \times 10^7$)

$\frac{\langle \Delta E_{in} \rangle}{\langle \Delta E_{th} \rangle}$	$\hat{F}_1$	$\hat{F}_2/\hat{F}_1$	$\hat{F}_3/\hat{F}_1$	$\hat{F}_4/\hat{F}_1$
2^{-1}	0.908	0.359	0.186	0.260
2^{-2}	1.238	0.600	0.395	0.351
2^{-3}	1.359	0.746	0.579	0.475
2^{-4}	1.399	0.785	0.658	0.568
2^{-6}	1.413	0.799	0.675	0.578
2^{-9}	1.414	0.803	0.675	0.573

TABLE 3

Rms energy spread at threshold and the ratio of the number of particles within this spread to the total number of particles, for an initial energy distribution of a squared cosine shape and of a Gaussian shape

$\frac{\langle \Delta E_{in} \rangle}{\langle \Delta E_{th} \rangle_l}$	$\frac{\langle \Delta E_{th} \rangle_{nl}}{\langle \Delta E_{th} \rangle_l}$		$\frac{\langle \Delta E_{th} \rangle_{sc}}{\langle \Delta E_{th} \rangle_G}$	μ_{sc}	μ_G
	sc	G			
0.065	1.185	1.180	1.004	0.6775	0.6688
0.327	1.104	1.076	1.026	0.5800	0.5825
0.490	1.021	1.039	0.983	0.5475	0.5525
0.654	0.975	1.017	0.959	0.5513	0.5813

Figure 2 - Stored energy after 10^3 and 10^4 turns as function of $\xi = \Omega/\omega'$ and Q ($N = 100$, $Z = 10^7$, $d = 0.5$, $h = 0.2$, $\gamma = 15$, $\eta = 0.1$, $\langle \Delta E_{in}/E \rangle = 10^{-5}$).

After 100

500

1000

2000

3000 (near threshold)

4000

5000

6000

7000 turns

Figure 3 - Phase space plot after different numbers of turns ($N = 800$, $Z = 5 \cdot 10^7$).

a) Stored energy

b) First and second Fourier component, $2Q/\omega\tau_\ell = 0.043$

c) Third Fourier component

d) Mean beam energy

e) Phase of cavity oscillation with respect to beam modulation

f) Energy variance in units of $(\langle \Delta E_{th} \rangle_\ell)^2$

Figure 4 - Output data as function of the number of turns ($N = 800$, $Z = 5 \cdot 10^7$).

a) $\kappa = [\sin(1.202 h/d)]/\sqrt{h}$, the results marked by "I" (or "I") have been obtained with $d = 0.5$ m (or $h = 0.2$ m)

b)

c)

d)

Figure 5 - Growth-time of the instability as function of the cavity dimensions d and h , the quality factor Q of the cavity, and the quantities γ and η , respectively.

Figure 6 - Rms energy spread at threshold and final rms energy spread as function of the initial spread; all values are given in units of the linear threshold width $\langle \Delta E_{th} \rangle_\ell$ ($N = 50$, $Z = 10^8$).

Figure 7 - Rms energy spread at threshold and final rms energy spread as function of the current $I_0 \sim NZ$; the energy spreads are given in units of the linear threshold width $\langle \Delta E_{th} \rangle_\ell$. The results marked by "o" (or " Δ ") have been obtained with $Z = 5 \cdot 10^7$ (or $N = 50$).

Figure 8 - Rms energy width at threshold as function of the cavity dimensions d and h , the quality factor Q of the cavity, and the quantities γ and η , respectively.

Figure 9 - Rms energy spread at threshold, and after 10^5 turns where $t/\tau_\ell = 8$, as function of the initial energy spread for a high quality factor Q of the cavity. All values are given in units of the linear threshold spread $\langle \Delta E_{th} \rangle_\ell$ ($N = 50$, $Z = 10^8$, $2Q/(\omega\tau_\ell) = 1.07$).

APPENDIX

Calculation of the cavity impedance at resonance

By conservation of energy the power dissipated in a cavity in ohmic losses is the negative of the time rate of change of stored energy. The latter can be related to the stored energy of a damped cavity

$$W = \frac{1}{2}(\dot{q}^2 + \omega'^2 q^2) \quad (\text{A } 1)$$

by differentiating (A 1) with respect to time and substituting q and its derivatives by the solution (15) of the damped cavity oscillation. One finds after averaging with respect to time

$$\left\langle \frac{dW}{dt} \right\rangle = - \frac{\omega}{Q} \langle W \rangle . \quad (\text{A } 2)$$

Since this result agrees with what follows directly from the definition of the Q of a cavity ²²⁾, defined as 2π times the ratio of the time-averaged energy stored in the cavity to the energy loss per cycle, the damping factor Q introduced in equation (14) is identical with the quality factor Q of a cavity. For the cylindrical cavity shown in Figure 1 one finds ²³⁾

$$Q = \frac{d}{\delta} \frac{h}{h + d}$$

where δ is the skin depth.

At resonance the angular frequency Ω of the particles circulating on the mean orbit of the accelerator equals the frequency ω' . Thus by replacing the differentials of equation (A 2) by difference quantities, $dW/dt = \omega' \delta W / 2\pi$ with δW given by equation (13) for N particles, we obtain

$$\langle W \rangle = - \frac{\omega' \omega N Q D^2}{4\pi} .$$

The coupling impedance is now defined ²³⁾ as the negative ratio of the perturbed voltage - obtained by integrating the average electric field along the trajectories of the particles over one period - to the amplitude of the perturbed current, which is

$$I_o = \frac{N Z e \omega'}{2\pi} \quad (A \ 3)$$

Hence

$$Z_c = - \frac{2\langle W \rangle}{Ze I} = \frac{\omega Q D^2}{(Ze)^2}$$

or

$$Z_c = \frac{4 Z_o Q}{\pi v_{01}^3 J_1^2(v_{01})} \frac{d}{h} \beta^2 \sin^2\left(\frac{v_{01} h}{2\beta d}\right) \quad (A \ 4)$$

where $\beta = v/c$ and $Z_o = 120\pi$ Ohm is the impedance of free space. The cavity impedance at resonance, Z_c , is a real quantity which becomes for highly relativistic particles

$$Z_c = 128Q \frac{d}{h} \sin^2(1.202 \frac{h}{d}) \text{ Ohm} .$$

This result is identical with that derived in ref. 23 where a different approach has been used.