DYNAMICAL SYMMETRY BREAKDOWN IN PURE YANG-MILLS FIELD THEORY

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A B S T R A C T

It is shown, in a simple approximation consistent with gauge invariance, that Yang-Mills fields associated with a compact simple Lie group, may acquire mass through their self-interactions; this phenomenon leads to a breakdown of the original group symmetry and to an eigenvalue condition for the coupling constant of the self-interaction. It is conjectured, from an analysis of Feynman graphs, that these facts are still true in the full theory. The relevance of these conclusions to the problem of fermions interacting with Yang-Mills mesons is pointed out.

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1. INTRODUCTION

It is well known that Yang-Mills vector mesons ¹⁾ may acquire mass when coupled to scalar fields in a gauge invariant Lagrangian; this happens when the original symmetry is spontaneously broken through non-vanishing vacuum expectation values of some scalar fields ^{2),3)}.

Such inclusion of scalar mesons appears, however, quite arbitrary and presumably of a phenomenological character; one is tempted, from a more fundamental point of view, to look for a dynamical symmetry breakdown which leads to mass generation without recourse to these scalar mesons ^{2),4)}. Along this line of thought, one is naturally led to inquire if a system of pure Yang-Mills fields associated to a compact simple Lie group is stable against mass generation. Such a phenomenon could arise because of the Yang-Mills self-interaction and would provide the "purest" example of a dynamical breakdown of symmetry.

If mass generation does in fact occur, it would be desirable to prove that it originates from the infra-red divergences of the massless theory as suggested by the work of Coleman and Weinberg ⁵⁾. This is, however, a difficult problem and we have tried to solve a more simple one, namely: is mass generation consistent with the field theory of pure Yang-Mills fields? The hope in this approach is that if such a solution of the field equations exists, it is in fact a stable one.

While the consistency of such a symmetry breaking solution with the Ward identities is straightforward ^{6),7)}, it is, however, far from obvious that this would still lead to finite results in the renormalized version of the theory. It is also of importance to understand if such a breakdown of symmetry may arise for arbitrary values of the coupling constant. The answer to these problems given in this paper rely on an assumption made in Section 4, which we can justify only by a plausibility argument. Nevertheless, we feel that our conclusions are of sufficient interest to motivate their presentation, especially because we are led to believe that spontaneous mass generation may indeed arise, but only for specific values of the coupling constant. This means that mass generation fixes the scale of the Yang-Mills fields with respect to the group parameters, a perhaps unexpected result.

The self-consistency of mass generation is suggested from a simple qualitative argument. Let us consider the matrix elements of the conserved current operator $J_\mu^a(0)$ between two vector meson states $|\epsilon^{(\lambda)b}, p\rangle$ and $|\epsilon^{(\lambda')c}, p'\rangle$. The indices b and c label here the conserved currents J_μ^b

and J_μ^a coupled to the mesons b and c . Defining $P_\mu = p_\mu + p'_\mu$, $Q_\mu = p_\mu - p'_\mu$, one has, in absence of symmetry breaking,

$$\langle \epsilon^{(q)b}, p_n | J_\mu^a(0) | \epsilon^{(q')c}, p'_m \rangle \xrightarrow{Q \rightarrow 0} N \epsilon^{(q)b} \cdot \epsilon^{(q')c} f^{abc} P_\mu \quad (1.1)$$

where N is a normalization factor ; indeed all other invariant form factors are multiplied by vanishing kinematical factors in this limit. One verifies, by taking the divergence of the current in (1.1), that the right-hand side vanishes ; explicitly $Q^\mu P_\mu$ is equal to zero on the mass shell of both mesons. If we now tentatively assume that the mesons b and c acquire unequal masses m_b and m_c , $Q^\mu P_\mu = m_b^2 - m_c^2 \neq 0$. As J_μ^a is still conserved, one must add to the right-hand side of (1.1) singular form factors as $Q^2 \rightarrow 0$ to ensure zero divergence in that limit ; the simplest solution is

$$\begin{aligned} \langle \epsilon^{(q)b}, p | J_\mu^a(0) | \epsilon^{(q')c}, p' \rangle \xrightarrow{Q^2 \rightarrow 0} & N \epsilon^{(q)b} \epsilon^{(q')c} [f^{abc} P_\mu - \\ & - f^{abc} [(m_b)^2 - (m_c)^2] \frac{Q_\mu}{Q^2} + \text{Regular terms as } Q^2 \rightarrow 0] \end{aligned} \quad (1.2)$$

This means that the current J_μ^a is coupled to a Nambu-Goldstone (N-G) singularity which in turn is expected to generate in the usual way a mass $m_a \neq 0$. Of course the above argument does not prove that such mass generation is consistent with the dynamics. Note that the conserved current used above is not the observable one because the singularity in (1.2) is only a gauge effect ; neither is it the one we shall use in the Feynman rules. Therefore the above argument should serve only as a guide to ease the way through the subsequent formalism of Yang-Mills field theory.

Our study is presented as follows. In Section 2 we review the known Ward identities for Yang-Mills fields⁸⁾. We emphasize the usefulness of the Landau gauge where the N-G singularities are less affected than in other gauges by the intrinsic complications of non-Abelian gauge field quantization.

In Section 3 we present a simple approximation to the Schwinger-Dyson equations for self-energies at zero external momentum. In this approximation no ultra-violet (u-v) divergences are present and therefore no renormalization program is used. A simple system of algebraic equations determining induced

mass ratios is obtained and it is shown that the bare coupling constant of the theory obeys a consistency equation. The approximation can be interpreted diagrammatically as a type of ladder approximation and shown to be consistent with the Ward identities implied by gauge invariance. The proof of this consistency of our ladder approximation requires tedious calculations; these are relegated to Section 5. The reader interested only in the general aspects of the problem may omit the computations of Section 5; we feel, however, that the results established there greatly clarify the connection of our approximation with perturbation theory.

In Section 4, the qualitative conclusions of Section 3 are extended to the full theory under some assumptions made plausible by the previous analysis. Namely, we show, if these assumptions are correct, that symmetry breaking does not introduce new divergences in the renormalized theory. Moreover, the occurrence of an eigenvalue equation for the renormalized coupling constant appears as a property of the full theory and not only of the approximation used in Section 3. We conclude the Section by pointing out the relevance of these conclusions to the case where coupling to fermions is included in a gauge invariant way; the results of a previous investigation on low energy eigenvalue equations for such systems ⁹⁾ may be extended to take into account non-perturbative effects of the non-linear Yang-Mills meson self-interactions.

2. WARD IDENTITIES

The Ward identities pertaining to a Lagrangian exhibiting gauge invariance of the second kind are well known ⁸⁾. We shall briefly review them and discuss the simple relations they imply in presence of spontaneous symmetry breaking when the Yang-Mills fields are described in the Landau gauge.

Let us consider a system of pure Yang-Mills fields associated with a compact simple Lie group G . They are characterized, in the set of covariant gauges, by the Lagrangian ¹⁰⁾

$$L = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{2\eta} \partial_\mu A^{*\mu} \partial_\nu A_\nu - \partial_\mu \varphi^{*a} \partial^\mu \varphi_a - g f_{abc} \partial_\mu \varphi^{*a} \varphi^b A^{c\mu} \quad (2.1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c \quad (2.2)$$

Here, as throughout the text, Latin letters refer to group indices and Greek letters to Lorentz indices. The A_μ^a are the Yang-Mills fields and $\varphi_a(\varphi_a^*)$ the Faddeev-Popov ghost ¹¹⁾ fields; η is a gauge parameter, g the bare coupling constant and f_{abc} the completely antisymmetric structure constants.

The Feynman graphs are drawn in terms of bare propagators corresponding to the unperturbed Lagrangian ⁸⁾.

$$L_0 = -\frac{1}{4} [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a] [\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] - \frac{1}{2\eta} \partial_\mu A^{*\mu} \partial_\nu A_a^\nu \quad (2.3)$$

This introduces not only three-point but also four-point vertices. We shall find it convenient to factorize the latter using a tensor propagator

$$A_{\mu\nu,\sigma\tau}^{a,b} = \frac{1}{2} [g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}] \left[\frac{i}{(2\pi)^4} \delta^{ab} \right] \quad (2.4)$$

This propagator couples to Yang-Mills propagators through the bare interaction vertex (Fig. 1)

$$V_{\mu\nu,\sigma\tau}^{ab,c} = \frac{g}{\sqrt{2}} [g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}] f^{abc} \quad (2.5)$$

It is readily verified that the exchange of (2.4) in the three channels restores to order g^2 the four-point vertex (Fig. 1). In higher orders, as (2.4) is dressed both by longitudinal and transverse self-energy insertions, it will be useful to separate the bare propagator (2.4) into a transverse and a longitudinal part. Thus we write

$$A_{\mu\nu,\sigma\tau}^{a,b} = T_{\mu\nu,\sigma\tau}^{a,b} + L_{\mu\nu,\sigma\tau}^{a,b} \quad (2.6)$$

with

$$T_{\mu\nu,\sigma\epsilon}^{a,b}(q) = \frac{1}{2} [G_{\mu\sigma}(q)G_{\nu\epsilon}(q) - G_{\mu\epsilon}(q)G_{\nu\sigma}(q)] \frac{i}{(2\pi)^4} \delta^{ab} \quad (2.7)$$

$$G_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \quad (2.8)$$

In this formalism the bare Yang-Mills propagator

$$D_{\mu\nu}^{ab}(q) = \frac{-i}{(2\pi)^4} \left[\frac{G_{\mu\nu}(q)}{q^2} + \eta \frac{q_\mu q_\nu}{q^2} \right] \delta^{ab} \quad (2.9)$$

couples to $A_{\rho\lambda,\sigma\tau}^{c,d}$ as exemplified in Fig. 2. However, from Lorentz invariance the coupling has the form

$$C_{\nu,\rho\lambda}^{b,c}(q) = [q_\rho g_{\nu\lambda} - q_\lambda g_{\nu\rho}] f^{bc}(q^2) \quad (2.10)$$

Therefore only the coupling of the transverse part of (2.9) to the longitudinal part of (2.6) survives.

The Ward identities give constraints on longitudinal parts of Green's functions, that is on the divergences of the tensors

$$G_{\mu_1\mu_2\ldots\mu_n}^{a_1a_2\ldots a_n}(x_1, x_2, \ldots, x_n) = \langle T^* A_{\mu_1}^{a_1}(x_1) \ldots A_{\mu_n}^{a_n}(x_n) \rangle \quad (2.11)$$

with respect to one or more space-time variables. In particular for $n > 2$ one has, contracting all indices

$$\partial^{\mu_1} \partial^{\mu_2} \ldots \partial^{\mu_n} \bar{G}_{\mu_1 \ldots \mu_n}^{a_1 \ldots a_n} = 0 \quad (n > 2) \quad (2.12)$$

where $\bar{G}_{\mu_1 \dots \mu_n}^{a_1 \dots a_n}$ is the connected part of $G_{\mu_1 \dots \mu_n}^{a_1 \dots a_n}$. The corresponding statement for $n = 2$ is the transversality of the self-energy matrix $(2\pi)^4 i \Pi_{\mu\nu}$. We write $\Pi_{\mu\nu} = \Pi_{\mu\nu}^{(1)} + \Pi_{\mu\nu}^{(2)}$ where $\Pi_{\mu\nu}^{(2)}$ contains all contributions to $\Pi_{\mu\nu}$ which are one particle reducible with respect to the tensor propagator (2.6). It follows kinematically from (2.10) that

$$\Pi_{\mu\nu}^{(2)}(q) = G_{\mu\nu}(q) q^2 \tilde{\Pi}(q^2) \quad (2.13)$$

Thus, the dynamical content of the Ward identity for $n = 2$ is embodied in the transversality of $\Pi_{\mu\nu}^{(1)}(q)$; we write therefore

$$\Pi_{\mu\nu}^{(1)}(q) = G_{\mu\nu}(q) q^2 \Pi(q^2) \quad (2.14)$$

and the dressed propagator matrix $\tilde{D}_{\mu\nu}$ has the form

$$\tilde{D}_{\mu\nu}(q) = \frac{-i}{(2\pi)^4} \left[\frac{G_{\mu\nu}(q)}{q^2 - q^2 \Pi(q^2) - q^2 \tilde{\Pi}(q^2)} + \eta \frac{q_\mu q_\nu}{q^2} \right] \quad (2.15)$$

We may use (2.15) if $\eta \neq 0$ to amputate all external vector propagators in (2.12). We then obtain for the proper vertex functions

$$\partial^{\mu_1} \partial^{\mu_2} \dots \partial^{\mu_n} \Gamma_{\mu_1 \dots \mu_n}^{a_1 \dots a_n} = 0 \quad (2.16)$$

By defining $\Gamma_{\mu_1 \mu_2} = \Pi_{\mu_1 \mu_2}$ we shall extend (2.16) for $n = 2$ also; as was done for $\Pi_{\mu_1 \mu_2}$, one can in general separate out kinematical effects due to tensor propagation but we shall not need to do so. We shall refer to (2.16) as the transversality relations.

It is important to note that because (2.12) and (2.15) are valid for all values of η , (2.16) is also true by continuity for $\eta = 0$ despite the fact that in this case (2.15) admits no inverse and that (2.12) is trivially satisfied. Thus the Landau gauge $\eta = 0$ does not play a special role in the transversality relations. Its usefulness will now appear in the remaining Ward identities where not all the Lorentz indices are contracted; we shall,

however, restrict ourselves to the three-point vertex function because the relevant features of the N-G bosons may be obtained from it.

Consider the three-point improper vertex function $\gamma_{\mu, \nu\sigma}^{a, bc}$ where only the vector propagator attached to the μ index has been amputated. This function satisfies the following Ward identity in momentum space

$$\begin{aligned} (1/q) q^\mu \gamma_{\mu; \nu\sigma}^{a; bc}(p+q/2; p-q/2) &= i q^\mu \tilde{G}_\mu^a(q^2) \{ \\ & f^{bxv} \tilde{D}_{\nu\sigma}^c(p+q/2) - \tilde{D}_{\nu, u\sigma}^b(p-q/2) f^{uxc} \} \\ & + \Phi_{\nu\sigma}^{a; bc}(p, q) \end{aligned} \quad (2.17)$$

The function Φ contains graphs where at least one external Yang-Mills propagator is connected to an external ghost propagator $\tilde{G}_x^a(q^2)$ (Fig. 3). Thus, from the Feynman rules in the Landau gauge, Φ will contain a momentum factor q in that gauge. As there seems to be no reason for Φ to develop a pole at $q^2 = 0$ we may assume

$$\lim_{q \rightarrow 0} \Phi(p, q) = 0 \quad (\text{Landau gauge}) \quad (2.18)$$

Thus, in the Landau gauge, as $q \rightarrow 0$ (2.17) reduces to

$$(1/q) q^\mu \gamma_{\mu; \nu\sigma}^a \xrightarrow{q \rightarrow 0} i [q^\mu \tilde{G}_\mu^a(q^2) F^x, \tilde{D}_{\nu\sigma}(p)] \quad (2.19)$$

where we have used a matrix notation with $(F^x)_{bc} = f_{bc}^x$. From now on, we shall restrict ourselves to the Landau gauge.

From (2.19) the singularity of $\gamma_{\mu; \nu\sigma}^a$ at $q = 0$ follows and one has

$$\begin{aligned} (1/q) \gamma_{\mu; \nu\sigma}^a \xrightarrow{q \rightarrow 0} i q_\mu [\tilde{G}_\mu^a(q^2) F^x, D_{\nu\sigma}(p)] \\ + \text{regular terms} \end{aligned} \quad (2.20)$$

This shows that if $[F^x, D_{\nu\sigma}(p)] \neq 0$ a singularity develops in $\gamma_{\mu;\nu\sigma}^a$ as $q \rightarrow 0$ and that the singularity of this "N-G boson" is identical to that of a Faddeev-Popov ghost. If one neglected the radiative corrections in the ghost propagator $[\tilde{G}_x^a(q^2) = \delta_x^a(-i)/(2\pi)^4 q^2]$, (2.19) would resemble the more conventional type Ward identities and would lead to the (incorrect) result

$$(1/q) \gamma_{\mu;\nu\sigma}^a \xrightarrow{q \rightarrow 0} (q_\mu/q^2) \frac{1}{(2\pi)^4} [F^a, D_{\nu\sigma}(p)] \quad (2.21)$$

+ regular terms

Comparing (2.21) to (2.20) one sees that for non-Abelian gauge fields, the N-G boson is affected by the same radiative corrections as the Faddeev-Popov ghost and this is presumably a consequence of unitarity ¹²⁾. Note, however, that the simplicity of the generalization from (2.20) to (2.21) occurs only in the Landau gauge. Note also that (2.20) is valid only for the improper vertex function because the proper vertex may develop singular form factors proportional to p_ν (or p_σ) which are projected out in the improper vertex due to the transversality of the external propagators in the Landau gauge. Such form factors do indeed appear, as will be seen in Section 5, and are implied in order to satisfy the transversality relations (2.16) for the three-point function. In fact, these transversality relations do allow us to write the equation analogous to (2.19) for the proper vertex function ⁷⁾, namely (in matrix notation)

$$(1/q) q^\mu \Gamma_{\mu;\nu\sigma}^a \xrightarrow{q \rightarrow 0} -(2\pi)^4 [q^2 \tilde{G}_x^a(q^2) F^x, \pi_{\nu\sigma}(p)] \quad (2.22)$$

in close analogy to (1.2). The equation corresponding to (2.20) is, however, more complicated because the transversality of (2.22) in p does not originate in a simple kinematical factor, as it does in (2.19). The Ward identities (2.16), (2.19) and (2.22) will provide a convenient tool for studying spontaneous symmetry breaking solutions in pure Yang-Mills field theory through the occurrence of N-G singularities.

We now show how $u-v$ divergences may be removed from (2.19) by a renormalization program so that the N-G singularity persists in the renormalized theory. We assume that the only counterterms required are those of the symmetric theory; thus no mass counter terms are used for Yang-Mills fields.

This convergence condition must ultimately be verified, as the symmetry of the unrenormalized Lagrangian must be the same as the symmetry underlying the finite theory.

If $Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3$ are respectively the vertex and wave function renormalization constants for the Yang-Mills and the ghosts fields, we may write (2.19) as

$$(1/g_R) q^\mu \gamma_{\mu, \nu\sigma}^{\prime a} \xrightarrow{q \rightarrow 0} \frac{Z_1}{Z_3} \tilde{Z}_3 i q^2 [\tilde{G}_\mu^{\prime a}(q^2) F^\mu, \tilde{D}_{\nu\sigma}^{\prime}(p)] \quad (2.23)$$

where $g_R = g Z_3^{\frac{3}{2}} Z_1^{-1}$ is the renormalized coupling constant and the prime denotes renormalized quantities. Now, if one chooses the vertex renormalization constants in such a way as to ensure that the renormalized coupling constant of the ghost is equal to g_R , one has the usual relation ¹³⁾

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} \quad (2.24)$$

Thus the renormalization constant in (2.23) is in fact $\tilde{Z}_1$ as seen from (2.24). If the subtractions are made at zero external momentum we will have $\tilde{Z}_1 = 1$ in the Landau gauge ; this follows from the same argument that led to (2.18) as applied to the ghost-ghost-Yang-Mills vertex. If one chooses another subtraction point in order to avoid spurious infra-red divergences in the renormalization program, $\tilde{Z}_1$ remains ultra-violet finite. We therefore see that all u-v infinities cancel in (2.19) in the renormalized theory.

The preceding discussion has been limited to pure Yang-Mills fields associated to a simple group. The extension to more general gauge invariant Lagrangians, including coupling to fermions, is, however, straightforward. The Ward identities for all vertices coupled to a Yang-Mills propagator take a simple form in the Landau gauge, in analogy to (2.19); as in the present case, all u-v divergences should be removed from Ward identities by terms of the symmetric theory.

3. SPONTANEOUS SYMMETRY BREAKING IN PURE YANG-MILLS FIELD THEORY : A SIMPLE APPROXIMATION

One may look for symmetry breaking solutions in pure Yang-Mills field theory by assuming that the Yang-Mills self-energy matrix $\Pi_{\mu\nu}(p)$ acquires a symmetry violating part, namely

$$[F^a, \Pi_{\mu\nu}(p)] \neq 0 \quad \text{for some } a \quad (3.1)$$

This means in general that as $q^2 \rightarrow 0$, $\Pi(q^2)$ as defined in (2.14) becomes a singular matrix with a $1/q^2$ singularity (up to log terms). Indeed, (3.1) implies, using (2.19), (2.20) or (2.22) that such a singularity will develop in $\Gamma_{\mu\sigma\tau}^{abc}$, the proper vertex function. As this vertex contributes to the Schwinger-Dyson equation for $\Pi_{\mu\nu}^{(1)}$ (Fig. 4), its singularity will generate, as $q \rightarrow 0$, a non-vanishing contribution to the coefficient of $q_\mu q_\nu / q^2$ in $\Pi_{\mu\nu}^{(1)}$. From gauge invariance this is a contribution to $q^2 \Pi(q^2)$ in (2.14) which does not vanish as $q^2 \rightarrow 0$. As a non-zero $\Pi_{\mu\nu}(0)$ matrix implies that at least some Yang-Mills fields acquire mass, we see that (3.1) is in fact equivalent to the assumption that some Yang-Mills fields acquire spontaneously a mass as a consequence of N-G bound state formation.

As already mentioned in Section 2, the consistency of assumption (3.1) hinges upon the proof that renormalized symmetry breaking terms are finite without recourse to symmetry violating counter terms. This problem will be discussed in the next Section but it is desirable, in order to study qualitative aspects of the symmetry breaking, to try a simple approximation in which no renormalization at all is necessary. In order to be significant, satisfy the Ward identities of Section 2 ; this is indeed necessary to ensure, in a way consistent with gauge invariance, the dynamical character of self-generated masses as is made clear in the above discussion of (3.1). We now construct such an approximation.

To order g^2 , the only diagrams contributing to $\Pi_{\mu\nu}$ are shown in Fig. 5. When regularized in a gauge invariant way, for instance through dimensional regularization¹⁴⁾, they yield $\Pi_{\mu\nu} \rightarrow 0$ as $q^2 \rightarrow 0$. To obtain a self-consistent non-vanishing $\Pi_{\mu\nu}(0)$ one may insert a non-vanishing mass term M^2 in the Yang-Mills propagator of Figs. 5a and 5b. Here M^2 is to be interpreted as

$$M^2 = \lim_{p^2 \rightarrow 0} p^2 \Pi(p^2) \quad (3.2)$$

and we shall restrict ourselves, for simplicity, to cases where M^2 is diagonalizable in the space of the regular representation. Thus

$$(M^2)_{ab} = m_a^2 \delta_{ab} \quad (3.3)$$

The consistency of approximating self-energies in the virtual loops, 5a, 5b by their zero momentum value (3.2) will eventually be justified by the fast convergence of the integrals at zero external momentum q . With (3.2), the Yang-Mills propagators become

$$\bar{D}_{\mu\nu}(p) = \frac{-i}{(2\pi)^4} \frac{g_{\mu\nu} - p_\mu p_\nu / p^2}{p^2 - M^2} \quad (3.4)$$

As $q \rightarrow 0$, Figs. 5a and 5b now yield a non-vanishing result while Fig. 5c remains equal to zero in that limit (see Section 5); it is understood that all divergent integrals are evaluated with the dimensional regularization technique.

The resulting $\Pi_{\mu\nu}(0)$, however, not only diverges as the space-time dimension n goes to 4, but it is proportional to $g_{\mu\nu}$ and therefore inconsistent with gauge invariance. In fact, as discussed above, a contribution to $\Pi_{\mu\nu}(0)$ proportional to $q_\mu q_\nu / q^2$ as $q \rightarrow 0$ can only arise if the Yang-Mills fields couple to a N-G bound state at $q^2 = 0$. From (2.20), we see that (3.4) indeed implies that such a bound state is developed if M^2 is not proportional to the unit matrix. To evaluate the contribution of this bound state to $\Pi_{\mu\nu}$, we approximate (2.20) by (2.21) in accordance with our neglect of radiative corrections to the ghost propagator. From (2.21) and (3.4) we find the simple result (1.2) in the form

$$(1/q) \Gamma_{\mu\sigma\tau}^{abc}(p-q/2; p+q/2) \xrightarrow{q \rightarrow 0} i g_{\sigma\tau} \frac{q_\mu}{q^2} f^{abc} [(m^b)^2 - (m^c)^2] + \dots \quad (3.5)$$

Here, as already discussed in Section 2, the missing form factors are not only the regular ones as $q \rightarrow 0$, but also those singular ones which contain longitudinal kinematical factors p_0 (or p_T) that were projected out in the improper vertex function $\gamma_{\mu;\sigma\tau}^{a;b\bar{c}}$. They do not contribute to $\Pi_{\mu\nu}(0)$ in the Landau gauge. The contribution of the singular factor (3.5) in Fig. 4 is shown in Fig. 6c. As will be seen below by direct evaluation, this integral is convergent, the leading logarithmic divergence vanishing as a consequence of Lorentz and group symmetries.

Thus the inclusion of diagram 6c, while required by the Ward identity (3.5), is insufficient to ensure the transversality of the approximate $\Pi_{\mu\nu}$ in the $q \rightarrow 0$ limit, as the $g_{\mu\nu}$ term was divergent at $n=4$. However, it appears possible, as will be discussed below, to restore transversality, and thus convergence, by introducing a suitably chosen self-energy to the tensor propagator. Explicitly, we introduce in Fig. 5b the modified tensor propagator

$$\bar{A}_{\mu\nu,\sigma\tau} = \bar{T}_{\mu\nu,\sigma\tau} + L_{\mu\nu,\sigma\tau} \quad (3.6)$$

$$\bar{T}_{\mu\nu,\sigma\tau} = \frac{n-1}{n-2} T_{\mu\nu,\sigma\tau} \quad (3.7)$$

where the limit $n=4$ will be taken at the end. Thus we finally retain for $\Pi_{\mu\nu}(0)$ the sum of the three contributions pictured in Fig. 6.

Clearly, our approximation summarized in Fig. 6 cannot be taken seriously before making contact with perturbation theory. This is studied in detail in Section 5 for a particular group (SU_3), and a particular self-consistent breaking mechanism ($M^2 \sim Y^2$). We shall summarize here the main results of Section 5, retaining our present more general notation, since we tentatively surmise their general validity.

If the diagram of Fig. 6c has really a dynamical origin, it may be rewritten in the form shown in Fig. 7 where the factorization of the residue at the N-G pole is made explicit. The resulting vertices, approximated at zero external momenta, may be determined from the Ward identity (3.5) as shown in Fig. 8. On the other hand, they should also be derivable from a homogeneous Bethe-Salpeter equation; they satisfy indeed the equation shown in Fig. 9. Thus our approximation may be identified with a "ladder approximation" for Green's functions at zero external momenta, using the propagators

(3.4) and (3.6). This permits us to prove not only the transversality and the convergence of $\Pi_{\mu\nu}$ as $q \rightarrow 0$, but also to verify diagrammatically the Ward identities (2.17) ^{$\mu\nu$} for three- and four-point vertices in our "ladder approximation". Note, however, that while the use of (3.4) can be easily understood, this is not so for (3.6); we shall return to this point in Section 5.

We now evaluate $\Pi_{\mu\nu}(0)$ from the diagrams of Fig. 6. It suffices to compute the $q_\mu q_\nu / q^2$ term which comes from diagram 6c. Retaining only terms linear in q in the integrand, one gets

$$\begin{aligned} [q_\mu q_\nu / q^2]_{q=0} m_a^2 \delta_{aa'} &= \lim_{q \rightarrow 0} 3q^2 \sum_{b,c} f_{abc} f_{a'bc} \cdot \\ & \frac{(m_b^2 - m_c^2)(q_\mu / q^2)}{(2\pi)^4} \int p_\nu \frac{1}{(p+q/2)^2 - m_b^2} \cdot \frac{1}{(p-q/2)^2 - m_c^2} d^4 p \\ &= [q_\mu q_\nu / q^2]_{q=0} \frac{3}{4} q^2 \sum_{b,c} [f_{abc} (m_b^2 - m_c^2)^2 f_{a'bc}] \cdot \\ & \frac{i}{(2\pi)^4} \int \frac{p^2}{(p^2 - m_b^2)^2 (p^2 - m_c^2)^2} d^4 p \end{aligned} \quad (3.8)$$

The integral is easily evaluated upon performing a Wick rotation. We obtain

$$\begin{aligned} \delta_{aa'} m_a^2 &= \frac{3}{64\pi^2} q^2 \sum_{b,c} f_{abc} f_{a'bc} \frac{(m_b^2 - m_c^2)}{m_b^2 + m_c^2} J(x_{bc}) \\ x_{bc} &= m_b^2 / m_c^2 \end{aligned} \quad (3.9)$$

$$J(x) = \left(\frac{x+1}{x-1} \right)^2 \left[1 - \frac{2x}{x^2-1} \log x \right]$$

One may check that $J(x) = J(1/x)$ and that $J(x)$ is a smoothly varying function in the relevant interval $0 \leq x \leq 1$ with $J(0) = 1$ and $J(1) = 2/3$.

Equation (3.9) is expected, in general, to yield a solution only for specific values of the coupling constant g . This is because (3.9) provides a non-trivial equation for each non-vanishing mass, but as from scale invariance (3.9) can determine only mass ratios, one of these equations becomes in fact an equation for g . We shall illustrate this by a simple example where all non-vanishing masses are equal.

Consider an arbitrary compact simple Lie group. We can always find a partition (A,B) such that if α_i is a generator belonging to class A and β_j a generator belonging to class B

$$\begin{aligned} [\alpha_i \alpha_j] &\sim \alpha_k \\ [\alpha_i \beta_j] &\sim \beta_k \\ [\beta_i \beta_j] &\sim \alpha_k \end{aligned} \tag{3.10}$$

For instance, in SU_3 , one may choose the α_i 's as the non-strange generators F_1, F_2, F_3, F_8 and the β_j 's as the strange ones F_4, F_5, F_6, F_7 . Equation (3.5) [or (1.2)] suggests the possibility of a spontaneous breaking of symmetry where all the mesons transforming like the β_j 's ("strange mesons") require a common mass m , while all the others ("non-strange mesons") remain massless. Indeed for such a solution it follows from (3.5) [or (1.2)], using the solution (3.10), that the only mesons which would be coupled to N-E singularities are the "strange" ones. Let us therefore verify that such a solution does indeed satisfy the dynamical constraint (3.9). We see that (3.9) is non-trivial only if a is a "strange" index, in which case it reduces to

$$\begin{aligned} \delta_{aa'} m^2 &= \frac{3}{64\pi^2} g^2 \sum_{b,c} f_{abc} f_{a'bc} m^2 \\ &= \frac{3}{64\pi^2} g^2 C m^2 \delta_{aa'} \end{aligned} \tag{3.11}$$

where C is a characteristic of the group [C is the value of the Casimir operator $\sum_a (F^a)^2$ in the regular representation]. The proposed solution does therefore satisfy the dynamical equation for spontaneous breakdown in our approximation, provided g^2 takes the value $64\pi^2/3C$. This number, should, of course, not be taken seriously, but what will be relevant for

further discussion, is the convergence of the loop diagrams of Fig. 6 which led to such an eigenvalue condition. As seen from (3.8), this arises because the singular vertex is symmetric in the indices b and c , while the bare vertex is antisymmetric in the same indices. A non-vanishing result is then obtained by expanding the inside propagator to first order in $p \cdot q$, thereby reducing the apparently logarithmically divergent integral to a convergent one. The convergence of the sum of diagrams 6a and 6b follows then from gauge invariance; the same is true for the diagrams of Fig. 9 which by power counting look as if they would diverge logarithmically. The generalization of these features, to all orders in perturbation theory and their relevance to the possible emergence of our eigenvalue condition in the full theory, will be discussed in the next section.

4. LOW-ENERGY EIGENVALUE CONDITIONS IN GAUGE FIELD THEORIES

4.1 General considerations

Before discussing the possible generalization of the results of Section 3, it might be worth while to review how eigenvalue conditions of a similar type may appear in presence of a dynamically broken symmetry ⁹⁾.

Consider a scale invariant Lagrangian admitting gauge invariance of the second kind, and suppose that a broken symmetry solution exists and can be characterized by scalar functions $\phi_i(p^2)$ of non-zero physical dimension. In the present case, such $\phi_i(p^2)$ may be identified with the matrix elements of $[T^a, p^2 \Pi(p^2)]$; here we have included the p^2 factor to avoid, as $p^2 \rightarrow 0$, singularities in $\phi_i(p^2)$ other than logarithmic ones. In the well-known model of mass generation of Baker, Johnson and Willey ¹⁵⁾, and in its generalization to include axial vector gauge fields ¹⁶⁾, the scalar part $\Sigma_2(p^2)$ of the fermion self-energy $\Sigma(p) = \not{p}\Sigma_1(p^2) + \Sigma_2(p^2)$ is such a $\phi(p^2)$.

The self-consistent $\phi_i(p^2)$ may be determined in the following way. Expand all Green's functions of the theory in Feynman graphs, using bare vertices and dressed propagators. If, as we suppose, $\phi_i(p^2)$ enters these propagators, they will obey coupled non-linear integral equations either from Schwinger-Dyson equations for self energies, or equivalently, from the vertex equations at zero-momentum transfer used in conjunction with Ward identities of the type (2.22). A simple example of such an equation is given in Ref. 9), Eq. (3); it determines $\Sigma_2(p^2)$ when chirality is dynamically broken by vector and axial vector Abelian gauge field interactions ¹⁶⁾.

In general, we shall suppose that $\phi_i(p^2)$ obeys an equation of the type

$$\phi_i(p) = \int \sum_{j,k} K_{ij}(p^2, p'^2; \{\phi_k\}) G_{jk}(p'^2; \{\phi_k\}) \phi_k(p'^2) d^4 p' \quad (4.1)$$

This is represented schematically in Fig. 10; G_{jk} stands for the product of the two dressed propagators, and K_{ij} is a Bethe-Salpeter kernel. All the contractions of Lorentz indices have been performed so that (4.1) involves only Lorentz scalars; from the Feynman rules, one has generally $K_{ij}(p^2, p'^2) = K_{ji}(p'^2, p^2)$ in Euclidean Wick rotated space; as $(p^2) \rightarrow \infty$, $G_{jk}(p^2; \{\phi_k\})$ tends to a symmetric limit which we shall denote $\delta_{jk} G(p^2)$. Equation (4.1) arises from a homogeneous vertex equation, since the inhomogeneity drops out in the singular contribution at $q=0$ which appears in Ward identities; therefore, Eq. (4.1) is also valid for renormalized quantities. Note that the renormalization program required can be carried out diagrammatically for our Feynman graphs: one replaces the dressed unrenormalized propagators and the bare coupling constants by their renormalized counterpart; the remaining divergences in subdiagrams are then taken out by the corresponding subtracted subdiagram, evaluated in the symmetric theory. We shall thus assume that such a renormalization procedure has been used¹⁷⁾. Equations (4.1) and (4.2) will thus refer to renormalized quantities unless otherwise stated.

A straightforward generalization of the argument of Ref. 9) leads to a relation between the dimensionless renormalized coupling constants of the theory if K_{ij} is a bounded kernel, that is, if

$$\int \sum_{ij} K_{ij}^2(p^2, p'^2; \{\phi_k\}) G(p^2) G(p'^2) d^4 p d^4 p' < \infty \quad (4.2)$$

in Euclidean space. This is so because the boundedness of the kernel permits the mapping of (4.1) onto a finite system of algebraic equations analogous to those determining masses in the preceding section. As in Section 3, it follows from scale invariance that only dimensionless ratios can be determined and that there remains one consistency equation for coupling constants. Thus if (4.2) were verified in general for a pure Yang-Mills field theory, the "dimensional transmutation"⁵⁾ found in Section 3 would persist in the full theory. We now investigate this problem.

4.2 Pure Yang-Mills theory: "ladder approximation"

To appreciate the relevance of the results of Section 3 for the full problem, we shall first rephrase our "ladder approximation" in the form

of Eq. (4.1) and show how the boundedness condition (4.2) arises in that case; remember that no renormalization is required in the present approximation.

From the equation in Fig. 9 and the factorization of the Goldstone pole exhibited in Fig. 8, one may formulate Eq. (3.9) as shown in Fig. 11 where we have used the Ward identity (3.5). We multiply both sides of the equation by $g^{\mu\nu} - p^\mu p^\nu / p^2$ and define

$$\begin{aligned} \phi_i(p^2)_{(i=s,t)} &= \phi_{st}^*(p^2) = [\tau^a, q^2 \pi(q^2)]_{st} \\ G_{jk}(p'^2)_{(j=l,m)} &= \left[\frac{1}{p'^2 - M^2} \right]_{ll'} \left[\frac{1}{p'^2 - M^2} \right]_{mm'} \\ K_{ij}^L(p^2, p'^2) &= K_{stlm}(p^2, p'^2) = \left[g_{\sigma\tau} - \frac{p_\sigma p_\tau}{p^2} \right] \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p'^2} \right] K_{stlm}^{\mu\nu\sigma\tau}(p, p') \end{aligned} \quad (4.3)$$

where $K_{stlm}^{\mu\nu\sigma\tau}$ is the irreducible Bethe-Salpeter kernel in our "ladder approximation". With these definitions the self-consistent equation for masses takes the form

$$\phi_i(0) = \int \sum_j K_{ij}^L(0, p'^2; \{q_k(0)\}) G_{jk}(p'^2; \{q_k(0)\}) q_j(0) d^4 p' \quad (4.4)$$

We recall that the reason why this integral converges is that the leading logarithmic divergence which would be expected from power counting vanishes as a consequence of gauge invariance; Eq. (4.4) is indeed equivalent through Ward identities to the convergent integral (3.8). By power counting, however, it now follows that the integral

$$I \equiv \int K_{ij}^L(p^2, p'^2; \{q_k(0)\}) G_{jk}(p'^2; \{q_k(0)\}) d^4 p' \quad (4.5)$$

is convergent, even if p^2 is not zero and that in general K_{ij}^L is a convergent kernel in the sense of (4.2). Thus one could generalize the algebraic equation (4.4) to an integral equation for $\phi_i(p)$ using K_{ij}^L and still end up with an eigenvalue condition. Such a generalization would not be of interest in itself, as there is no reason to believe that it is consistent with gauge invariance. It is, however, interesting to note that the fact that our "ladder approximation" leads to an eigenvalue condition

is traceable to the fact that the "ladder" kernel is bounded; it is not simply due to the fact that the approximation of $q^2 \Pi(q^2)$ by a constant M^2 in (3.4) has reduced at the outset the self-consistent problem to a system of algebraic equations.

4.3 Pure Yang-Mills theory: the general case

In general, the quantity $q^\mu \Gamma_{\mu\nu\sigma}^{abc}$ as $q \rightarrow 0$ does not satisfy a single integral equation, but a system of coupled integral equations. This is represented in Fig. 12. Formally, one may solve this system by an iterative procedure where the "unperturbed" value of $q^\mu \Gamma_{\mu\nu\sigma}^{abc}$ is the one obtained by taking into account in Fig. 12 only the diagrams 12a. This is then corrected by substituting for the other vertices the solutions obtained by solving their respective equations iteratively, considering the terms involving $q^\mu \Gamma_{\mu\nu\sigma}^{abc}$ as an inhomogeneity. The result is depicted in Fig. 13 where the sum of irreducible Bethe-Salpeter kernels of the diagrams 12a has been replaced by a more general kernel $\tilde{K}_{bcfg}^{\nu\sigma\lambda\rho}(p, p')$. This method is valid only if $\tilde{K}$ does not itself exhibit a N-G singularity at $q=0$; the latter should therefore already appear in the "unperturbed" problem defined by the irreducible kernels of the diagrams 12a, or by some approximation to it. Such a "bare N-G boson" appearing in the vertex $\Gamma_{\mu\nu\sigma}^{abc}$ would then drive the mass instability and the remainder of the kernel $\tilde{K}$ would provide its "dressing". This situation would be in close analogy to what happens when the symmetry is broken through non-vanishing expectation values of scalar fields. In this case the bare N-G boson appears already at the stage of tree diagrams, which are not present here, and is coupled to lowest order in g only to $\Gamma_{\mu\nu\sigma}^{abc}$. Higher-order radiative corrections dress this bare propagator $-i/(2\pi)^4 1/q^2$ to all orders in g .

We cannot prove that the mechanism described above is the correct one; however, it appears plausible especially in view of the fact that in our "ladder approximation" (Fig. 12a) we have indeed obtained a "bare" N-G boson. We shall therefore assume that the procedure leading to the equation shown in Fig. 13 is indeed convergent. With this assumption, we shall now show why one expects in general an eigenvalue condition for the low-energy renormalized coupling constant.

As we have now departed from our "ladder approximation", u-v divergences will occur, at least in individual graphs. As discussed in subsection 4.1, these may be handled by including counter terms of the unbroken symmetric theory and the equation shown in Fig. 13 is still valid for

renormalized quantities. Using the Ward identity (2.23) and factoring out the N-G (or ghost) singularity as well as the finite renormalization constant, we may pass strictly to the limit $q=0$. After contracting both sides of the equation with $g^{\mu\nu} - p^\mu p^\nu / p^2$, we obtain Eq. (4.1). The kernel K_{ij} is now

$$K_{ij}(p, p'; \{\phi_k\}) = \underset{\substack{(i \neq st) \\ (j \neq tm)}}{=} \left[g_{st} - \frac{p_s p_t}{p'^2} \right] \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \tilde{K}_{sttm}^{\mu\nu\sigma\tau}(p, p') \quad (4.6)$$

and we wish to show that (4.2) is satisfied for this kernel. In what follows, we assume the validity of the power counting argument.

However, as in subsection 4.2 above, this result cannot follow from power counting alone; indeed this would lead to logarithmic divergences in (4.2). We shall thus again compare (4.1) with the Schwinger-Dyson equations for the self-energy; these contain, among others, the term depicted in Fig. 4; when this term is renormalized, one obtains as usually for $\Pi'_{\mu\nu} (= Z_3 \Pi_{\mu\nu})$, a result proportional to Z_1 . This means that the integrand in $\Pi'_{\mu\nu}$ would lead to divergences when expressed in terms of renormalized quantities, even when the external momentum goes to zero. To cancel these divergences against those appearing in Z_1 , one should retain the cut-off in the renormalized propagators and vertices, thereby losing all benefit of the renormalization program. An alternative way is to use the following identity which follows from the transversality of $\Pi_{\mu\nu}$

$$\Pi_{\mu\nu}(p) = -p_\sigma \frac{\partial}{\partial p^\sigma} \Pi^\sigma{}_\nu(p) \quad (4.7)$$

When $\partial/\partial p^\mu \Pi^\sigma{}_\nu(p)$ is then expanded in Feynman graphs¹⁸⁾, the contribution of Fig. 4 becomes converted into that of Fig. 14. Here we have, for simplicity, kept only those contributions to $\Gamma_{\mu\nu\sigma}^{abc}$ which arise from the truncated vertex equation of Fig. 15. To obtain the renormalized counterpart of Fig. 14, it now suffices to replace, therein, all the unrenormalized quantities by renormalized ones; no explicit renormalization constant will appear in the resulting expression for $\Pi'_{\mu\nu}(p)$. We now perform the contraction with p_σ in (4.7) and use the renormalized version of the Ward identity (2.23); this is correct for evaluating the $g_{\mu\nu}$ part of $\Pi_{\mu\nu}(0)$ for which only the regular part of the other vertex at $p=0$ contributes¹⁹⁾.

Thus as $p \rightarrow 0$, the value of $p^2 \Pi^{aa'}(p^2)$ obtained from Fig. 14 and the Ward identity has the form $\lim_{p \rightarrow 0} \int p'^2 \Pi'(p'^2) F(p'^2, p) d^4 p'$ in matrix notation. Here $p'^2 \Pi'(p'^2)$ arises from the Ward identity applied to the left-hand vertices and the kernel contains that part of the right-hand vertex which is regular as $p \rightarrow 0$, except for logarithmic singularities¹⁹⁾. Thus, as $p'^2 \rightarrow \infty$, the kernel F should go to the symmetric limit. This means that by power counting this kernel appears to drop asymptotically as $1/p^4$ up to log terms. As in Section 3, however, we see that the group symmetry enhances this decrease by at least two powers of p .

A similar analysis may be carried out for all the other vertices if we retain the above assumption that the symmetry breaking is driven by the induced singularity in $\Gamma_{\mu\nu\sigma}^{abc}$. This simply amounts to replacing F by a more complicated kernel having, by power counting and group symmetry, the same asymptotic behaviour. Now we may relate this general $F(p'^2, 0)$ to $K_{ij}(0, p'^2) G_j(p'^2)$ in (4.4). This leads again, as in subsection 4.2, to the boundedness condition (4.2).

We then see that the mechanism that led to an eigenvalue condition in our "ladder approximation" may survive in the full theory. Spontaneous mass generation in pure Yang-Mills theory would then occur only for specific values of the (renormalized) coupling constant.

4.4 Coupling to fermions

For the sake of completeness, we briefly recall our previous conclusions⁹⁾ concerning "minimal" Lagrangians. These are Lagrangians containing only fermions coupled minimally to the Yang-Mills fields associated to some subgroup of the fermion kinetic energy. It was shown in Ref. 9) that when each separate Yang-Mills field couples in the Lagrangian to massless fermions of definite chirality, a relation arises between the dimensionless coupling constants of the theory. Such a relation, or eigenvalue condition, is unrelated to the Gell-Mann-Low equation²⁰⁾, but arises for much the same reason as for pure Yang-Mills fields. Namely, the self-consistent equation for $\Sigma_2(p^2)$ has a bounded kernel. This stems from the fact that for such Lagrangians, there is asymptotically no communication between left- and right-handed fermions in the graphs, therefore nothing can drive a mass instability in the asymptotic region. For the low-energy region, however, left-right communication in the kernel is possible due to non-linear effects of the mass generation itself; the fast drop of the kernel in the asymptotic region was responsible in that case, just as in the present work, for the low-energy eigenvalue condition.

The argument of Ref. 9) was quite general, but in the case where the gauge groups were non-Abelian, we had to limit ourselves to cases where the meson symmetry breaking terms were expressible through Feynman graphs in terms of $\Sigma_2(p^2)$ alone. Thus we assumed that the instability was driven by the Fermion-Yang-Mills vertex Γ_μ in the same sense as here it is driven by $\Gamma_{\mu\nu\sigma}^{abc}$. We see now that this result is more general because if we use (4.1) where $\phi_1(q^2)$ refers both to fermion and meson symmetry breaking terms, the boundedness condition (4.2) still follows. This, of course, remains contingent upon the validity of the assumption of subsection 4.3 for the pure mesonic couplings.

5. CONSISTENCY OF THE "LADDER APPROXIMATION"

5.1 Preliminaries

We shall verify that the approximation carried out in Section 3 can be diagrammatically interpreted as a type of ladder approximation, and that it is consistent with the gauge invariance of the theory in the sense that all the relevant Ward identities are satisfied. It will be convenient from now on, to associate a specific group to our Yang-Mills theory. We choose SU_3 and a spontaneous breakdown $M^2 \sim Y^2$. The strange components 4, 5, 6, 7 will thus acquire a common mass m while 8 and 1, 2, 3 remain massless.

The $[q_\mu q_\nu / q^2]_{q \rightarrow 0}$ contribution to $\Pi_{\mu\nu}^L(0)$, which we shall call $\Pi_{\mu\nu}^L$ has been evaluated, in general, in (3.8); in the present case the resulting eigenvalue equation reduces to (3.11) with $C=3$. We may re-write (3.8), which corresponds to Fig. 6c, in the form

$$\begin{aligned} \Pi_{\mu\nu}^{aa'} &= [q_\mu q_\nu / q^2]_{q \rightarrow 0} \frac{n-1}{n} g^2 \sum_{b,c} f_{abc} f_{a'bc} m^* \frac{i}{(2\pi)^4} I \\ &\quad (a=a' = 4, 5, 6, 7) \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{5.1}$$

with

$$I = \int d^4 p \frac{1}{p^2 (p^2 - m^2)^2} \tag{5.2}$$

We do not set $n=4$ as yet. The reason for this will become clear below.

5.2 Transversality of $\Pi_{\mu\nu}^{aa'}$

In evaluating the $g_{\mu\nu}$ contribution to $\Pi_{\mu\nu}^{a'a}(0)$ we shall be confronted with divergent integrals whose divergences will eventually cancel out. To handle these divergences correctly, we shall use the dimensional regularization procedure ¹⁴⁾. A useful tool in this procedure is the property that all linearly, quadratically, and so on, divergent integrals having no dependence on masses and external momenta may be set equal to zero. This is why the diagram of Fig. 5c has not been retained in Fig. 6.

The diagram of Fig. 6a gives, if a is 4, 5, 6, 7

$$(6a) \equiv \sum_{b,c} f_{abc} f_{a'bc} 2(n-1) g^2 \frac{i}{(2\pi)^4} \int d^4 p \frac{p_\mu p_\nu}{p^2} \frac{1}{p^2 - m^2} \quad (5.3)$$

By the property stated above, we may symbolically pose

$$\int \frac{d^4 p}{p^2} = 0 \quad (n > 2) \quad (5.4)$$

Hence (5.3) may be written as

$$(6a) \equiv \sum_{b,c} f_{abc} f_{a'bc} 2(n-1) g^2 \frac{im^2}{(2\pi)^4} \int d^4 p \frac{p_\mu p_\nu}{p^4} \frac{1}{p^2 - m^2} \quad (5.5)$$

which can easily be shown to be equivalent to

$$(6a) \equiv - \sum_{b,c} f_{abc} f_{a'bc} (n-1) g^2 \frac{i}{(2\pi)^4} \frac{4m^4}{4-n} \int d^4 p \frac{p_\mu p_\nu}{p^4} \frac{1}{(p^2 - m^2)^2} \quad (5.6)$$

To motivate the introduction of the propagator (3.6) in Fig. 6b, we first evaluate this diagram with the original tensor propagator (2.6). Calling this contribution $(6b)^0$, we have for $a=4, 5, 6, 7$

$$(6b)^0 = \sum_{b,c} f_{abc} f_{a'bc} g^2 \frac{i}{2 \cdot (2\pi)^4} \cdot \left[\int (n-1) g_{\mu\nu} \frac{d^4 p}{p^2 - m^2} + \int (p_\mu p_\nu - g_{\mu\nu}) \frac{d^4 p}{p^2 - m^2} \right] \quad (5.7)$$

Note that all diagrams where the internal vector propagator is non-strange vanish in view of (5.4); this explains the factor of $\frac{1}{2}$ in (5.7). The first term in (5.7) may, in the same manner as above, be transformed to

$$(6b)^{0,1} \equiv \sum_{b,c} f_{abc} f_{a'bc} \frac{g^2 m^4}{2} \frac{2}{4-n} \frac{i}{(2\pi)^4} \int \frac{g_{\mu\nu} d^4 p}{p^2 (p^2 - m^2)^2} \quad (5.8)$$

Combining (5.6) and (5.8) we get

$$(6a) + (6b)^{0,1} \equiv \sum_{b,c} (-f_{abc} f_{a'bc}) \frac{m^4 (n-1)}{n} g^2 \cdot \frac{i}{(2\pi)^4} g_{\mu\nu} \int \frac{d^4 p}{p^2 (p^2 - m^2)^2} \quad (5.9)$$

Comparing with (5.1), we see that (5.9) is precisely the term we need to obtain a transverse expression for $\lim_{q \rightarrow 0} \Pi_{\mu\nu}^{aa'}(q)$ if $a = a' = 4, 5, 6, 7$. However, the last term in (5.7) destroys the transversality and the convergence of the resulting $\Pi_{\mu\nu}$. It can, however, be eliminated if we replace (5.6) by (3.6), as seen by direct evaluation. We shall therefore tentatively use (3.6) and investigate its implication on the Ward identities before trying to interpret it. Thus we see that for $a = 4, 5, 6, 7$, $(6b) \equiv (6b)^{0,1}$, therefore the self-energy of a strange vector meson is indeed transverse at $q = 0$, if we use the diagrams of Fig. 6.

It remains to be verified that for $a = 1, 2, 3$ or 8 where the diagram of Fig. 6c does not contribute, we indeed find $\Pi_{\mu\nu}(0) = 0$. We have

$$(6a) \equiv \sum_{b,c \text{ strange}} f_{abc} f_{a'bc} g^2 2(n-1) \frac{i}{(2\pi)^4} \int d^4 p \frac{p^\mu p^\nu}{(p^2 - m^2)^2} \quad (5.10)$$

$$(6b) \equiv \sum_{b,c \text{ strange}} f_{abc} f_{a'bc} g^2 (n-1) \frac{(-i)}{(2\pi)^4} \int \frac{d^4 p}{p^2 - m^2} \quad (5.11)$$

The reason why non-strange b and c do not contribute to (5.10) and (5.11) is again (5.4). By direct evaluation of the integrals we indeed find that $(6a) + (6b) = 0$. Note that this result again depends critically on the replacement of (2.6) by (3.6).

5.3 Identification with a "ladder approximation"

We first evaluate the coupling introduced in Fig. 7 by trying to solve the equation represented in Fig. 8. This coupling must be of the form

$$A_{xbc}^{\sigma\tau} = \kappa D_{xbc} g^{\sigma\tau} \quad (5.12)$$

where κ has the dimension of a mass and the constants D_{xbc} are dimensionless. Inserting (5.12) in the diagrams of Fig. 8, one has for strange a

$$g F_a^{\sigma\tau} i(q^\mu/q^2) m^2 g_{\sigma\tau} = g \sum_{e,f,x} F_a^{ef} D_x^{ef} D_{xbc} \cdot \kappa^2 m^2 i(q^\mu/q^2) g_{\sigma\tau} \cdot \frac{3}{4} \frac{i}{(2\pi)^4} I \quad (5.13)$$

where I is the integral (5.2), and we have introduced

$$F_a^{\sigma\tau} = f_a^{\sigma\tau} (\lambda_b - \lambda_c) \\ \lambda_b = 1 \quad b \text{ strange} \\ \lambda_b = 0 \quad b \text{ non strange} \quad (5.14)$$

Thus the constant D_{xbc} may be chosen to satisfy

$$F_a^{\sigma\tau} = \sum_{e,f,x} D_{xbc} D_x^{ef} F_a^{ef} \quad (5.15)$$

One easily shows that this is solved by

$$D_x^{\sigma\tau} = (D_x^{\sigma\tau})_{bc} = \frac{2}{3} ([F_b, F_x]_+)^{\sigma\tau} \quad x \text{ strange} \\ = 0 \quad x \text{ non strange} \quad (5.16)$$

Using (5.15), (5.13) reduces to

$$1 = \frac{3}{4} \kappa^2 \frac{i}{(2\pi)^4} I \quad (5.17)$$

On the other hand, the eigenvalue condition (3.11) can be written in this case ($C=3$)

$$1 = 3 g^2 m^2 \left[\frac{3}{4} \frac{i}{(2\pi)^4} I \right] \quad (5.18)$$

so that

$$\kappa = \sqrt{3} g m \quad (5.19)$$

Thus a solution of the equation shown in Fig. 8 can indeed be found: it is (5.12) with D_{bc} and κ given, respectively, by (5.16) and (5.19). This means that the factorization exhibited in Fig. 7 is justified. To prove that our approximation is indeed the "ladder approximation" defined in Section 3, it remains to be shown that the equation of Fig. 9 is satisfied with $A_{xbc}^{\sigma\tau}$ given by (5.12).

The sum of the diagrams of Fig. 9 gives

$$(9) \equiv (n-1) g^3 m \sum_{e,f,g} \left[\sqrt{3} D_{ef} f_{ege} f_{cgt} \right] \frac{i}{(2\pi)^4} \int \frac{-4 p_e p_f + g_{ef} (p^2 - m_g^2) d^n p}{(p^2 - m_e^2)(p^2 - m_f^2)(p^2 - m_g^2)} \quad (5.20)$$

which may be written as

$$(9) \equiv (n-1) g^3 m G_1 \frac{i}{(2\pi)^4} \int (g_{ef} - \frac{4 p_e p_f}{p^2}) \frac{d^n p}{p^2 (p^2 - m^2)} + \\ + (n-1) g^3 m G_2 \frac{i}{(2\pi)^4} \int (g_{ef} - \frac{4 p_e p_f}{p^2 - m^2}) \frac{d^n p}{p^2 (p^2 - m^2)} \quad (5.21)$$

where

$$G_1 = \sum_{g \text{ non strange}} \sum_{e,f} \sqrt{3} D_{ef} f_{ege} f_{cgt} \\ G_2 = \sum_{g \text{ strange}} \sum_{e,f} \sqrt{3} D_{ef} f_{ege} f_{cgt} \quad (5.22)$$

The integrals in (5.21) are convergent when $n \rightarrow 4$ and this can be made explicit by using (5.4). One thus obtains

$$(9) \equiv (G_1 - G_2) m^3 g^3 \frac{2(n-1)}{n} g_{ef} \frac{i}{(2\pi)^4} I \quad (5.23)$$

Furthermore, it turns out that

$$G_1 - G_2 = \frac{3\sqrt{3}}{2} \frac{D}{z} \epsilon_c \quad (5.24)$$

so that using the eigenvalue condition (5.18), one indeed sees that (5.23) is equal to (5.12). Note the importance of the propagator (3.6) in ensuring the validity of the dynamical interpretation (Fig. 9) of the approximation of Section 3.

5.4 Transversality of the three-point functions

The graphs contributing to the three-point function in our approximation are represented in Fig. 16. This introduces two new couplings C_1 and C_2 , which in our "ladder approximation" obey the equations represented in Figs. 17 and 18. Let us first evaluate C_1 and C_2 .

The calculation of C_1 is a by-product of the equation (5.13) of Fig. 8. Using the eigenvalue equation (5.18) one has

$$(C_1)_{a\mu, x} = i m q_\mu \epsilon_{ax} \quad (5.25)$$

where

$$\epsilon_{ax} = \frac{1}{\sqrt{3}} \sum_{b,c} \frac{F}{a} \epsilon_c \frac{D}{z} \epsilon_c \quad (5.26)$$

Note that ϵ_{ax} takes only the values ± 1 , or 0.

To evaluate C_2 , we expand the integrand in Fig. 18 to first order in r and q to obtain the kinematic factor $(r-q)_\mu$. One has

$$(18) \equiv -3ig^2 m^2 \sum_{x,y,z} \left[\frac{3D_{yx} D_{xz}}{6} f_{axy} \right] \frac{i}{(2\pi)^4} \left\{ (r-q)_\mu \int \frac{d^4 p}{(p^2 - m_y^2)(p^2 - m_x^2)(p^2 - m_z^2)} \right. \\ \left. - \int 2p_\mu \left[\frac{2p \cdot r}{p^2 - m_z^2} - \frac{2p \cdot q}{p^2 - m_y^2} \right] \frac{d^4 p}{(p^2 - m_y^2)(p^2 - m_x^2)(p^2 - m_z^2)} \right\} \quad (5.27)$$

Explicitly, this may be written as

$$-(n-1)ig^2 m^2 G_3 (r-q)_\sigma \frac{i}{(2\pi)^4} \int \frac{d^4 p}{p^2 (p^2 - m^2)} \left(g_{\sigma\mu} - \frac{4p_\sigma p_\mu}{p^2} \right) \\ -(n-1)ig^2 m^2 G_4 (r-q)_\sigma \frac{i}{(2\pi)^4} \int \frac{d^4 p}{p^2} \left(g_{\sigma\mu} - \frac{4p_\sigma p_\mu}{p^2 - m^2} \right) \frac{1}{(p^2 - m^2)^2} \quad (5.28)$$

where

$$\begin{aligned} G_3 &= \sum_{\substack{xy \\ x \text{ strange}}} 3 D_{\epsilon}^{yx} D_{\epsilon}^{xz} f_{azy} \\ G_4 &= \sum_{\substack{xy \\ x \text{ nonstrange}}} 3 D_{\epsilon}^{yx} D_{\epsilon}^{xz} f_{azy} \end{aligned} \quad (5.29)$$

Both integrals in (5.28) turn out to be equal to $i/(2\pi)^4 \frac{1}{2} I g_{\sigma\mu}$; furthermore, one has

$$G_3 + G_4 = -\frac{3}{2} f_{abc} \quad (5.30)$$

so that one finds

$$(C_2)_{a\mu, bc} = 3ig^3 m^2 (r-q)_\mu \frac{n-1}{n} \frac{i}{(2\pi)^4} I f_{abc} \quad (5.31)$$

Using again the eigenvalue equation (5.18), one finds

$$(C_2)_{a\mu, bc} = ig(r-q)_\mu f_{abc} \quad (5.32)$$

Using (5.12), (5.25) and (5.32), the proof of transversality for the sum of the diagrams of Fig. 16 is straightforward. The bare Yang-Mills vertex is transverse by itself. So is the sum of the remaining three graphs; this is readily checked by multiplying each graph by $A^\alpha B^\beta C^\gamma$ and using the following identity for the group indices

$$\epsilon_{ax} \epsilon_{by} f^{xyz}_c = f_{abc} \quad (5.33)$$

4.5 Transversality of the four-point functions

If all external mesons are non-strange, then the only graphs to be considered are those of Fig. 19 (and the four similar ones in the crossed channels). The transversality relation is verified channel by channel: this is because on the one hand the intermediate propagator (3.4) coincides in this case with the bare propagator, on the other hand the transverse part of the tensor propagator does not contribute to the four-point transversality relation. Therefore, the graphs of Fig. 19 satisfy the transversality relations as do the graphs of ordinary perturbation theory up to order g^2 .

In other cases the investigation of the transversality relation is considerably more tedious because one should exchange not only vector and

tensor propagators, but also N-G bosons. In the case where two external mesons are strange and two non-strange, one obtains in this way 23 one-particle reducible graphs to be contracted with $A^\alpha B^\beta C^\gamma D^\delta$. The hard labouring reader will find that the result is not zero and he might wish to determine the form of a "contact term" between two N-G and two Yang-Mills vectors, in order to cancel the effect of the 23 other graphs. He might guess that this contact term should be given by the sum of the diagrams of Fig. 20 at zero external momentum; this is indeed the case! When all external mesons are strange, then the transversality relations are saturated by the reducible graphs alone; we have not studied in this case the numerous possible contact terms.

5.6 Remarks

While the diagrammatic interpretation of the approximation of Section 3 has been clarified, the consistency achieved in this way rests strongly on the use of the tensor propagator (3.6). As the difference between (3.6) and (2.6) is mass independent, it cannot be understood directly as a symmetry breaking effect. Rather the propagator (3.6) seems to be a more convenient unperturbed propagator to study symmetry breakdown than (2.6); this suggests that one should look for a formalism where the separation of the tensor propagator A in T and L is present ab initio in the Lagrangian. This is the case in the so-called linear formalism of Faddeev and Popov ²¹⁾ which we suspect might provide a better understanding of this problem. The use of (3.6) might also be related to the problem of gauge invariance: the value of $\lim_{q \rightarrow 0} q^2 \Pi(q^2)$ need not be gauge invariant and therefore the interpretation of M^2 as a "mass matrix" in our approximation is not necessarily correct. However, when M^2 arises from coupling to scalars φ_i with $\langle \varphi_i \rangle \neq 0$, one can show that the use of (3.6) instead of (2.6) singles out the gauge invariant contribution to M^2 in lowest order in g^2 .

Finally, we wish to point out that even within this simple approximation, the phenomenon described in this paper admits no classical analogue; this is because our ladder summation which leads to our bare Goldstone singularity involves an infinite number of loops. This means that the mass generated thereby cannot be expanded in a power series in $\hbar$ ⁵⁾ and therefore our dynamical symmetry breaking mechanism may be interpreted as arising from quantum mechanical fluctuations alone. This situation is distinct from the case where masses of Yang-Mills mesons are generated through expectation values of scalar mesons wherein the classical limit is recovered in free approximation. Also, our mechanism is unrelated to a possible mass instability in the classical theory of massless Yang-Mills fields ²²⁾.

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FIGURE CAPTIONS

Fig. 1 : Factorization of the four-point bare vertex .

$$\frac{(a, \mu)}{\text{---}} \frac{(b, \nu)}{\text{---}} = \text{vector bare propagator } D_{\mu, \nu}^{a, b}$$

$$\frac{(a, \mu \nu)}{\text{---}} \frac{(b, \sigma \tau)}{\text{---}} = \text{tensor bare propagator } A_{\mu \nu, \sigma \tau}^{a, b}$$

$$\begin{array}{c} (a, \mu) \\ \diagup \\ (b, \nu) \end{array} \begin{array}{c} (c, \sigma \tau) \\ \diagdown \end{array} = \text{bare vertex (3.5)}$$

$$\begin{array}{c} (a, \alpha) \quad (b, \beta) \\ \diagdown \quad \diagup \\ (d, \delta) \quad (c, \gamma) \end{array} = \text{four-point bare vertex} = g^2 \{ f_{hac} f_{hed} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma}) + f_{had} f_{hec} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta}) + f_{hab} f_{hed} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \}$$

Fig. 2 : A contribution to $C_{\nu, \rho \lambda}^{b, c}(q)$

$$\begin{array}{c} (b, \nu) \\ \diagdown \\ \square \\ \diagup \\ (c, \rho \lambda) \end{array} \xrightarrow{q} = \text{vector-tensor transition matrix } C_{\nu, \rho \lambda}^{b, c}$$

Fig. 3 : A diagram contributing to $1/iq^2 \phi_{\nu\sigma}^{a;bc}(p, q)$; the explicit q_λ factor is exhibited on the diagram.

$$\bar{a} \rightarrow \bar{x} = \text{ghost bare propagator. } G_{ax}(q^2) = \frac{-i}{(2\pi)^4} \frac{1}{q^2} \delta_{ax}$$

Fig. 4 : A contribution to $\Pi_{\mu\nu}^{aa'}(1)$

$$\begin{array}{c} (a, \mu) \\ \diagup \\ \bigcirc \\ \diagdown \\ (b, \sigma) \quad (c, \tau) \end{array} = \Gamma_{\mu\sigma\tau}^{abc}$$

$$\text{---} // \text{---} = \text{dressed Yang-Mills propagator}$$

Fig. 5 : $\Pi_{\mu\nu}^{aa'}(q)$ to order g^2 .

Fig. 6 : Contribution to $\Pi_{\mu\nu}^{aa'}(a)$ as $q \rightarrow 0$ in our "ladder approximation"

$$\begin{array}{c} (a, \mu) \quad (b, \sigma) \\ \diagdown \quad \diagup \\ \bigcirc \\ \diagup \quad \diagdown \\ (c, \tau) \end{array} \xrightarrow{q} = i g g_{\sigma\tau} (q_\mu/q^2) f_{abc} (m_b^2 - m_c^2)$$

$$\text{---} = \text{vector propagator (3.4).}$$

$$\text{---} // \text{---} = \text{tensor propagator (3.6).}$$

Fig. 7 : Diagrammatic expression of (6c) in terms of a N-G boson propagator

$$----- = \text{N-G boson propagator } -i/(2\pi)^4 1/q^2$$

$$\begin{array}{c} (d, \sigma) \\ \diagdown \\ \text{---} \otimes \text{---} x \\ \diagup \\ (c, \tau) \end{array} = \text{Coupling of the N-G boson to Yang-Mills vector mesons.}$$

Fig. 8 : Expansion of the singular $\otimes$ vertex in terms of the N-G boson propagator.

Fig. 9 : "Ladder approximation" for the coupling of the N-G boson to Yang-Mills vector mesons. All external lines have zero momentum.

Fig. 10 : Symbolic representation of Eq. (4.1)

$$\begin{array}{c} p \nearrow \text{---} i \searrow p \\ \text{---} p' \nearrow \text{---} j \searrow p' \\ \text{---} p \nearrow \text{---} i \searrow p \end{array} = \phi_i(p^2)$$

$$\begin{array}{c} p \nearrow \text{---} j \searrow p' \\ \text{---} p \nearrow \text{---} i \searrow p \end{array} = K_{ij}(p^2, p'^2; \{\phi_k\})$$

$$\text{---} = \text{dressed propagator}$$

Fig. 11 : Reformulation of the eigenvalue equation in the "ladder approximation"

$$\begin{array}{c} a \\ \text{---} \text{---} \text{---} \\ p \nearrow (s, \mu) \searrow (t, \nu) \end{array} = [T^a, \pi_{\mu\nu}]_{s,t}$$

$$\begin{array}{c} (l, \sigma) \nearrow p' \\ (s, \mu) \nearrow \text{---} \text{---} \text{---} \searrow p' (m, \tau) \\ (t, \nu) \searrow p \end{array} = K_{stlm}^{\mu\nu\tau\sigma}(p, p')$$

Fig. 12 : Coupled integral equations for $q^\mu \Gamma_{\mu\nu\sigma}^{abc}$. The dressed (renormalized) propagators are doubly slashed; the irreducible Bethe-Salpeter kernels are represented by rectangular boxes and the divergences of vertex functions by circles. The letters A, B, C, ... refer to the various vertex functions, in particular

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