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TRANSFORMATION PROPERTIES OF THE SUPERCURRENT

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A B S T R A C T

It is shown that the spinor current, if correctly defined, belongs to a supermultiplet, together with the energy-momentum tensor and the axial-vector current. The transformation properties of this supermultiplet under both restricted and general supersymmetry transformations are given. The generators of special supersymmetry transformations can be obtained as first moments of the time component of the spinor current. The transformation laws of the supermultiplet containing the spinor current provide the local version of the supersymmetry algebra.

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1. INTRODUCTION

Supersymmetry transformations [1] form an extended Lie algebra involving commutators and anticommutators. The algebraic structure was first given in ref. [1] (where the supersymmetry transformations were called supergauge transformations). More explicit formulae were later given in ref. [2] and, for the special case of restricted (constant parameter) supersymmetry transformations, in ref. [3].

As shown in a special field theory example [4, 5] the generators of supersymmetry transformations are the three-dimensional integrals of the time components of spinor currents. Spinor currents for other field theory examples can be found in ref. [6].

The spinor currents given in refs. [4, 5] were obtained from the Lagrangian by using Noether's construction, adapted to the case of anticommuting group parameters. As explained in sect. 2, one finds in this way two different spinor currents, one giving by integration the generators of restricted supertransformations and the other giving the remaining (special) generators of the larger algebra. These currents do not have simple transformation properties. Our first step is to show that it is possible to define an "improved" spinor current from which all spinor generators can be constructed by taking moments of order zero and one. This is perfectly analogous to the introduction of the "improved" energy-momentum tensor [7], from which one can construct, by taking suitable moments, not only the generators of the Poincaré group, but also those of the scale and special conformal transformations. Just as in that case, no improvement is needed in the spinor current if only spinor and vector fields occur in the theory; scalar fields, however, require the new definition of the current.

The improved spinor current has also the advantage of having simple transformation properties under both restricted and general supersymmetry transformations. For the case of vanishing masses, in which the larger algebra is, at least formally, valid, the spinor current belongs to a supermultiplet of which the other members are the (traceless) improved energy-momentum tensor and the (conserved) axial-vector current. The transformation laws of this supermultiplet under general supersymmetry transformations are given in sect. 4. If the masses do not vanish and only the restricted algebra is valid the multiplet is larger. Its transformation properties under restricted supertransformations are given in sect. 3. In this case of finite masses it is also useful to modify the currents and the energy-momentum tensor by adding terms linear in the fields, so as to have, for instance, a simpler expression for the trace of the energy-momentum tensor. The general formulae of sects. 3 and 4 are abstracted from a particular model. In appendix A we verify their validity in two other examples.

Some of the formulae given in this paper are presumably invalid in perturbation theory, as a consequence of the well-known scale and conformal anomalies. This is true, in particular, of formulae involving the trace of the energy-momentum tensor and of equations like (13) below. Consequently, the transformation laws under the larger algebra given in sect. 4 are only formal. Nevertheless, just as in the purely conformal case, and for the same reasons, the occurrence of anomalies does not render the formal equations any less interesting.

The conservation laws for the spinor current and the energy-momentum tensor should, of course, be anomaly-free. The (partial) conservation equation for the axial-vector current is also anomaly-free as one can see most simply for the model of sect. 2 by observing that it is still valid in the regularized version of the theory [5].

The formulae given in sect. 4 are interesting for another reason. As shown in appendix B, the multiplet given in sect. 4 can be used to describe massive particles of spin 2, $\frac{3}{2}$, $\frac{3}{2}$ and 1 (the last being an axial vector). For the case of zero mass, the corresponding multiplet contains only a particle of spin 2 and one of spin $\frac{3}{2}$. This is probably the multiplet to which one should assign the graviton in a supersymmetric generalization of Einstein's theory of gravitation [6].

2. THE IMPROVED AND THE MODIFIED SPINOR CURRENT

We begin with the field theory example of refs. [4] and [5]. The Lagrangian is

$$\begin{aligned} L = & -\frac{1}{2}(\partial A)^2 - \frac{1}{2}(\partial B)^2 - \frac{i}{2}\bar{\Psi}\gamma\partial\Psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ & + m(FA + GB - \frac{i}{2}\bar{\Psi}\Psi) \\ & + g(FA^2 - FB^2 + 2GAB - i\bar{\Psi}\Psi A + i\bar{\Psi}\gamma_5\Psi B) \end{aligned} \quad (1)$$

and the corresponding equations of motion are

$$\begin{aligned} \square A + mF + 2gFA + 2gGB - i\bar{\Psi}\gamma\Psi &= 0 \\ \square B + mG - 2gFB + 2gGA + i\bar{\Psi}\gamma_5\Psi &= 0 \\ \gamma\partial\Psi + m\Psi + 2g(A - \gamma_5 B)\Psi &= 0 \\ F + mA + g(A^2 - B^2) &= 0 \\ G + mB + 2gAB &= 0 \end{aligned} \quad (2)$$

The Lagrangian (1) is invariant under restricted supersymmetry transformations and, for $m = 0$, also under the larger algebra. Applying Noether's method to (1), one finds the spinor current associated with the restricted supertransformations

$$J^\mu = [\gamma^\lambda \partial_\lambda (A - \gamma_5 B)] \gamma^\mu \psi + m \gamma^\mu (A - \gamma_5 B) \psi + g \gamma^\mu (A - \gamma_5 B)^2 \psi \quad (3)$$

and the second spinor current associated with the remaining generators of the special supertransformations

$$I^\mu = -\gamma^\lambda x_\lambda J^\mu - 2(A - \gamma_5 B) \gamma^\mu \psi \quad (4)$$

They satisfy, respectively

$$\partial_\mu J^\mu = 0 \quad (5)$$

and

$$\partial_\mu I^\mu = -2m(A - \gamma_5 B) \psi \quad (6)$$

Using the equations of motion (2), J^μ can also be written as

$$J^\mu = [\gamma^\lambda \partial_\lambda (A - \gamma_5 B)] \gamma^\mu \psi - (F + \gamma_5 G) \gamma^\mu \psi \quad (7)$$

The above formulae were already given in ref. [5].

The current I^μ is not expressible in terms of J^μ alone. This can be remedied by defining an improved spinor current

$$J_{\text{impr.}}^\mu = J^\mu + \frac{1}{3} (\gamma^\mu \gamma^\lambda \partial_\lambda - \gamma^\lambda \partial_\lambda \gamma^\mu) [(A + \gamma_5 B) \psi] \quad (8)$$

Clearly $J_{\text{impr.}}^\mu$ is also conserved and the three-dimensional integral of its time component has the same value as for J^0 . Furthermore, using the equations of motion (2) one can easily verify that

$$\gamma_\mu J_{\text{impr.}}^\mu = 2m(A - \gamma_5 B) \psi \quad (9)$$

We can now define as improved second spinor current simply

$$I_{\text{impr.}}^\mu = -\gamma^\lambda x_\lambda J_{\text{impr.}}^\mu \quad (10)$$

Clearly it satisfies the same partial conservation equation (6) as I^μ . Furthermore, the three-dimensional integral of $I_{\text{impr.}}^0$ is the same as that of I^0 . We see, therefore, that it is sufficient to introduce one local operator current, namely $J_{\text{impr.}}^\mu$, and that both the restricted and the special supersymmetry generators can be obtained from it as moments of order zero and one.

In the presence of interaction, it is preferable to introduce the "modified" current, defined by

$$J_{\text{mod}}^{\mu} = J_{\text{impr}}^{\mu} + \frac{m}{6g} (\gamma^{\mu} \gamma^{\lambda} \partial_{\lambda} - \gamma^{\lambda} \partial_{\lambda} \gamma^{\mu}) \psi \quad (11)$$

Clearly, it is conserved

$$\partial_{\mu} J_{\text{mod}}^{\mu} = 0 \quad (12)$$

but the extra term in (11) changes Eq. (9) into

$$\gamma_{\mu} J_{\text{mod}}^{\mu} = -\frac{m^2}{g} \psi \quad (13)$$

as seen using once more the equations of motion (2). Therefore, the modified second current

$$I_{\text{mod}}^{\mu} = -\gamma^{\lambda} x_{\lambda} J_{\text{mod}}^{\mu} \quad (14)$$

satisfies

$$\partial_{\mu} I_{\text{mod}}^{\mu} = \frac{m^2}{g} \psi \quad (15)$$

The three-dimensional integral of J_{mod}^0 is again the same as that of J^0 , the generator of restricted transformations

$$Q = \int J_{\text{mod}}^0 d^3x \quad (16)$$

The integral of I_{mod}^0 differs from that of I^0 , but only if $m \neq 0$, since I_{mod}^{μ} and I_{impr}^{μ} are equal for $m = 0$. Since the larger algebra is uniquely fixed only for $m = 0$, this is not a difficulty.

Using the equations of motion (2), it is not difficult to verify that

$$\begin{aligned} 3 J_{\text{impr}}^{\mu} &= [\gamma^{\lambda} \partial_{\lambda} (A - \gamma_5 B)] \gamma^{\mu} \psi + (F + \gamma_5 G) \gamma^{\mu} \psi \\ &\quad - 2 (A + \gamma_5 B) \overleftrightarrow{\partial}^{\mu} \psi + 2m (A + \gamma_5 B) \gamma^{\mu} \psi \end{aligned} \quad (17)$$

where

$$\overleftrightarrow{\partial}_{\mu} = \overrightarrow{\partial}_{\mu} - \overleftarrow{\partial}_{\mu} \quad (18)$$

and that

$$3 J_{\text{mod}}^{\mu} = [\gamma^{\lambda} \partial_{\lambda} (A - \gamma_5 B)] \gamma^{\mu} \psi + (F + \gamma_5 G) \gamma^{\mu} \psi - 2 (A + \gamma_5 B) \overleftrightarrow{\partial}^{\mu} \psi - \frac{m}{g} \partial^{\mu} \psi - \frac{m^2}{g} \gamma^{\mu} \psi \quad (19)$$

As we shall see in sect. 3, this last form has a very simple meaning in terms of superfields.

Since we shall need them later, we now recall [5] the expressions for the energy-momentum tensor and for the axial-vector current associated with the Lagrangian (1). The canonical energy-momentum tensor is

$$T_{\mu\nu} = \partial_{\mu} A \partial_{\nu} A + \partial_{\mu} B \partial_{\nu} B + \frac{i}{4} (\bar{\psi} \gamma_{\mu} \partial_{\nu} \psi + \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi) + \eta_{\mu\nu} L \quad (20)$$

while the improved energy-momentum tensor is

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6} (\partial_{\mu} \partial_{\nu} - \eta_{\mu\nu} \square) (A^2 + B^2) \quad (21)$$

and the modified energy-momentum tensor is

$$\Theta_{\mu\nu \text{ mod}} = \Theta_{\mu\nu} - \frac{m}{6g} (\partial_{\mu} \partial_{\nu} - \eta_{\mu\nu} \square) A \quad (22)$$

They are all three conserved and symmetric, for instance

$$\partial^{\mu} \Theta_{\mu\nu \text{ mod}} = 0 \quad (23)$$

but the modified tensor satisfies the simplest trace relation

$$\Theta_{\mu}^{\mu \text{ mod}} = - \frac{m^2}{2g} F \quad (24)$$

Finally, the axial current is

$$J_{\mu}^{(5)} = A \partial_{\mu} B - B \partial_{\mu} A - \frac{i}{4} \bar{\psi} \gamma_5 \gamma_{\mu} \psi \quad (25)$$

and the modified axial current

$$J_{\mu \text{ mod}}^{(5)} = J_{\mu}^{(5)} + \frac{m}{2g} \partial_{\mu} B \quad (26)$$

It satisfies

$$\partial^{\mu} J_{\mu \text{ mod}}^{(5)} = - \frac{m^2}{2g} G \quad (27)$$

as a consequence of the equations of motion.

Before closing this section we observe that the modified currents and energy-momentum tensor are obtained from their unmodified counterparts in the forms (7), (8), (21) and (25), simply by the replacement $A \rightarrow A + (m/2g)$. In terms of this new field the Lagrangian (1) takes a different form, in which the bilinear mass term is absent and has been replaced by a term $-(m^2/4g)F$.

This explains the emergence of the simpler equations (13), (24) and (27).

3. RESTRICTED SUPERSYMMETRY TRANSFORMATIONS

In this section we follow the notation of refs. [8] and [9]. Written as superfield equations the equations of motion (2) take the compact form

$$\begin{aligned}\frac{1}{4} \bar{D} \bar{D} \phi^* &= m \phi + 2g \phi^2 \\ \frac{1}{4} D D \phi &= m \phi^* + 2g (\phi^*)^2 \\ \bar{D}_{\dot{\alpha}} \phi &= D_{\alpha} \phi^* = 0\end{aligned}\tag{28}$$

where D_{α} and $\bar{D}_{\dot{\alpha}}$ are the covariant derivatives. They satisfy

$$\{D_{\alpha}, D_{\beta}\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0\tag{29}$$

and

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2i \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu}\tag{30}$$

from which it follows that

$$[\bar{D}_{\dot{\alpha}}, D D] = 4i D^{\alpha} \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu}\tag{31}$$

We now define the real superfield

$$V_{\alpha\dot{\alpha}} = i \phi \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu} \phi^* + \frac{1}{2} D_{\alpha} \phi \bar{D}_{\dot{\alpha}} \phi^* - \frac{m}{4g} i \sigma_{\alpha\dot{\alpha}}^{\mu} \partial_{\mu} (\phi - \phi^*)\tag{32}$$

which is endowed with one dotted and one undotted index (or equivalently one four-vector index). As we shall see below, the component fields of (32) are related

in a simple way to the modified currents and energy-momentum tensor of sect. 2. Therefore, we shall call $V_{\alpha\dot{\alpha}}$ the supercurrent. Observe that the last term of $V_{\alpha\dot{\alpha}}$ can be generated* by adding a constant to the superfield Φ . Using the relations (29), (30) and (31), it is easy to verify that the equations of motion (28) imply

$$D^\alpha V_{\alpha\dot{\alpha}} = -\frac{m^2}{4g} \bar{D}_{\dot{\alpha}} \Phi^* \quad (33)$$

together with its complex conjugate. From (33) we abstract the equations

$$D^\alpha V_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} S, \quad D_\alpha S^* = 0 \quad (34)$$

While (33) depends upon the particular model used to derive it, (34) (with their complex conjugate) turn out to be of general validity. Other examples of supercurrents are given in appendix A, where it is verified that they satisfy (34), although in general the left-handed superfield S is not purely linear in the fundamental fields. Equations (34) can also be written in the equivalent form

$$\bar{D}_{\dot{\beta}} D D V_{\alpha\dot{\alpha}} = 4i \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu D^\beta V_{\beta\dot{\beta}} \quad (35)$$

In real four-component notation the component fields of $V_{\alpha\dot{\alpha}}$ are defined by**

$$\begin{aligned} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} &= C_\mu + i \bar{\theta} \gamma_5 \chi_\mu + \frac{i}{2} \bar{\theta} \gamma_5 \gamma^\lambda \theta J_{\lambda\mu} \\ &+ \frac{i}{2} \bar{\theta} \gamma_5 \theta M_\mu - \frac{i}{2} \bar{\theta} \theta N_\mu + \frac{1}{2} \bar{\theta} \theta \bar{\theta} \gamma_5 (\lambda_\mu + \frac{1}{2} \gamma \cdot \partial \chi_\mu) \\ &- \frac{1}{2} \bar{\theta} \gamma_5 \theta \bar{\theta} (\lambda_\mu + \frac{1}{2} \gamma \cdot \partial \chi_\mu) - \frac{1}{2} (\bar{\theta} \theta)^2 \left(\frac{1}{2} D_\mu + \frac{1}{4} \square C_\mu \right) \end{aligned} \quad (36)$$

* The first two terms of (32) give a supercurrent which corresponds to the improved, rather than the modified, quantities of sect. 2. If only the first two terms are kept, (33) must be replaced by

$$D^\alpha V_{\alpha\dot{\alpha}} = \frac{m}{2} \bar{D}_{\dot{\alpha}} (\Phi^*)^2$$

but (34) and (35) are still valid. Clearly this is the procedure to be followed, in particular, when $g = 0$.

** Equation (36) corresponds to eq. (1) of ref. [9], except for the additional vector index μ and for the use of real four-component notation.

where all tensor fields are real and all spinors are real four-component Majorana spinors. One can easily verify that (35), together with its conjugate, is equivalent to the constraints

$$\partial_\lambda M_\mu - \partial_\mu M_\lambda = \partial_\lambda N_\mu - \partial_\mu N_\lambda = 0 \quad (37)$$

$$\partial^\lambda (\sigma_{\lambda\mu} - \eta_{\lambda\mu} \sigma_\rho{}^\rho) = \partial^\mu (\sigma_{\lambda\mu} - \eta_{\lambda\mu} \sigma_\rho{}^\rho) = 0 \quad (38)$$

$$\sigma_{\lambda\mu} - \sigma_{\mu\lambda} = -\frac{1}{2} \epsilon_{\lambda\mu}{}^{\nu\rho} (\partial_\nu C_\rho - \partial_\rho C_\nu) \quad (39)$$

$$\lambda_\mu + \gamma \cdot \partial \chi_\mu - \partial_\mu \gamma \cdot \chi = 0 \quad (40)$$

$$\gamma \cdot \lambda = 0 \quad (41)$$

$$D_\mu + \square C_\mu - \partial_\mu \partial \cdot C = 0 \quad (42)$$

which are therefore invariant under restricted supertransformations. Observe that (40) and (41) imply

$$\partial \cdot \chi = \gamma \cdot \partial \gamma \cdot \chi \quad (43)$$

while (37) means that one can introduce a scalar field $\hat{A}$ and a pseudoscalar field $\hat{B}$ such that

$$M_\mu = \partial_\mu \hat{A} \quad , \quad N_\mu = \partial_\mu \hat{B} \quad (44)$$

Using the invariant constraints (37) to (44) one can eliminate the fields λ_μ and D_μ , as well as the antisymmetric part of the field $v_{\lambda\mu}$, from the transformation law* of the supermultiplet (36). Therefore, the fields C_μ , χ_μ , $\hat{A}$, $\hat{B}$ and the symmetric part $t_{\lambda\mu}$ of $v_{\lambda\mu}$

$$t_{\lambda\mu} = \frac{1}{2} (\sigma_{\lambda\mu} + \sigma_{\mu\lambda}) \quad (45)$$

form a supermultiplet which transforms as

* This transformation law is given explicitly by eq. (10) of ref. [1] simply by endowing all fields with an additional vector index and, for the case of restricted supertransformations, by taking the parameter α to be coordinate independent.

$$\delta \hat{A} = i \bar{\alpha} \gamma \cdot X$$

$$\delta \hat{B} = i \bar{\alpha} \gamma_5 \gamma \cdot X$$

$$\delta C_\mu = i \bar{\alpha} \gamma_5 X_\mu$$

$$\delta X_\mu = \gamma^\lambda t_{\lambda\mu} \alpha - \frac{1}{2} \gamma^\lambda \varepsilon_{\lambda\mu}{}^{\nu\rho} \partial_\nu C_\rho \alpha - \partial_\lambda C_\mu \gamma_5 \gamma^\lambda \alpha + \partial_\mu (\hat{A} + \gamma_5 \hat{B}) \alpha \quad (46)$$

$$\delta t_{\lambda\mu} = \frac{i}{2} \bar{\alpha} (\partial_\lambda X_\mu + \partial_\mu X_\lambda) + \frac{i}{2} \bar{\alpha} \gamma_\lambda (\partial_\mu \gamma \cdot X - \gamma \cdot \partial X_\mu) + \frac{i}{2} \bar{\alpha} \gamma_\mu (\partial_\lambda \gamma \cdot X - \gamma \cdot \partial X_\lambda)$$

As mentioned above, the components of the superfield (36) are related in a simple way to the modified currents and energy-momentum tensor of sect. 2. Expanding (32) and comparing with (19), (22), (26) and (13), one can check that the connection is given by

$$J_\mu^{(5)} \text{ mod } = 2 C_\mu \quad (47)$$

$$J_\mu \text{ mod } = -\frac{4}{3} (X_\mu - \gamma_\mu \gamma \cdot X) \quad (48)$$

$$\Theta_{\lambda\mu} \text{ mod } = -\frac{2}{3} (t_{\lambda\mu} - \eta_{\lambda\mu} t_\rho{}^\rho) \quad (49)$$

From (46) and (43) one sees that the fields $\hat{A}$, $\hat{B}$, $\gamma \cdot X$, $t_\mu{}^\mu$ and $\partial_\mu C^\mu$ form a "scalar" (left-handed) multiplet in real notation. In general, this multiplet corresponds to the left-handed field S of eqs. (34). Using (13), (24), (27) and (47) to (49) we see that, in the particular model given by the Lagrangian (1), this multiplet equals $-m^2/4g$ times the multiplet A , B , ψ , F , G of the fundamental fields.

The transformation laws (46) where $t_{\lambda\mu}$ is symmetric and satisfies

$$\partial^\lambda (t_{\lambda\mu} - \eta_{\lambda\mu} t_\rho{}^\rho) = 0 \quad (50)$$

and X_μ satisfies (43), together with the identifications (47), (48) and (49), are independent of the particular model we have used to derive them. They provide the

desired transformation law of the multiplet containing the spinor current under restricted supertransformations*.

4. GENERAL SUPERSYMMETRY TRANSFORMATIONS

For the case of zero mass, the formulae of sects. 2 and 3 become simpler. At the same time, the larger supersymmetry algebra, including special transformations, becomes relevant.

Setting $m = 0$ in (33) we see that the left-handed superfield S vanishes, so that $V_{\alpha\dot{\alpha}}$ satisfies not only (34) but the more restrictive condition

$$D^{\alpha} V_{\alpha\dot{\alpha}} = 0 \quad (51)$$

In terms of the component fields given by (36) this means that the invariant constraints (37) to (42) are now replaced by the stronger conditions

$$M_{\mu} = N_{\mu} = 0 \quad (52)$$

$$\partial^{\lambda} \psi_{\lambda\mu} = \partial^{\mu} \psi_{\lambda\mu} = 0 \quad (53)$$

$$\psi_{\rho}^{\rho} = 0 \quad (54)$$

$$\psi_{\lambda\rho} - \psi_{\rho\lambda} = -\frac{1}{2} \varepsilon_{\lambda\rho}{}^{\nu\rho} (\partial_{\nu} C_{\rho} - \partial_{\rho} C_{\nu}) \quad (55)$$

$$\lambda_{\mu} + \gamma \cdot \partial \chi_{\mu} = 0, \quad \gamma \cdot \lambda = 0 \quad (56)$$

* It is amusing to observe that (46), (48) and (49) can be used to express the energy-momentum tensor locally in terms of the spinor current as

$$T_{\mu\nu}^{\lambda\mu} = -\frac{1}{16} \pi_2 (\gamma^{\lambda} \{ J_{\mu\alpha\dot{\alpha}}^{\mu}, \bar{Q} \} + \gamma^{\mu} \{ J_{\mu\alpha\dot{\alpha}}^{\lambda}, \bar{Q} \})$$

This expression is local, because the spinor charge is conserved and any two components of the spinor current anticommute for space-like separation. Observe that

$$\pi_2 (\gamma^{\lambda} \{ J_{\mu\alpha\dot{\alpha}}^{\mu}, \bar{Q} \} - \gamma^{\mu} \{ J_{\mu\alpha\dot{\alpha}}^{\lambda}, \bar{Q} \}) = -\frac{8}{3} \varepsilon_{\lambda\rho}{}^{\mu\nu} \partial^{\rho} J_{\mu\alpha\dot{\alpha}}^{(5)\rho}$$

$$\gamma \cdot \chi = 0 \quad (57)$$

$$D_\mu + \square C_\mu = 0, \quad \partial \cdot C = 0 \quad (58)$$

while (43) becomes

$$\partial \cdot \chi = 0 \quad (59)$$

The identifications (47) to (49) are now

$$J_\mu^{(s)} \text{ mod} = 2 C_\mu \quad (60)$$

$$J_\mu \text{ mod} = -\frac{4}{3} \chi_\mu \quad (61)$$

$$\Theta_{\lambda\mu} \text{ mod} = -\frac{2}{3} t_{\lambda\mu} \quad (62)$$

where $t_{\lambda\mu}$ is, as before, the symmetric part of $v_{\lambda\mu}$, given by (45) (clearly, for $m = 0$, the improved and the modified quantities are identical). The fields $\hat{A}$ and $\hat{B}$ vanish in the case $m = 0$. Therefore, $J_\mu^{(s)}$, $J_\mu \text{ mod}$ and $\Theta_{\lambda\mu} \text{ mod}$ just transform into each other by a restricted supersymmetry transformation, as given by (46) with the appropriate simplifications.

We now wish to find out how the supercurrent multiplet $J_\mu^{(s)}$, $J_\mu \text{ mod}$ and $\Theta_{\lambda\mu} \text{ mod}$ transforms under the larger supersymmetry algebra. We observe that the conditions (51) or (52) to (59) are invariant under the larger algebra. The desired transformation law can be obtained from the transformation of the superfield $V_{\alpha\dot{\alpha}}$ under the larger algebra, which is known. Alternatively, they can be obtained by combining the transformation law of the supercurrent multiplet under restricted supertransformations with its transformation law under conformal transformations*, using the algebra given explicitly in ref. [2]. The result of these calculations is

* It is essential here that the modified (improved) currents and energy-momentum tensor transform irreducibly under conformal transformations when $m = 0$. This is not true of the original J_μ and $T_{\mu\nu}$.

$$\begin{aligned}
 \delta C_\mu &= i \bar{\alpha} \gamma_5 \chi_\mu \\
 \delta \chi_\mu &= \gamma^\lambda t_{\lambda\mu} \alpha - \frac{1}{2} \gamma^\lambda \varepsilon_{\lambda\mu}{}^{\nu\rho} \partial_\nu C_\rho \alpha \\
 &\quad - \partial_\lambda C_\mu \gamma_5 \gamma^\lambda \alpha - C_\mu \gamma_5 \gamma^\lambda \partial_\lambda \alpha - 2 C_\lambda \gamma_5 \gamma^\lambda \partial_\mu \alpha \quad (63) \\
 \delta t_{\lambda\mu} &= \frac{i}{2} \bar{\alpha} (\partial_\lambda \chi_\mu + \partial_\mu \chi_\lambda) - \frac{i}{2} \bar{\alpha} (\gamma_\lambda \gamma^\nu \partial_\nu \chi_\mu + \gamma_\mu \gamma^\nu \partial_\nu \chi_\lambda) \\
 &\quad + 5i (\partial_\lambda \bar{\alpha} \chi_\mu + \partial_\mu \bar{\alpha} \chi_\lambda)
 \end{aligned}$$

where the coordinate dependent parameters α are defined in ref. [1]. As a check one can calculate the commutator of two supertransformations of parameters α_1 and α_2 . With some algebra one verifies that (63) implies

$$\begin{aligned}
 [\delta_2 \delta_1] C_\mu &= \xi^\lambda \partial_\lambda C_\mu + \frac{1}{2} \partial_\lambda \xi^\lambda C_\mu + \partial_\mu \xi^\lambda C_\lambda \\
 [\delta_2 \delta_1] \chi_\mu &= \xi^\lambda \partial_\lambda \chi_\mu + \frac{7}{8} \partial_\lambda \xi^\lambda \chi_\mu + \frac{1}{4} (\partial_\lambda \xi_\rho - \partial_\rho \xi_\lambda) \Sigma^{\lambda\rho} \chi_\mu + \frac{1}{2} (\partial_\mu \xi_\lambda - \partial_\lambda \xi_\mu) \chi^\lambda - \frac{3}{4} \eta \gamma_5 \chi_\mu \\
 [\delta_2 \delta_1] t_{\mu\nu} &= \xi^\lambda \partial_\lambda t_{\mu\nu} + \frac{1}{2} \partial_\lambda \xi^\lambda t_{\mu\nu} + \partial_\mu \xi^\lambda t_{\lambda\nu} + \partial_\nu \xi^\lambda t_{\lambda\mu} \quad (64)
 \end{aligned}$$

which are the correct transformation formulae under conformal transformations. Here, as in ref. [1]

$$\begin{aligned}
 \xi_\mu &= 2i \bar{\alpha}_1 \gamma_\mu \alpha_2 \\
 \eta &= i \partial_\mu \bar{\alpha}_1 \gamma_5 \gamma^\mu \alpha_2 - i \partial_\mu \bar{\alpha}_2 \gamma_5 \gamma^\mu \alpha_1 \quad (65)
 \end{aligned}$$

Using (16) and

$$Q^{(1)} = - \int \gamma^\lambda \chi_\lambda J_{\text{mod}}^0 d^3x \quad (66)$$

it is also easy to see that (63), for $\mu = 0$, gives rise, by three-dimensional integration, to the algebra written out explicitly in ref. [2]. Therefore, (60) to (63) provide the local equations corresponding to the larger supersymmetry algebra.

APPENDIX A

In this appendix we give two more examples of supercurrents, and show that they satisfy, respectively, (34) and (51). The first example* is that of the free massive "vector" multiplet. This multiplet is described by a real superfield V satisfying the equation of motion

$$\frac{1}{16} (D^\alpha \bar{D} \bar{D} D_\alpha + \bar{D}_\alpha D D \bar{D}^\alpha) V + m^2 V = 0 \quad (A.1)$$

In real notation the components of the superfield V are defined by an expansion perfectly analogous to the right-hand side of (36), with the omission of the index μ . Using (30) and (31) (A.1) can be transformed into

$$\left\{ \frac{1}{16} (D D \bar{D} \bar{D} + \bar{D} \bar{D} D D) - \square + m^2 \right\} V = 0 \quad (A.2)$$

On the other hand, from (31) we see that

$$[D D, \bar{D} \bar{D}] = 16 \square + 8i \bar{D} \sigma^\mu D \partial_\mu = -16 \square - 8i D \sigma^\mu \bar{D} \partial_\mu \quad (A.3)$$

Therefore, from (A.2) it follows that

$$D D V = \bar{D} \bar{D} V = 0 \quad (A.4)$$

which in turn implies that one has separately

$$\frac{1}{8} D^\alpha \bar{D} \bar{D} D_\alpha V + m^2 V = 0 \quad (A.5)$$

$$\frac{1}{8} \bar{D}_\alpha D D \bar{D}^\alpha V + m^2 V = 0 \quad (A.6)$$

If we now define the (real) supercurrent to be (up to an over-all proportionality factor)

$$\begin{aligned} V_{\alpha\dot{\alpha}} &= (\bar{D} \bar{D} D_\alpha V) (D D \bar{D}_{\dot{\alpha}} V) \\ &+ \frac{16m^2}{3} (V D_\alpha \bar{D}_{\dot{\alpha}} V - V \bar{D}_{\dot{\alpha}} D_\alpha V - D_\alpha V \bar{D}_{\dot{\alpha}} V) \end{aligned} \quad (A.7)$$

* The results on this example are based on work done in collaboration with J. Wess.

we can easily verify that it satisfies eqs. (34) and their conjugate, with

$$S = \frac{4}{3} m^2 DDV^2 \quad (A.8)$$

The second example is that of the massless self-interacting Yang-Mills multiplet discussed in ref. [9]. We recall that the Yang-Mills multiplet is described by a superfield V which is also a hermitian $N \times N$ matrix. The generalized Yang-Mills transformations are defined by

$$e^V \rightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda}, \quad e^{-V} \rightarrow e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger} \quad (A.9)$$

where Λ is a (matrix) left-handed superfield and $\Lambda^\dagger$ its right-handed hermitian conjugate

$$D_\alpha \Lambda^\dagger = \bar{D}_{\dot{\alpha}} \Lambda = 0 \quad (A.10)$$

The superfield

$$W_\alpha = \bar{D}\bar{D} (e^{-V} D_\alpha e^V) \quad (A.11)$$

is left-handed

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0, \quad D_\alpha \bar{W}_{\dot{\alpha}} = 0 \quad (A.12)$$

and corresponding to (A.9), transforms as

$$W_\alpha \rightarrow e^{-i\Lambda} W_\alpha e^{i\Lambda}, \quad \bar{W}_{\dot{\alpha}} \rightarrow e^{-i\Lambda^\dagger} \bar{W}_{\dot{\alpha}} e^{i\Lambda^\dagger} \quad (A.13)$$

The equations of motion

$$\begin{aligned} D^\alpha (e^V W_\alpha e^{-V}) &= 0 \\ \bar{D}^{\dot{\alpha}} (e^{-V} \bar{W}_{\dot{\alpha}} e^V) &= 0 \end{aligned} \quad (A.14)$$

are supersymmetric and also invariant under (A.9). If we now define the supercurrent, up to a proportionality constant, to be

$$V_{\alpha\dot{\alpha}} = \text{Tr} (e^V W_\alpha e^{-V} \bar{W}_{\dot{\alpha}}) = \text{Tr} (W_\alpha e^{-V} \bar{W}_{\dot{\alpha}} e^V) \quad (A.15)$$

we see immediately that it satisfies (51) and the complex conjugate, as a consequence of (A.12) and (A.14). In the special gauge described in ref. [9], the spinor current contained in (A.15) agrees with that given by eq. (41) of ref. [6]. Clearly, the present example contains as a special case that of the free massless "vector" multiplet, in which case there is no matrix multiplication and eqs. (A.9) to (A.15) take a slightly simpler form.

APPENDIX B

As mentioned in sect. 1, the formulae of sect. 4 can be used to describe the supermultiplet containing free fields of mass m and spin 2, $3/2$, $3/2$ and 1. One can use the superfield equations

$$D^\alpha V_{\alpha\dot{\alpha}} = 0 \quad (B.1)$$

and

$$(\square - m^2) V_{\alpha\dot{\alpha}} = 0 \quad (B.2)$$

Indeed, as we already know, (B.1) gives, for

$$t_{\lambda\mu} = t_{\mu\lambda} \quad (B.3)$$

the equations

$$t_\rho{}^\rho = 0, \quad \partial^\lambda t_{\lambda\mu} = 0 \quad (B.4)$$

and for $\hat{C}_\mu = mC_\mu$ the equation

$$\partial^\mu \hat{C}_\mu = 0 \quad (B.5)$$

Together with the Klein-Gordon equation for each component, (B.3) to (B.5) describe free fields of spin 2 and 1. Furthermore, (B.1) gives

$$\lambda_\mu + \gamma \cdot \partial \chi_\mu = 0 \quad (B.6)$$

Applying $\gamma \cdot \partial$ to this equation and using the Klein-Gordon equation for χ_μ , one finds

$$\gamma \cdot \partial \lambda_\mu + m^2 \chi_\mu = 0 \quad (B.7)$$

Therefore, $\hat{\lambda}_\mu = \lambda_\mu + m\chi_\mu$ and $\hat{\chi}_\mu = \gamma_5(\lambda_\mu - m\chi_\mu)$ both satisfy the Dirac equation for mass m . At the same time (56), (57) and (59) insure that the Rarita-Schwinger conditions

$$\partial \cdot \hat{\lambda} = \partial \cdot \hat{\chi} = \gamma \cdot \hat{\lambda} = \gamma \cdot \hat{\chi} = 0 \quad (B.8)$$

are satisfied. Therefore, $\hat{\lambda}_\mu$ and $\hat{\chi}_\mu$ are each a free field of spin $3/2$. The other fields of the multiplet are not independent, for instance

$$D_\mu = -m \hat{C}_\mu \quad (B.9)$$

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