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EQUIVALENCE OF SUM RULES FOR K_L-K_S DECAY *)

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There has been recently some discussion ^{1),2),3),4)} about sum rules in neutral K decay, in particular about the Bell-Steinberger sum rule ⁵⁾, in connection with the fact that sum rules can be obtained by treating K_L and K_S as overlapping resonances in an S matrix. The open channels of this matrix, naturally, are the 2π , 3π , etc., channels into which K_0 and $\bar{K}_0$ can decay. The B.S. sum rule is, on the other hand, an integral part of the sum rules obtainable by the standard Lee-Oehme-Yang treatment ⁶⁾ of K_0 , $\bar{K}_0$ decay in the Wigner-Weisskopf approximation. One would expect the "standard" and the S matrix approach (in the small width approximation) to be entirely equivalent and we will show in the following that this is so. We hope thereby to lay to rest some of the doubts remaining, after the statement ¹⁾ that the B.S. sum rule and S matrix sum rules do not agree, and its partial refutation ²⁾ in the statement that they agree "in their phase information". Specifically we show that : (a) the sum rules agree completely ; (b) the standard method when applied to the S matrix problem yields all the results of the pure S matrix approach and (c) one can go conversely from the pure S matrix formulae back to all the results of the usual treatment.

Let us first recall, referring for notation to McVoy ²⁾ and Durand and McVoy ³⁾, the results of the pure S matrix treatment. The $N \times N$ S matrix (N open channels, labelled $c, c', \dots$) is obtained in the form

$$S(E) = B - i\Gamma_L \frac{g_L \tilde{h}_L}{E - M_L} - i\Gamma_S \frac{g_S \tilde{h}_S}{E - M_S} \quad (1)$$

where M_L and M_S are complex masses, and $g_L \tilde{h}_L$, $g_S \tilde{h}_S$ are non-symmetric dyads. We may assume

$$g_L^\dagger g_L = g_S^\dagger g_S = 1 \quad (2)$$

i.e., $\sum_c g_{Lc}^* g_{Lc} = 1$, etc. The unitarity conditions for $S(E)$ are given by D.MV., Eqs. (4),(5),(6). They are valid in the small width approximation, when the non-resonant part of the S matrix, B , may be regarded as a constant. B is then unitary, [D.MV., Eq. (4)]. With the abbreviation

$$i(\Gamma_L \Gamma_S)^{\frac{1}{2}} (g_L^\dagger g_S) (M_L^* - M_S)^{-1} = \rho e^{i\phi} \quad (3)$$

where ρ and ϕ are real, the remaining unitarity conditions, [D.MV., Eqs. (5), (6)], may be written, assuming (2)

$$\begin{aligned} (1-\rho^2) B h_L^* &= g_L - \sqrt{\frac{\Gamma_S}{\Gamma_L}} \rho e^{-i\phi} g_S \\ (1-\rho^2) B h_S^* &= -\sqrt{\frac{\Gamma_L}{\Gamma_S}} \rho e^{i\phi} g_L + g_S \end{aligned} \quad (4)$$

We now sketch the proof of our statements (a), (b), (c). Taking up (b) first, we recall that in the standard approach one introduces : (i) "in" and "out" eigenstates of the strong Hamiltonian labelled, e.g., $|c, \text{out}\rangle$, $|c', \text{out}\rangle$, ..., etc., in correspondence with the decay channels $c, c', \dots$; (ii) the weak Hamiltonian H_w and its matrix elements $\langle c, \text{out} | H_w | K_0 \rangle$ etc. ; (iii) the non-Hermitian mass matrix in the $(K_0, \bar{K}_0)$ subspace, with right-eigenvectors $|L\rangle$ and $|S\rangle$, and dual left-eigenvectors $\langle L'|$, $\langle S'|$, related to the former (with suitable normalizations) by

$$\langle L'|L\rangle = \langle S'|S\rangle = 1 \quad ; \quad \langle L'|S\rangle = \langle S'|L\rangle = 0 \quad (5)$$

The most common application of this formalism is to the decay problem, but it is an easy matter ⁷⁾ to extend the treatment to the S matrix problem and to obtain precisely the result of Eq. (1), with more detailed specifications for the quantities involved. Namely, M_L and M_S are now, of course, the eigenvalues of the mass matrix, and the components of the g_L and $\tilde{h}_S$ vectors are the matrix elements

$$g_{Lc} = (2\pi/\Gamma_L)^{1/2} \langle c, out | H_w | L \rangle \quad (6)$$

$$h_{Lc} = (2\pi/\Gamma_L)^{1/2} \langle L' | H_w | c, in \rangle \quad (7)$$

Similar formulae hold with $L \rightarrow S$. Furthermore, B is now the transformation between "in" and "out" states

$$|c, in\rangle = \sum_{c'} B_{c'c} |c', out\rangle \quad (8)$$

Now suppose $|L\rangle$ and $|S\rangle$ are normalized conventionally, $\langle L|L\rangle = \langle S|S\rangle = 1$; then the standard formulae for the widths Γ_L, Γ_S imply that the g vectors are normalized as in Eq. (2). Then, on the other hand, the dual vectors are not of unit norm, in general, since $\langle L'| = N \{ \langle L| - \langle L|S\rangle \langle S| \}$, etc., where

$$N = \{1 - |\langle L|S\rangle|^2\}^{-1} = \langle L'|L'\rangle = \langle S'|S'\rangle \quad (9)$$

Using (8) one can now evaluate (6)

$$\begin{aligned} h_{Lc} &= (2\pi/\Gamma_L)^{1/2} N \{ \langle L|H_w|c, in\rangle - \langle L|S\rangle \langle S|H_w|c, in\rangle \} \\ &= N \sum_{c'} B_{c'c} \left\{ g_{Lc'}^* - \sqrt{\frac{\Gamma_S}{\Gamma_L}} \langle L|S\rangle g_{S c'}^* \right\} \end{aligned} \quad (10)$$

and similarly for $\tilde{h}_S$. Finally, the standard treatment includes the B.S. rule (obtained from the defining equations for the imaginary part of the mass matrix)

$$(\Gamma_L \Gamma_S)^{1/2} g_L^\dagger g_S = -i \langle L|S\rangle (M_L^* - M_S) \quad (11)$$

The proof of our statement (b) is now complete since comparing (3) and (11) we see that

$$\langle L | S \rangle = \rho e^{i\phi} \quad (12)$$

whereupon (10) immediately yields the first of the unitarity conditions (4). The second one is obtained similarly.

Turning now to point (a), consider, e.g., McVoy's Eq. (8) or (9). Rewriting the latter without the approximations made by McVoy, and to agree with our normalization of the h vectors, it becomes

$$-i(M_L^* - M_S)[g_L^\dagger g_S - (1 - e^2)h_S^\dagger h_L] = i(\Gamma_S + \Gamma_L)g_L^\dagger g_S \quad (13)$$

It is an obvious consequence of the previous calculations that when (10) is substituted into this, it must ultimately reduce to the B.S. equation, as one can in fact verify. In order to avoid some tedious algebra, and illustrate the main point, let us neglect terms of order ρ^2 , so that, in Eq. (10), $N = 1$. Then substituting into (13), terms of zero order in ρ cancel on the left-hand side, since

$$h_S^\dagger h_L = g_L^\dagger g_S - (\Gamma_L \Gamma_S)^{-\frac{1}{2}}(\Gamma_S + \Gamma_L)\langle L | S \rangle + \mathcal{O}(\rho^2)$$

What is left reduces to (11). Q.E.D.

Finally turning to point (c), we seek a definition of the decaying states in a pure S matrix approach. To this end, let us introduce new "in" and "out" states, which are eigenstates of the full Hamiltonian, so that their scalar product defines the exact S matrix. The notation for these new states will be the same as before, except that the value of the total energy E in the centre-of-mass system will be indicated explicitly. According to the principle that the asymptotic out states, for example, form a complete system, it must be possible to express the $|L\rangle$ and $|S\rangle$ states as linear combinations of asymptotic states. Assuming the normalization

$$\langle E, c, \text{out} | E', c', \text{out} \rangle = \delta(E - E') \delta_{cc'} \quad (14)$$

and guided by the results of decay theory we define $|L\rangle$, for example, as the Hilbert space vector

$$|L\rangle = (\Gamma_L / 2\pi)^{\frac{1}{2}} \int \frac{dE}{E - M_L} \sum_c g_{Lc} |E, c, \text{out}\rangle \quad (15)$$

which at $t = +\infty$ decays into a superposition of channel states with amplitudes $g_{Lc}/(E - M_L)$. The state $|S\rangle$ is given by an analogous formula. In the spirit of the small width approximation one can compute the norm of the state (15) as if E could be integrated from $-\infty$ to $+\infty$ (while the integral actually has a finite lower limit). One then finds $\langle L | L \rangle = 1$. Similarly one finds

$$\begin{aligned} \langle L | S \rangle &= (g_L^\dagger g_S) (\Gamma_L \Gamma_S)^{\frac{1}{2}} (2\pi)^{-1} \int \frac{dE}{(E - M_S)(E - M_L^*)} = \\ &= i (g_L^\dagger g_S) (\Gamma_L \Gamma_S)^{\frac{1}{2}} (M_L^* - M_S)^{-1} \end{aligned} \quad (16)$$

in agreement with the B.S. rule.

Having thus acquired a handle to deal with the states $|L\rangle$ and $|S\rangle$, one can construct by linear superposition the states K_0 and $\bar{K}_0$. The consequences of CP or CPT invariance in this scheme have been studied by McVoy and Durand ^{2), 3)} and need not be repeated here. Thus all the features of the standard approach can be reconstructed. Let us therefore mention only one last point. A natural, although slightly unfamiliar, question to ask is what happens if one follows the decay of the state $|L\rangle$, Eq. (15), "backwards in time". To this end we expand in terms of "in" states, using the relation

$$\langle E, c, \text{out} | E', c', \text{in} \rangle = \delta(E - E') S_{cc'}(E) \quad (17)$$

where $S(E)$ is given by (1). The result is, after a little algebra

$$|L\rangle = (\Gamma_L / 2\pi)^{\frac{1}{2}} \int dE \sum_c |E, c, in\rangle \cdot \left\{ \frac{h_{Lc}^*}{E - M_L^*} + \sqrt{\frac{\Gamma_S}{\Gamma_L}} \langle S|L\rangle \frac{h_{Sc}^*}{E - M_S^*} \right\} \quad (18)$$

While this complicated result may seem surprising, at first sight, we obtain a simpler result by calculating

$$|L'\rangle = (\Gamma_L / 2\pi)^{\frac{1}{2}} \int \frac{dE}{E - M_L^*} \sum_c h_{Lc}^* |E, c, in\rangle \quad (19)$$

which is clearly the counterpart of Eq. (15). This shows, in agreement with (7), that in order to form the state $|L'\rangle$ at $t = 0$, we must start at $t = -\infty$ with amplitudes $h_{Lc}^* / (E - M_L^*)$ in the channel states. These equations form a natural point of departure to examine questions of time-reversal or CPT.

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