The Rest-Frame Darwin Potential from the Lienard-Wiechert Solution in the Radiation Gauge

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Abstract

In the semiclassical approximation in which the electric charges of scalar particles are described by Grassmann variables \( Q_i^2 = 0, \ Q_i Q_j \neq 0 \), it is possible to re-express the Lienard-Wiechert potentials and electric fields in the radiation gauge as phase space functions, because the difference among retarded, advanced, and symmetric Green functions is of order \( Q_i^2 \). By working in the rest-frame instant form of dynamics, the elimination of the electromagnetic degrees of freedom by means of suitable second classs contraints leads to the identification of the Lienard-Wiechert reduced phase space containing only \( N \) charged particles with mutual action-at-a-distance vector and scalar poten-
tials. A Darboux canonical basis of the reduced phase space is found. This allows one to re-express the potentials for arbitrary $N$ as a unique effective scalar potential containing the Coulomb potential and the complete Darwin one, whose $1/c^2$ component agrees for with the known expression. The effective potential gives the classical analogue of all static and non-static effects of the one-photon exchange Feynman diagram of scalar electrodynamics.

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I. INTRODUCTION

Recently the rest-frame Wigner-covariant instant form of dynamics has been developed in Ref. [1] for isolated systems in Minkowski spacetime $M^4$ starting from the case of $N$ scalar charged particles plus the electromagnetic field. The charges of the particles are described by bilinears in Grassmann variables following the scheme (called pseudo-classical mechanics) which uses a semiclassical approximation to quantum operators with a finite discrete spectrum like spin [2,3], that otherwise would have no strict classical limit; Grassmann variables give fermionic oscillators after quantization. The extension of this scheme to the electric charge is based on the experimental fact that all measurable charges are multiples of $\pm e$, the electron and positron charges. Therefore, even if it is not clear whether the electric charge has to be considered as a quantum operator in the standard sense (except in the case of the existence of magnetic charges; in this case there is the Dirac quantization rule for the product of the electric and magnetic charges), one can consider it as a two-level system [which becomes a six-level system ($\pm e, \pm \frac{1}{3}e, \pm \frac{2}{3}e$) at the quark-lepton level] described by an operator with quantum $e$ instead of $\hbar$. Then one can define a semiclassical approximation with Grassmann variables like in the case of spin. As shown in Ref. [1], this semiclassical approximation automatically implies the regularization of the Coulomb self-energies (the $i \neq j$ rule). Therefore, this semiclassical approximation may be considered as an alternative to the extended electron models, which were introduced for regularization aims.

The idea leading to the rest-frame instant form is to consider an arbitrary 3+1 splitting of Minkowski spacetime by means of a foliation with spacelike hypersurfaces $\Sigma(\tau)$ diffeomorphic to $R^3$. The parameter $\tau$ labelling the leaves is used as a Lorentz scalar mathematical time parameter. For each $\tau$ the leaf $\Sigma(\tau)$ is defined through the embedding $R^3 \hookrightarrow \Sigma(\tau) \subset M^4$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, where $\vec{\sigma}$ are curvilinear coordinates on $R^3$. Then one considers the Lagrangian describing the coupling of the given isolated system to an external gravitational field and replaces the 4-metric with the induced metric on $\Sigma(\tau)$, which is a functional of
\( z^\mu(\tau, \vec{\sigma}) \). In this way one gets the Lagrangian for the description of the isolated system on arbitrary spacelike hypersurfaces (i.e. in arbitrary accelerated reference frames in Minkowski spacetime) with the embedding functions \( z^\mu(\tau, \vec{\sigma}) \) as extra configuration variables describing the hypersurface. However, there are four first class constraints at each point implying the independence of the description from the chosen 3+1 splitting. Thus the \( z^\mu(\tau, \vec{\sigma}) \)'s are gauge variables. Therefore, one can restrict the description of the isolated system to spacelike hyperplanes \( \Sigma_H(\tau) \), \( z^\mu(\tau, \vec{\sigma}) = x^\mu_s(\tau) + b^\mu_r(\tau)\sigma^r \) (inertial reference frames in Minkowski spacetime; \( x^\mu_s(\tau) \) is an arbitrary origin).

Then, if one selects all the configurations of the isolated system with total timelike 4-momentum (they are dense in the space of all configurations), one finds that each timelike configuration identifies a privileged family of hyperplanes: those orthogonal to its total 4-momentum [Wigner hyperplanes \( \Sigma_W(\tau) \)]. At this stage one has obtained the analogue of the nonrelativistic center-of-mass separation and the definition of a new instant form of dynamics [4], the rest-frame one [1]. There is a decoupled point \( \tilde{x}^\mu_s(\tau) \) on each Wigner hyperplane describing the “external” center of mass of the isolated system (thus serving as a decoupled point particle clock) with conjugate momentum \( p^\mu_s \). \( \tilde{x}^\mu_s(\tau) \) is a canonical variable, but it is not covariant like the Newton-Wigner position operator: it has only covariance under the little group of timelike Poincaré orbits.

After the restriction to the Wigner hyperplane only four first class constraints are left:

i) three of them say that the total 3-momentum of the isolated system vanishes (rest-frame condition): the natural gauge fixing for these constraints is the requirement that the “internal” 3-center of mass of the isolated system inside the Wigner hyperplanes coincides with the origin \( x^\mu_s(\tau) \) of the coordinates \( \vec{\sigma} \) in it (see Refs. [5,6], where the group-theoretical results of Ref. [7] are used). In this way only “internal” relative variables describe the isolated system: they are either Lorentz scalars or Wigner spin 1 3-vectors.

ii) the fourth one identifies the invariant mass of the isolated system as the Hamiltonian of the evolution in \( \tau \) when, with a gauge fixing, \( \tau \) is made to coincide with the Lorentz scalar
rest-frame time \( T_s = p_s \cdot \hat{x}_s / \sqrt{p_s^2} = p_s \cdot x_s / \sqrt{p_s^2} \) of the decoupled external center of mass.

In this description the standard manifestly covariant fields like the Klein-Gordon field \( \tilde{\phi}(z') \) are replaced by the new fields \( \phi(\tau, \vec{\sigma}) = \tilde{\phi}(z'(\tau, \vec{\sigma})) \), which know the embedding and have the non-local information about the equal time hypersurfaces \( \Sigma(\tau) \) built-in. In the case of gauge theories, one can make a canonical reduction to a canonical basis of Dirac’s observables in the radiation gauge (or Coulomb or generalized Coulomb; the literature is ambiguous about the terminology to be used), with the only universal breaking of manifest covariance connected with the external center of mass, since the relative motions are Wigner covariant.

See Ref. [8] for a complete review of the research program aiming to give a unified description of the four interactions in terms of Dirac-Bergmann’s observables in the framework of the rest-frame instant form of dynamics, which is the classical background of the Tomonaga-Schwinger formulation of quantum field theory.

The description of scalar (or spinning [9]) particles on arbitrary spacelike hypersurfaces requires the choice of the sign of the energy of the particle. This happens because the position of the particle on \( \Sigma(\tau) \) is identified by 3 numbers \( \vec{\sigma} = \vec{\eta}(\tau) \) and not by 4: this implies that the mass-shell first class constraints of the standard manifestly covariant approach have been solved and that one of the two disjoint branches of the mass spectrum has been chosen. In this way, one gets a different Lagrangian for each branch of the mass spectrum of an isolated system of particles (in the standard manifestly covariant description the branches are topologically disjoint for free particles): there is no possibility of crossing of the branches (the classical background of pair production) when interactions are present inside the isolated system (for instance charged particles plus the electromagnetic field) as happens in the manifestly covariant approach. While there is no problem in the coupling to magnetic fields of these particles with a definite sign of the energy, the minimal coupling in the Lagrangian will miss those couplings to the electric fields which are at the basis of the non-diagonalizability of both the Feshbach-Villars description of the Klein-Gordon field and of
the Dirac equation through the Foldy-Wouthuysen transformation (in the case of spinning particles these couplings will have to be extracted from the iterative diagonalization of these theories and added non-minimally). However, this description of particles with, say, positive energy (whose quantization requires pseudodifferential operators [10]) seems suited for the description of the asymptotic Tomonaga-Schwinger states and will be used to introduce a notion of particle in a future quantization of classical fields on Wigner hyperplanes in the rest-frame instant form. These asymptotic states will replace the Fock ones of the standard manifestly covariant theory, which are the main source of problems in the theory of relativistic quantum bound states (the spurious solutions of the Bethe-Salpeter equation; see Ref. [8,11]). This framework should allow the introduction of bound states among the asymptotic states.

Coming back to the isolated system formed by \( N \) scalar charged particles of positive energy plus the electromagnetic field, we recall that in Ref. [1], after canonical reduction to the radiation gauge, the final invariant mass of the reduced system is a function only of Dirac’s observables (gauge invariant particles dressed with a Coulomb cloud and transverse radiation field) and contains:

i) the kinetic energy of the radiation field;

ii) the kinetic energy for the particles with minimal coupling to the radiation field;

iii) the instantaneous action-at-a-distance Coulomb potential among the charges with the Coulomb self-energies regularized (the \( i \neq j \) rule, \( Q_i Q_j \neq 0 \)) due to the Grassmann character of the electric charges \( Q_i \), i.e. \( Q_i^2 = 0 \) [at this semiclassical level we have \( Q_i = e\theta^*_i \theta_i \); this does not imply the vanishing of the fine structure constant \( \alpha = e^2/4\pi \approx 1/137 \), since it gets contributions from \( Q_i Q_j \); therefore, we are retaining effects of order \( \alpha \) but not of higher order, because, as it will be shown, we do not have many-body forces].

Then, in Ref. [12] there was the evaluation of the retarded Lienard-Wiechert potentials of the charged particles in the radiation gauge in the rest-frame instant form. Since the Lienard-Wiechert potentials and fields are linearly dependent on the charges \( Q_i \), the semiclassical regularization \( Q_i^2 = 0 \) eliminates the radiation coming from a single particle
(the electromagnetic energy-momentum tensor contains only terms in \( Q_i Q_j, i \neq j \)) and the causal problems of the Abraham-Lorentz-Dirac equation of each charged particle [since the radiation reaction term has the coefficient \( \tau_o = 2Q^2/3mc^2 \) (\( \tau_o \) is proportional to the time needed for light to travel across a classical electron radius), which vanishes if \( Q^2 = 0 \), there are neither the Schott term nor the Larmor one but only the Lorentz forces produced by the other particles]. In Refs [13] one may find an extended discussion about this equation and in Ref. [14] a recent review on its derivation and its causal problems (preacceleration, runaway solutions). See Ref. [15] for modern attempts to extract the subset of causal solutions of this equation with the requirement of selecting only its solutions which admit a smooth limit for \( \tau_o \to 0 \) (the runaway solutions are singular in this limit) and to find an effective second order equation for this subset of solutions.

Even if at the semiclassical level the “single charged particle” has no acausal behaviour, because, notwithstanding it produces a Grassmann-valued vector potential, it does not irradiate, we can recover the asymptotic Larmor formula for a system of charges (considered as external sources of the electromagnetic field) due to the interference radiation from \( Q_i Q_j, i \neq j \), terms (this result is in accord with macroscopic experimental facts). See Ref. [9] for the extension to spinning particles.

Being in the radiation gauge, at each \( \tau \) the retarded Lienard-Wiechert potentials evaluated in Ref. [12] contain also a non-local (in \( \vec{\sigma} \)) term (coming from the transverse projector; see Eqs.(5.13), (5.22) in Section V): since this term involves all the points of \( \Sigma(\tau) \), the Lienard-Wiechert potential receives contributions from “all” the retarded times before \( \tau \), namely from the whole past history of the particles. In the absence of incoming radiation one could put these retarded Lienard-Wiechert potentials inside the Lagrangian given in Eq.(74) of Ref. [12] for the description of the isolated system in the rest-frame radiation gauge on the Wigner hyperplanes: one would get a Fokker-like action in the radiation gauge, replacing the standard Fokker-Tetrode one in the Lorentz gauge of the manifestly covariant description, and would have to face the problem of how to find a Hamiltonian description when there are integro-differential equations of motion with delay. The existing attempts
are based on the idea of replacing retarded particle coordinates and velocities with instantaneous coordinates and accelerations of all the orders (see for instance Refs. [16–20]). In this way one replaces integro-differential equations of motion with an infinite set of coupled differential ones. Since it is unknown how to formulate the Cauchy problem for these integro-differential equations (see Ref. [21]; exceptions are the 1-dimensional case [22] or the time-asymmetric case [23,24]), there have been complicated attempts to find conditions for extracting a set of effective second order differential equations from the infinite set (see for instance Refs. [21,20]). In Ref. [25] there was an attempt to study the Dirac constraints originating from actions depending on accelerations of all orders, following previous attempt of Kerner [26] of defining a Hamiltonian approach. See also the recent approach of Ref. [27].

Moreover, one should face a problem similar to the one raised in Ref. [28], that only with symmetric Green’s functions like $\frac{1}{2}(\text{retarded} + \text{advanced})$ can the Fokker-Tetrode action corresponding to the Lorentz gauge give rise to a variational principle whose extremals are equivalent to the subspace of extremals of the original action defined by the symmetric Lienard-Wiechert solutions without incoming radiation, i.e. the adjunct Lienard-Wiechert fields (otherwise there are problems with the boundary terms). This is compatible with the Feynman-Wheeler [29] starting point for their theory of the absorbers (see Ref. [30] for the definition of radiation in this theory). A noncovariant justification of the results of Ref. [28] is given in Ref. [19]; by ignoring the self-interactions and assuming a Lagrangian for two charged particles at equal times in which each particle interacts with the retarded Lienard-Wiechert potential of the other one, one obtains in the equations of motion Lorentz forces which correspond to $\frac{1}{2}(\text{retarded} + \text{advanced})$ interactions, because in this Lagrangian the transition from retarded to $\frac{1}{2}(\text{retarded} + \text{advanced})$ interactions is a total time derivative. However, self-reaction is ignored in these calculations and it is not clear how to arrive at a covariant formulation of these results. Let us remark that from the point of view of quantum field theory its regularization and renormalization require the use of the complex Feynman Green function (which does not vanish outside the lightcone; the solutions with the retarded Green function cannot be regularized at the distributional level): while its imaginary part is
connected to absorption in other channels, its real part is just the \( \frac{1}{2} (\text{retarded} + \text{advanced}) \) Green function like in Feynman-Wheeler [see Ref. [31] for the extraction of a Fokker-Tetrode action with \( \frac{1}{2} (\text{retarded} + \text{advanced}) \) kernel from the particle limit of QED (it does not work in QCD)].

However, both in the Dirac derivation [32] of the Abraham-Lorentz-Dirac equation through the evaluation of the near zone self-field (with the same results obtainable with balance equations using the far zone fields; see Refs. [33,14]) and in the Feynman-Wheeler approach with the assumption of complete absorbers [13] the radiation is determined by the radiative Green function \( \frac{1}{2} (\text{retarded} - \text{advanced}) \) [at the conceptual level this introduces the acausal advanced Green function and interpretational problems]. Indeed, the regularization of the self-energy divergence due to radiation reaction is done by rewriting \( \text{retarded} = \frac{1}{2} (\text{retarded} + \text{advanced}) + \frac{1}{2} (\text{retarded} - \text{advanced}) \), by noting that the non radiative Coulomb piece of the fields (which does not influence the motion of the particle only giving a divergent electromagnetic contribution to the mass) is in \( \frac{1}{2} (\text{retarded} + \text{advanced}) \) and by discarding this term as a regularization.

However, till now all the calculations have been done in the Lorentz gauge and it is not clear whether the previous statements are “gauge invariant”.

Since we now have the results of Refs. [1,12] in the radiation gauge and a new type of regularization with the semiclassical approximation, it is interesting to revisit all these problems.

The aim of this paper is to show that, starting from the rest-frame instant form description of \( N \) charged scalar particles plus the electromagnetic field with Grassmann-valued electric charges \( Q_i \), the semiclassical regularization \( Q_i^2 = 0 \) allows one to transform the subspace of Lienard-Wiechert solutions (without or with incoming radiation) into a symplectic submanifold of the space of all solutions. This takes place since all the higher accelerations coming from an equal time development of the delay decouple, being of order \( Q_i^2 \) on the solution of the particle equations of motion. As a consequence, at this semiclassical level the retarded, advanced and symmetric Lienard-Wiechert solutions coincide by using the equa-
tions of motion, so that there is only one sector of semiclassical Lienard-Wiechert solutions (modulo the incoming radiation). The semiclassical Lienard-Wiechert potential and fields can be expressed as phase space functions and it is possible to eliminate the electromagnetic degrees of freedom by means of second class constraints added to the reduced phase space of the radiation gauge in the rest frame. Having gone to Dirac brackets with respect to these constraints, we get a reduced phase space containing only particles with mutual instantaneous action-at-a-distance interactions. We can find new canonical variables for the particles corresponding to a Darboux basis for these brackets. We can evaluate the final Hamiltonian, showing that besides the Coulomb potential there are vector potentials under the particle kinetic energy square roots (coming from the minimal coupling to the radiation field) and a scalar potential outside them (coming from the energy of the radiation field): due to $Q_i^2 = 0$ one can extract the vector potentials from under the square roots and write a unique effective scalar potential added to the Coulomb one. This can be done both in the original (no longer) canonical variables and in the final canonical basis. The effective scalar potential is the complete Darwin potential: at the lowest order in $1/c^2$ we obtain the known form of the Darwin potential.

We find that in the framework of Maxwell (not Feynman-Wheeler) theory, the semiclassical regularization $Q_i^2 = 0, Q_i Q_j \neq 0$ extracts automatically the instantaneous action-at-a-distance potential hidden in the delay, which as mentioned above turns out to be the same in the retarded and advanced solutions, because the difference is in the $Q_i^2$ terms which depend on the higher accelerations (in QFT these effects are hidden in the radiative corrections coming out from the regularization of the ultraviolet divergences, and this is possible only with the Feynman Green function).

This means that at this semiclassical level, the elimination of the electromagnetic degrees of freedom produces a system of particles with instantaneous action at a distance given by the Coulomb and Darwin potentials. Since $\frac{1}{2}(\text{retarded} - \text{advanced}) = 0$, all the effects now come from the regularized $\text{retarded} = \text{advanced} = \frac{1}{2}(\text{retarded} + \text{advanced})$ solution and there is no mass renormalization. Even if the transverse projector implies contributions
from the whole past (or future) history of the particles, in the semiclassical approximation only the instantaneous action-at-a-distance effects on $\Sigma_W(\tau)$ survive. Each particle feels only the action of the other $N - 1$ [thus giving us an effective Abraham-Lorentz-Dirac equation with no self-reaction and with the Lorentz forces of the other particles replaced by action-at-a-distance interactions]. Like in the Feynman-Wheeler approach [30], we can now speak of radiation only as the effect of the other $N - 1$ particles on the one chosen as a detector of radiation when it is far away from the other particles: equations of motion), one has accord with the Larmor formula coming from the $Q_iQ_j$ interference terms; (the Larmor formula gives zero at the semiclassical level due to the particle equations of motion).

An important and new feature of this formalism is the possibility to find the final canonical variables for the particles after the introduction of the Dirac brackets: their use introduce new higher order contributions to the Darwin potential coming from the kinetic energy square roots. These contributions lead to a substantial cancellation with corresponding terms coming from what began as the electromagnetic energy integral. Our final generalized Darwin interaction naturally divides into two portions. The first portion has the same form either as the original lowest order correction derived originally by Darwin in the retarded case or as the lowest order $1/c^2$ well known form of the Darwin potential in the case of symmetric [$1/2(\text{retarded} + \text{advanced})$] Lienard-Wiechert potentials but with the masses replaced by the kinetic energies, $m_i \to \sqrt{m_i^2 + \kappa_i^2}$, and this is a new result (strictly speaking this is a higher order correction, but it shows that we are using the correct relativistic kinematics without $1/c^2$ expansions). The second portion, a double infinite series, is, like that of the generalization of the $1/c^2$ Darwin potential, is also new. It is of higher order in $1/c^2$ than the more familiar first portion. Furthermore, our generalized Darwin interaction for $N$ bodies is equal to the pairwise sum of two body pieces.

For the restricted case of two bodies considerable simplifications result when one evaluates the series in the center-of-mass rest frame. It then can be written in closed form.

The Darwin potential we obtain can be regarded as the classical analogue of the full effects (complete transverse as well as longitudinal) of the single photon exchange in the
Bethe-Salpeter equation since it is the same order in the coupling constants. The effect of the semiclassical regularization $Q_i^2 = 0$ is to truncate out the classical analogue of the numerous higher order ladder and cross ladder diagrams. To the extent that the Darwin potential we obtain has a low order ($1/c^2$) portion that agrees with the standard result, it would be expected to contribute correctly to the spectral results in a quantized formalism.

For two particles there is the problem of the comparison of the semiclassical Lienard-Wiechert sector with the 2-particle models defined by two first class constraints with covariant instantaneous action-at-a-distance interactions (phenomenological approximations of the Bethe-Salpeter equation). Several authors beginning with [34–36] and continued by [37–39] have developed pairs of commuting generalized mass shell conditions, first class constraints with instantaneous potentials in the center-of-mass system, whose quantization gives coupled Klein-Gordon equations for two spin zero particles [see Refs. [38–41] for similar equations for Dirac particles deriving from pairs of first class constraints for spinning particles]. Some of these models were generated as approximations to the Bethe-Salpeter equation, by reducing it in a covariant instantaneous approximations to a 3-dimensional equation (with the elimination of the spurious abnormal sectors of relative energy excitations) of the Lippmann-Schwinger type and then to the equation of the quasipotential approach [see the bibliography of the quoted references], which Todorov [35] reformulated as a pair of first class constraints at the classical level. In Ref. [39] it is directly shown how the normal sectors of the Bethe-Salpeter equation are connected with the quantization of pairs of first class constraints with instantaneous (in general nonlocal, but approximable with local) potentials like in Todorov’s examples. Ref. [38] shows how to derive the Todorov potential for the electromagnetic and world scalar case from Tetrode-Fokker-Feynman-Wheeler dynamics with scalar and vector potentials [this theory is connected with $\frac{1}{2}(\text{retarded} + \text{advanced})$ solutions with no incoming radiation (adjunct Lienard-Wiechert fields) of Maxwell equations with particle currents in the Lorentz gauge]; besides the Coulomb potential, at the order $1/c^2$ one gets the standard Darwin potential (becoming the Breit one at the quantum level when spin is added), which is known to be phenomenologically correct.
What are the connections between the center-of-mass rest frame form of the two-body interaction Hamiltonian that we develop in this paper to all orders in $1/c^2$ and that obtained in the above references? The Darwin potential becomes a common overlap of the two approaches and thus an important testing ground for the approach we develop in this paper. Furthermore, when in future papers pseudoclassical spin is introduced (extending as in [9] to interacting systems of particles and fields) the types of tests we perform in this paper will be relevant (note, however, there are difficulties with other categories of Darwin type of interactions brought on by the introduction of transverse spin-dependent electric field effects which unlike magnetic fields cannot be diagonalized by a Foldy-Wouthuysen transformation [45,9]).

In Section II we give a review of parametrized Minkowski theories on arbitrary spacelike hypersurfaces.

In Section III we apply this formalism to the isolated system of $N$ charged scalar particles, with Grassmann-valued electric charges, plus the electromagnetic field and we arrive at its rest-frame instant form on the Wigner hyperplanes. We also make the canonical reduction to the radiation gauge.

In Section IV we study the “internal” Poincaré algebra and the “internal” center of mass on the Wigner hyperplanes and we derive the Hamiltonian and Lagrange equations of motion for fields and particles. Also the energy-momentum tensor is evaluated.

In Section V we evaluate the Lienard-Wiechert potentials, we show that at the semi-classical level they depend only on particle coordinates and velocities (since at this level $retarded = advanced$) and we find their phase space expression. Then we eliminate the electromagnetic degrees of freedom by means of second class constraints which force them to coincide with the Lienard-Wiechert solution. We introduce the associated Dirac brackets and we find their canonical Darboux basis.

In Section VI we derive the physical Hamiltonian and the effective Darwin potential to all orders in $1/c^2$ in terms of the old (noncanonical) and of the new (canonical) variables. In
the two-particle case we get a closed form of this potential using the rest-frame condition. Also the final form of the energy-momentum tensor is given.

In the Conclusions there are some general considerations and hints for future developments.

Appendix A gives an explicit summation for the vector potential presented in Section V from the Lienard-Wiechert series in the rest-frame radiation gauge.

In Appendix B we evaluate the field energy and momentum integrals used in Section VI when the electric and magnetic fields are expressed in terms of the Lienard-Wiechert series for the vector potential.

We derive in Appendix C a general formula for a certain quantity, which is important for obtaining the closed form expression of the Darwin potential of Section VI for $N=2$ in the rest frame.

In Appendix D we use a technique similar to that developed by Kerner (and applied in [38] to obtain the Todorov quasipotential from the Wheeler-Feynman action) in the transformation of the Lagrangian expression for the invariant mass to $M$. In this proof, (done to all orders) we must use Dirac brackets since we have used the Lienard-Wiechert constraints as a strong condition on the dynamical variables. This necessitates the explicit expression for the field momentum integrals developed in the earlier appendix from the Lienard-Wiechert solutions.

In Appendix E we obtain a special solution of Hamilton’s equations for the two-body problem that is analogous to Schild’s solution for circular orbits [42].
II. PARAMETRIZED MINKOWSKI THEORY FOR N FREE SCALAR PARTICLES.

In this Section we shall review the description of \(N\) scalar free particles on arbitrary spacelike hypersurfaces \([1]\), leaves of the foliation of Minkowski spacetime associated with one of its 3+1 splittings following the suggestions of Refs. \([43,44]\). The scalar parameter \(\tau\) labelling the hypersurfaces \(\Sigma(\tau)\) allows one to introduce a covariant concept of “equal time”, which will be useful in the description of an isolated system of interacting particles and fields. It will be shown that in these parametrized Minkowski theories there are first class constraints implying the independence of the description of the isolated system from the chosen 3+1 splitting.

As said in the Introduction this requires the addition to the theory of an infinite number of new configuration variables \(z^\mu(\tau, \vec{\sigma})\) describing the spacelike hypersurfaces as an embedding of \(R^3\) into Minkowski spacetime. We use the notation \(\sigma^A = (\sigma^\tau = \tau, \sigma^r)\), i.e. \(\vec{A} = (\tau, \vec{r})\) [the notation \(A = (\tau, r)\) will be used for the Wigner indices on the Wigner hyperplane, see Section III].

In the manifestly Lorentz covariant approach the worldlines of scalar particles are described by 4-vector coordinates \(x_\mu^i(\tau_i)\), where the \(\tau_i\)'s are affine parameters (often they are restricted to be the proper times of the particles). Even if one uses a unique affine parameter \(\tau\) for all the particles, \(x_\mu^i(\tau)\), there is the problem that the particles times \(x_0^i(\tau)\) are gauge variables due to the presence of the first class mass-shell constraints \(p_i^2 - m_i^2 \approx 0\). For each free particle the constraint manifold is the union of two disjoint submanifolds \(p^0 = \pm \sqrt{m_i^2 + \vec{p}_i^2}\). The gauge nature of the \(x_0^i\)'s is connected with: i) the arbitrariness in the choice of the center-of-mass time; ii) the arbitrariness in the choice of how to trigger the \(N\) particles (at equal times or with any conceivable mutual delay; this is the gauge freedom of relative times). Given the foliation of Minkowski spacetime with leaves \(\Sigma(\tau)\) we can give a covariant description of the particles at “equal times” (covariant zero relative times condition) by parametrizing the worldlines as \(x_\mu^i(\tau) = z^\mu(\vec{\eta}_A^i(\tau)) = z^\mu(\tau, \vec{\eta}_i(\tau))\), with
with the Lorentz scalar coordinates $\eta^A_i(\tau) = (\tau, \tilde{r}_i(\tau))$. Only 3 Lorentz scalar coordinates $\tilde{r}_i(\tau)$ identify the intersection of the particle worldline with $\Sigma(\tau)$. This implies that in this description there are no mass-shell constraints, namely that we are describing particles with a well defined sign of the energy: $\eta_i = \text{sign} p^0_i = \pm 1$. In this paper we shall consider only positive energy particles, so that $\eta_i = 1$ for every $i$. There will be a conjugate momentum $\tilde{\kappa}_i(\tau)$ for each particle and the standard momentum $p^\mu_i(\tau)$ will be a derived quantity which satisfies $p^2_i = m_i^2$.

The metric induced on $\Sigma(\tau)$ from the Minkowski metric $\eta^{\mu\nu} = (+-++)$ is

$$g_{AB}(z(\tau, \vec{\sigma})) := \eta_{\mu\nu} z^\mu_A z^\nu_B, \quad (2.1)$$

where we have used the notation

$$z^\mu_A(\tau, \vec{\sigma}) = \frac{\partial z^\mu}{\partial \sigma^A} := \partial_A z^\mu. \quad (2.2)$$

If we define the quantity $z^A_\mu(\tau, \vec{\sigma})$ by means of

$$z^A_\mu z^\mu_B = \delta^A_B, \quad (2.3)$$

they satisfy

$$g_{AB} z^A_\mu z^B_\nu = \eta_{\lambda\kappa} \delta^A_\lambda \delta^B_\kappa z^\mu_\lambda z^\nu_\kappa = \eta_{\mu\nu}. \quad (2.4)$$

Therefore, the $z^A_\mu(\tau, \vec{\sigma})$ are a set of vierbeins, with the $z^\mu_A(\tau, \vec{\sigma})$ the inverse vierbeins.

Since we require $g_{\tau\tau} > 0$ as a condition on the embedding $z^\mu(\tau, \vec{\sigma})$, $z^\mu_\tau$ is a time-like 4-vector and the $z^\mu_\nu(\tau, \vec{\sigma})$’s are spacelike 4-vectors tangent to $\Sigma(\tau)$.

The determinant of the metric is

$$g = -\text{det} |g_{AB}| = (\text{det} z^\mu_A)^2. \quad (2.5)$$

The spatial part of the metric has an associated determinant which is defined by

$$\gamma = -\text{det} |g_{\tilde{r}\tilde{r}}|; \quad \Gamma = \sqrt{\gamma}. \quad (2.6)$$
We next define the inverse metric $g^{AB}$ by

$$g^{AB} g_{BC} = \delta^A_C.$$  

(2.7)

This implies

$$g^{\hat{A}\hat{B}} g_{\hat{A}\hat{B}} = 4 = g^{\hat{A}\hat{B}} z^\mu_A z^\nu_B \eta_{\mu\nu} = \eta^\mu\nu \eta_{\mu\nu},$$  

(2.8)

so that

$$\eta_{\mu\nu} = g^{\hat{A}\hat{B}} z^\mu_A z^\nu_B = g^{\tau\tau} z^\mu_\tau z^\nu_\tau + 2 g^{\tau\rho} z^\mu_\tau z^\rho_\tau + g^{s\bar{s}} z^\mu_s z^\nu_s.$$  

(2.9)

By definition of the element of an inverse matrix

$$g^{\tau\tau} = \frac{\Gamma^2}{g} = \frac{\gamma}{g},$$  

(2.10)

while the inverse of the spatial $g_{rs}$ is defined as $\gamma_{rs}$, that is,

$$\gamma_{rs} = \delta_{rs}.$$  

(2.11)

To find $g^{\tau\bar{u}}$ use the fact that

$$g^{\tau\bar{u}} g_{\tau\bar{s}} + g^{\tau\tau} g_{\tau\bar{s}} = \delta^\tau_{\bar{s}} = 0,$$  

(2.12)

and therefore

$$g^{\tau\bar{u}} g_{\tau\bar{s}} = - g^{\tau\tau} g_{\tau\bar{s}}.$$  

(2.13)

Multiplying by $\gamma^\bar{s}u$ leaves us with

$$g^{\tau\bar{u}} = - \frac{\Gamma^2}{g} g_{\tau\bar{s}} \gamma^\bar{s}u.$$  

(2.14)

Now using this consider

$$g^{\bar{s}u} g_{\bar{s}l} + g^{\bar{r}l} g_{\bar{r}r} = \delta_{\bar{l}l}.$$  

(2.15)

Multiply both sides by $\gamma^\bar{u}l$, use the definition of $\gamma^\bar{u}l$ and the above expression for $g^{\tau\tau}$, and we obtain
\[ g^{\tilde{u}\tilde{v}} = \gamma^{\tilde{u}\tilde{v}} + \frac{\Gamma^2}{g} g_{\tau\tilde{u}} g_{\epsilon \tau} \gamma^{\tilde{v}\epsilon}. \]  

(2.16)

Thus in summary we have expressed the inverse metric in terms of the metric and the inverse of its spatial parts

\[ g_{\tau\tau} = \frac{\Gamma^2}{g}, \]
\[ g^{\tau\tau} = -\frac{\Gamma^2}{g} g_{\tau\tilde{u}} \gamma^{\tilde{u} \epsilon}, \]
\[ g^{\tilde{s}\tilde{s}} = \gamma^{\tilde{s}\tilde{s}} + \frac{\Gamma^2}{g} g_{\tau\tilde{u}} g_{\tau\epsilon} \gamma^{\tilde{u} \epsilon}. \]  

(2.17)

Moreover, we have

\[ \eta_{\mu\nu} \tilde{A}_\mu \tilde{B}_\nu = g^{\tilde{A}\tilde{B}} z_{\mu\nu} \tilde{A}_\mu \tilde{B}_\nu = g^{\tilde{A}\tilde{B}}. \]  

(2.18)

The normal to \( \Sigma(\tau) \) at the point \( z^\mu(\tau, \tilde{\sigma}) \) is the Lorentz four vector

\[ l^\mu(\tau, \tilde{\sigma}) = \frac{1}{\Gamma(\tau, \tilde{\sigma})} \epsilon^{\mu\alpha\beta\gamma} z_{\alpha \beta}(\tau, \tilde{\sigma}) z_{\gamma}(\tau, \tilde{\sigma}). \]  

(2.19)

with the normalization \( l^2(\tau, \tilde{\sigma}) = 1 \). By construction we have \( l_\mu(\tau, \tilde{\sigma}) z^\mu(\tau, \tilde{\sigma}) = 0 \).

The evolution 4-vector \( z^\mu(\tau, \tilde{\sigma}) \) can be decomposed on the mutually orthogonal four vectors \( l^\mu \) and \( z^\mu_\tilde{s} \):

\[ z^\mu(\tau, \tilde{\sigma}) = N(\tau, \tilde{\sigma}) l^\mu(\tau, \tilde{\sigma}) + N^\tilde{s}(\tau, \tilde{\sigma}) z^\mu_\tilde{s}(\tau, \tilde{\sigma}). \]  

(2.20)

where \( N(\tau, \tilde{\sigma}) \) is the lapse and \( N^\tilde{s}(\tau, \tilde{\sigma}) \) the shift functions in the terminology of ADM general relativity. To determine the vector \( N^\tilde{s} \) we use the orthogonality of \( z_{\mu\tilde{s}} \) and \( l^\mu \). That is multiplying both sides of the above equation by \( z_{\mu\tilde{s}} \) and using the expression for \( g_{\tilde{A}\tilde{B}} \) and multiplying by \( \gamma^{\tilde{s}\tilde{u}} \) we obtain

\[ N^\tilde{u}(\tau, \tilde{\sigma}) = g_{\tau\tilde{u}}(\tau, \tilde{\sigma}) \gamma^{\tilde{s}\tilde{u}}(\tau, \tilde{\sigma}). \]  

(2.21)

Using this expression for \( N^\tilde{s} \) we determine the scalar \( N \) by first multiplying the previous equation by \( z_{\mu\tilde{s}} \). Then we use the definition of \( l^\mu \) and the determinant, multiply the result by \( g^{\tau\tau} \). Then use the definition of \( \gamma^{\tilde{s}\tilde{u}} \), and we obtain \( N = \Gamma / \sqrt{\gamma} \). Hence
\[ z_\mu^I(\tau, \vec{\sigma}) = \left[ \frac{\Gamma}{\sqrt{\gamma}} \right]^{\mu} + g_{\tau\tau} \gamma^{\tilde{r}\tilde{s}} z_\tilde{s}^I(\tau, \vec{\sigma}). \] (2.22)

Substituting this expression for \( z_\tau^\mu \) into
\[
\eta^{\mu\nu} = g^{AB} z_A^{\mu} z_B^{\nu} = g^{\tau\tau} z_\tau^{\mu} z_\tau^{\nu} + g^{\tilde{r}\tilde{r}} (z_\tilde{r}^{\mu} z_\tilde{r}^{\nu} + z_\tilde{s}^{\mu} z_\tilde{s}^{\nu}) + g^{\tilde{s}\tilde{s}} \]
(2.23)
together with those for \( g^{AB} \), we obtain after some algebra the following decomposition of the Minkowski metric
\[
\eta^{\mu\nu} = l^{\mu}(\tau, \vec{\sigma}) l^{\nu}(\tau, \vec{\sigma}) + \gamma^{\tilde{r}\tilde{r}}(\tau, \vec{\sigma}) z_\tilde{r}^{\mu}(\tau, \vec{\sigma}) z_\tilde{r}^{\nu}(\tau, \vec{\sigma}). \] (2.24)

Coming back to the scalar particles, the relation between their world line velocities in the two descriptions is
\[
\dot{x}_i^\mu(\tau) = z_\tau^{\mu}(\tau, \vec{\eta}_i(\tau)) + z_\tilde{r}^{\mu}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^{\tilde{r}}(\tau). \] (2.25)

Noting that
\[
\dot{x}_i^2 = \dot{x}_i^\mu \dot{x}_i^\nu \eta_{\mu\nu} = z_\tau^{\mu} z_\tau^{\nu} \eta_{\mu\nu} + 2 z_\tilde{r}^{\mu} z_\tilde{r}^{\nu} \eta_{\mu\nu} \dot{\eta}_i^{\tilde{r}} + z_\tilde{s}^{\mu} z_\tilde{s}^{\nu} \eta_{\mu\nu} \dot{\eta}_i^{\tilde{s}} \dot{\eta}_i^{\tilde{s}} =
\] (2.26)
the standard action for \( N \) free scalar particles becomes
\[
S = \int d\tau \sum_i^N \left[ - m_i \sqrt{\dot{x}_i^2} \right] = \int d\tau L(\tau) = \int d\tau d^3\sigma L(\tau, \vec{\sigma}), \] (2.27)
with the Lagrangian density
\[
L(\tau, \vec{\sigma}) = - \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) m_i \sqrt{g_{\tau\tau}(\tau, \vec{\sigma})} + 2 g_{\tau\tau}(\tau, \vec{\sigma}) \dot{\eta}_i^{\tau}(\tau) + g_{\tilde{r}\tilde{r}}(\tau, \vec{\sigma}) \dot{\eta}_i^{\tilde{r}}(\tau) \dot{\eta}_i^{\tilde{r}}(\tau), \] (2.28)
and the Lagrangian
\[
L(\tau) = - \sum_{i=1}^N m_i \sqrt{g_{\tau\tau}(\tau, \vec{\eta}_i(\tau)) + 2 g_{\tau\tau}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^{\tau}(\tau) + g_{\tilde{r}\tilde{r}}(\tau, \vec{\eta}_i(\tau)) \dot{\eta}_i^{\tilde{r}}(\tau) \dot{\eta}_i^{\tilde{r}}(\tau)}. \] (2.29)

The above action is invariant under separate \( \tau \) and \( \vec{\sigma} \) reparametrization. This leads naturally to constraints.
The canonical momenta are determined from the dependence on the “velocity” $z_{\tau}^{\mu}(\tau, \tilde{\sigma})$ associated with the hypersurface and the particle velocities $\dot{\eta}_{i}(\tau)$.

$$
\rho_{\mu}(\tau, \tilde{\sigma}) = - \frac{\partial L(\tau, \tilde{\sigma})}{\partial z_{\tau}^{\mu}(\tau, \tilde{\sigma})} = \sum_{i=1}^{N} \delta^{3}(\tilde{\sigma} - \tilde{\eta}_{i}(\tau)) \left( m_{i} \frac{z_{\tau \mu}(\tau, \tilde{\sigma}) + z_{\tau \mu}(\tau, \tilde{\sigma}) \dot{\eta}_{i}^{\mu}(\tau)}{\sqrt{g_{\tau \tau}(\tau, \tilde{\sigma}) + 2g_{\tau \tau}(\tau, \tilde{\sigma}) \dot{\eta}_{i}^{\mu}(\tau) + g_{\tau \tau}(\tau, \tilde{\sigma}) \dot{\eta}_{i}^{\mu}(\tau) \dot{\eta}_{i}^{\mu}(\tau)} \right) = [\rho_{\mu}^{L}\nu]_{\mu} + (\rho_{\nu}^{\nu} \gamma^{\tau \tilde{\sigma}} z_{\nu \mu}]_{\tau, \tilde{\sigma}},
$$

Using finally

$$
\kappa_{\nu \tau}(\tau) = - \frac{\partial L(\tau)}{\partial \dot{\eta}_{i}(\tau)} = m_{i} \frac{g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) + g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau)}{\sqrt{g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) + 2g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) + g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) \dot{\eta}_{i}(\tau)}}.
$$

(2.30)

Using the above we have

$$
m_{i}^{2} - \gamma^{\tau \tilde{\sigma}} \kappa_{\nu \tau} \kappa_{i \nu} = m_{i}^{2} \left[ 1 - \frac{\gamma^{\tau \tilde{\sigma}}(g_{\tau \tau} + g_{i \tilde{\sigma}} \delta^{\tau \tilde{\sigma}})}{g_{\tau \tau} + 2g_{\tau \tau} \gamma_{i \tilde{\sigma}} + g_{\gamma \gamma} \gamma_{i \tilde{\sigma}} \gamma_{i \tilde{\sigma}}} \right] = \\
= \left( \frac{m_{i}}{\sqrt{g_{\tau \tau} + 2g_{\tau \tau} \gamma_{i \tilde{\sigma}} + g_{\gamma \gamma} \gamma_{i \tilde{\sigma}} \gamma_{i \tilde{\sigma}}}} \right)^{2} \left( g_{\tau \tau} - \gamma^{\tau \tilde{\sigma}} g_{\tau \tau} \gamma_{\tau \tilde{\sigma}} \right).
$$

(2.31)

Use the following two forms

$$
\rho_{\mu}^{L} = \sum_{i=1}^{N} \frac{\delta^{3}(\tilde{\sigma} - \eta_{i}(\tau)) m_{i} z_{\tau \mu}^{\nu}}{\sqrt{g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) + 2g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) + g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) \dot{\eta}_{i}(\tau)},
$$

$$
z_{\tau} \cdot l = \frac{1}{\gamma^{\mu \alpha \beta \gamma}} z_{\tau \mu} z_{\alpha \beta} z_{\gamma} = \sqrt{g} \frac{1}{\Gamma},
$$

(2.32)

together with the square root of the above relation and $g^{\tau \tilde{\sigma}} = -\frac{g}{\gamma} g_{\tau \tau} \gamma^{\tau \tilde{\sigma}}$ and

$$
g^{\tau \tilde{\sigma}} g_{\tau \tilde{\sigma}} = g^{\tau \alpha} g_{\tau \alpha} - g^{\tau \tau} g_{\tau \tau} = 1 - \frac{\gamma}{g} g_{\tau \tau},
$$

(2.33)

to obtain

$$
\rho_{\mu}^{L} = \sum_{i=1}^{N} \delta^{3}(\tilde{\sigma} - \eta_{i}(\tau)) \sqrt{m_{i}^{2} - \gamma^{\tau \tilde{\sigma}} \kappa_{i \nu} \kappa_{i \nu}}.
$$

(2.34)

Using finally

$$
\rho_{\mu} z_{\nu}^{\mu} = \sum_{i=1}^{N} \frac{\delta^{3}(\tilde{\sigma} - \eta_{i}(\tau)) m_{i} (z_{\tau \mu}^{\nu} z_{\nu \mu} + z_{\nu \nu}^{\nu} \eta_{i}^{\nu})}{\sqrt{g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) + 2g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) + g_{\tau \tau}(\tau, \tilde{\eta}_{i}(\tau)) \dot{\eta}_{i}(\tau) \dot{\eta}_{i}(\tau)}} = \\
= \sum_{i=1}^{N} \delta^{3}(\tilde{\sigma} - \eta_{i}(\tau)) \kappa_{i \nu}.
$$

(2.35)
we obtain the form of the four primary first class constraints $H_\mu$ following from $\tau$ and $\bar{\sigma}$ reparametrization invariance:

$$H_\mu(\tau, \bar{\sigma}) = p_\mu(\tau, \bar{\sigma}) - l_\mu(\tau, \bar{\sigma}) \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))\sqrt{m_i^2 - \gamma^{\bar{r}\bar{s}}(\tau, \bar{\sigma})\kappa_{i\bar{r}}(\tau)\kappa_{i\bar{s}}(\tau)} -$$

$$- z_{\bar{r}\mu}(\tau, \bar{\sigma})\gamma^{\bar{r}\bar{s}}(\tau, \bar{\sigma})\sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))\kappa_{i\bar{s}} \approx 0.$$  \hspace{1cm} (2.36)

Assuming the following Poisson brackets

$$\{z^\mu(\tau, \bar{\sigma}), p_\nu(\tau, \bar{\sigma}')\} = -\eta_{\nu}^{\mu} \delta^3(\bar{\sigma} - \bar{\sigma}') ,$$

$$\{\eta_{i\bar{r}}(\tau), \kappa_{j\bar{s}}(\tau)\} = -\delta_{ij} \delta^3_{\bar{r}\bar{s}}.$$  \hspace{1cm} (2.37)

one can show the first class nature of the above constraints [1]. Since these constraints are solved in terms of one of the independent momenta $[p_\mu(\tau, \bar{\sigma})]$, their Poisson brackets are exactly zero:

$$\{H_\mu(\tau, \bar{\sigma}), H_\nu(\tau, \bar{\sigma}')\} = 0.$$  \hspace{1cm} (2.38)

These constraints imply that the description of the system is independent from the chosen 3+1 splitting of Minkowski spacetime.

The standard particle momenta $p_i^\mu$ are reconstructed as $p_i^\mu = (\sqrt{m_i^2 - \gamma^{\bar{r}\bar{s}}\kappa_{i\bar{r}}\kappa_{i\bar{s}}}; \kappa_i)$ and satisfy $p_i^2 = m_i^2$.

In the next Section we shall add Grassmann-valued electric charges to the scalar particles, we shall find the new action and constraints, eventually arriving at the rest-frame instant form.
III. N CHARGED SCALAR PARTICLES AND THE ELECTROMAGNETIC FIELD.

In this Section we will extend the formalism of the previous Section to the case of an isolated system of $N$ charged scalar particles plus the electromagnetic field. By using Lorentz scalar electromagnetic potentials and field strengths, which employ the covariant “equal time” concept associated with the spacelike hypersurfaces $\Sigma(\tau)$, we define the action for the combined field and particle system, that, like the free system, is separately invariant under $\tau$ and $\vec{\sigma}$ reparametrizations. As in the free particle case we obtain four primary first class constraints $\mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0$ at each point $\vec{\sigma}$ on each of the space-like surfaces $\Sigma(\tau)$, implying the independence from the chosen 3+1 splitting. Moreover, there are the two additional first class constraints describing the electromagnetic gauge invariance of the theory. By using a gauge fixing condition which restricts the $\Sigma(\tau)$ ’s to hyperplanes $\Sigma_H(\tau)$ and by using Dirac brackets, the embedding variables $z^\mu(\tau, \vec{\sigma})$ are reduced to only ten ones. The original constraints $\mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0$ are reduced to just 10 first class global constraints on each of the hyperplanes. We then specialize to hyperplanes orthogonal to the total timelike four-momentum of the system with 6 new gauge fixings depending on the standard Wigner boost for timelike Poincaré orbits. These hyperplanes, defined by the system configuration, are called the Wigner hyperplanes $\Sigma_W(\tau)$. After having defined the new Dirac brackets, we remain with a decoupled “external” canonical non-covariant center of mass, with Wigner-covariant particle and field degrees of freedom on the Wigner hyperplane and with only four first class contraints. One of these four constraints identifies the invariant mass of the system as the effective Hamiltonian, while the other three define the rest-frame condition $\mathcal{H}_p(\tau) \approx 0$ (vanishing of the total 3-momentum inside the Wigner hyperplane) for the combined system of particles and fields. Finally, we make the canonical reduction to eliminate the electromagnetic gauge degrees of freedom, placing the formalism (including the Poincaré generators) in the Wigner covariant rest-frame radiation gauge. Also we give the energy-momentum tensor of the full isolated system.
A. The Action, The Constraints and the Canonical Reduction.

Let us now review the isolated system of $N$ charged scalar particles plus the electromagnetic field following Ref. [1]. Just as on the hypersurface $\Sigma(\tau)$ the positive energy particles are described by coordinates $\vec{\eta}_i(\tau)$ such that $x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$, so the electric charge of each particle is described in a semiclassical way by means of a pair of complex conjugate Grassmann variables $\theta_i(\tau), \theta^*_i(\tau)$ [3] satisfying $[I_i = I^*_i = \theta^*_i \theta_i$ is the generator of the $U_{em}(1)$ group of particle $i]$.

$$\theta^2_i = \theta^{*2}_i = 0, \quad \theta_i \theta^*_i + \theta^*_i \theta_i = 0,$$

$$\theta_i \theta_j = \theta_j \theta_i, \quad \theta^*_i \theta^*_j = \theta^*_j \theta^*_i, \quad i \neq j. \quad (3.1)$$

As said in the Introduction, the formal quantization procedure sends $\theta^*_i, \theta_i$ into the Clifford algebra describing a two-level Fermi oscillator $b^\dagger_i, b_i$, and each Grassmann-valued electric charge $Q_i = e\theta^*_i \theta_i$ goes into $eb^\dagger_i b_i$ with eigenvalues $\pm e$.

The standard electromagnetic potential $A_\mu(z)$ and the field strength $F_{\mu\nu}(z)$ are replaced on $\Sigma(\tau)$ by Lorentz-scalar variables $A_\hat{A}(\tau, \vec{\sigma})$ and $F_{\hat{A}\hat{B}}(\tau, \vec{\sigma})$ respectively, defined by

$$A_\hat{A}(\tau, \vec{\sigma}) = z_\hat{A}^\mu(\tau, \vec{\sigma}) A_\mu(z(\tau, \vec{\sigma})),
F_{\hat{A}\hat{B}}(\tau, \vec{\sigma}) = \partial_{\hat{A}} A_{\hat{B}}(\tau, \vec{\sigma}) - \partial_{\hat{B}} A_{\hat{A}}(\tau, \vec{\sigma}) = z_\hat{A}^{\mu}(\tau, \vec{\sigma}) z_\hat{B}^\nu(\tau, \vec{\sigma}) F_{\mu\nu}(z(\tau, \vec{\sigma})). \quad (3.2)$$

The new potentials $A_\hat{A}(\tau, \vec{\sigma})$ have built-in the covariant concept of “equal time” through their implicit dependence on the embeddings $z^\mu(\tau, \vec{\sigma})$.

With $d^3 \Sigma^\mu$ the surface element of $\Sigma(\tau)$ we have the following volume element of Minkowski space-time

$$d^4 z = z_\mu^\nu d\tau d^3 \Sigma^\mu = d\tau z_\mu^\nu l_\mu \Gamma d^3 \sigma = \sqrt{g} d\tau d^3 \sigma. \quad (3.3)$$

The action now depends on the configuration variables $z^\mu(\tau, \vec{\sigma}), A_\hat{A}(\tau, \vec{\sigma}), \vec{\eta}_i(\tau), \theta_i(\tau)$ and $\theta^*_i(\tau), i = 1, \ldots, N$, and consists of a “kinetic” piece for the complex Grassmann charges $\int \frac{i}{2} [\theta^*_i(\tau) \dot{\theta}_i(\tau) - \dot{\theta}^*_i(\tau) \theta_i(\tau)] d\tau$, the same particle kinetic piece as in the previous Section, the
kinetic term $\int d^4z(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})$ for the electromagnetic field, and the field-particle interaction term $\int Q_iA_\mu(x_i(\tau))\dot{x}_i^\mu(\tau)\,d\tau$:

$$S = \int d\tau d^3\sigma L(\tau, \bar{\sigma}) = \int d\tau L(\tau),$$

$$L(\tau) = \int d^3\sigma L(\tau, \bar{\sigma}),$$

$$\mathcal{L}(\tau, \bar{\sigma}) = \frac{i}{2} \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))[\theta_i^*(\tau)\dot{\theta}_i(\tau) - \dot{\theta}_i^*(\tau)\theta_i(\tau)] -$$

$$- \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))[m_i \sqrt{g_{\tau\tau}(\tau, \bar{\sigma}) + 2g_{\tau\bar{\eta}}(\tau, \bar{\sigma})\dot{\eta}_i^2(\tau) + g_{\bar{\eta}\bar{\eta}}(\tau, \bar{\sigma})\tilde{\eta}_i^2(\tau)} +$$

$$+ Q_i(\tau)(A_\tau(\tau, \bar{\sigma}) + A_{\bar{\eta}}(\tau, \bar{\sigma})\dot{\eta}_i^2(\tau))] -$$

$$- \frac{1}{4} \sqrt{g(\tau, \bar{\sigma})}g^{i\bar{c}}(\tau, \bar{\sigma})g^{\bar{s}D}(\tau, \bar{\sigma})F_{\bar{A}B}(\tau, \bar{\sigma})F_{\bar{C}D}(\tau, \bar{\sigma}),$$

$$Q_i(\tau) = e\theta_i^*(\tau)\dot{\theta}_i(\tau).$$

(3.4)

Since $A_\tau(\tau, \bar{\sigma})$ transforms as a $\tau$-derivative the action is still invariant under separate $\tau$- and $\bar{\sigma}$-reparametrizations as in the free case. In addition it is invariant under the electromagnetic local gauge transformations and under the odd global phase transformations $\delta \theta_i \mapsto i \alpha \theta_i$, generated by the $I_i$'s. The $Q_i = eI_i$ are the constants of motion associated with this last symmetry $[\frac{d}{d\tau}Q_i(\tau) \overset{\text{ev}}{=} 0$, where '$\overset{\text{ev}}{=}$' means evaluated on the solutions of the Euler-Lagrange equations; from now on we shall write $Q_i$ instead of $Q_i(\tau)$].

Since the semiclassical approximation $Q_i^2 = 0$ regularizes the Coulomb self-energy, the criticism of Rohrlich [13] to this action principle [that the minimal coupling term and the electromagnetic field term are ill defined because they diverge on the worldlines of the particles] does not apply.

The canonical momenta are $[E_\tau = F_{\tau\tau} \text{ and } B_\tau = \frac{1}{2} \epsilon_{i\bar{s}\bar{t}}F_{\bar{s}\bar{t}} (\epsilon_{i\bar{s}\bar{t}} = \epsilon^{i\bar{s}\bar{t}})]$ are the “electric” and “magnetic” fields respectively; for $g_{AB} \rightarrow \eta_{AB}$ one gets $\pi^\tau = -E_\tau = E_\tau$]

$$\rho^\mu(\tau, \bar{\sigma}) = -\frac{\partial \mathcal{L}(\tau, \bar{\sigma})}{\partial \dot{\tau}_\mu(\tau, \bar{\sigma})} = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau))m_i$$

$$\frac{\bar{z}_{\tau\mu}(\tau, \bar{\sigma}) + \bar{z}_{\tau\mu}(\tau, \bar{\sigma})\dot{\eta}_i^2(\tau)}{\sqrt{g_{\tau\tau}(\tau, \bar{\sigma}) + 2g_{\tau\bar{\eta}}(\tau, \bar{\sigma})\dot{\eta}_i^2(\tau) + g_{\bar{\eta}\bar{\eta}}(\tau, \bar{\sigma})\tilde{\eta}_i^2(\tau)}} +$$

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The following Poisson brackets are assumed

\[ \{ \pi^\tau (\tau, \bar{\sigma}), \rho_\nu (\tau, \bar{\sigma}') \} = -\eta_{\nu}^{\mu} \delta^3 (\bar{\sigma} - \bar{\sigma}') , \]

\[ \{ A_A (\tau, \bar{\sigma}), \pi^B (\tau, \bar{\sigma}') \} = \eta^B_A \delta^3 (\bar{\sigma} - \bar{\sigma}') , \]

\[ \{ \pi^i (\tau), \pi^j (\tau) \} = -\delta_{ij} \delta^3 , \]

\[ \{ \theta_i (\tau), \pi_{\theta^j} (\tau) \} = -\delta_{ij} , \]

\[ \{ \theta_i (\tau), \pi_{\theta^*} (\tau) \} = -\delta_{ij} . \]  

The Grassmann momenta give rise to the second class constraints \( \pi_{\theta^i} + \frac{i}{2} \theta^i \approx 0, \pi_{\theta^*} + \frac{i}{2} \theta_i \approx 0 \) \[ \{ \pi_{\theta^i}, \pi_{\theta^*} + \frac{i}{2} \theta_i \} = -i \delta_{ij} ] \]. \( \pi_{\theta^i} \) and \( \pi_{\theta^*} \) are then eliminated with the help of Dirac brackets
\[ \{A, B\}^* = \{A, B\} - i[\{A, \pi_{\theta i} + \frac{i}{2} \theta^*_i\} \{\pi_{\theta^* i} + \frac{i}{2} \theta_i, B\} + \{A, \pi_{\theta^* i} + \frac{i}{2} \theta_i\} \{\pi_{\theta i} + \frac{i}{2} \theta^*_i, B\}] \]

(3.7)

so that the remaining Grassmann variables have the fundamental Dirac brackets [which we will still denote \{., .\} for the sake of simplicity]

\[ \{\theta_i(\tau), \theta_j(\tau)\} = \{\theta^*_i(\tau), \theta^*_j(\tau)\} = 0, \]

\[ \{\theta_i(\tau), \theta^*_j(\tau)\} = -i \delta_{ij}. \]

(3.8)

As in the free particle case of Section II, we obtain four primary constraints

\[
\mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma})[T_{\tau\tau}(\tau, \vec{\sigma}) + \sum_{i=1}^{\mathcal{N}} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) × \sqrt{m_i^2 - \gamma^{\bar{f}\bar{s}}(\tau, \vec{\sigma})[\kappa_{\bar{f}\bar{s}}(\tau) - Q_{\bar{f}}A_{\bar{s}}(\tau, \vec{\sigma})][\kappa_{\bar{s}\bar{f}}(\tau) - Q_{\bar{s}}A_{\bar{f}}(\tau, \vec{\sigma})]} - \zeta_{\bar{f}_{\bar{f}}}(\tau, \vec{\sigma})\gamma^{\bar{f}\bar{s}}(\tau, \vec{\sigma})[-T_{\bar{t}\bar{s}}(\tau, \vec{\sigma}) + \sum_{i=1}^{\mathcal{N}} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))[\kappa_{\bar{t}\bar{s}} - Q_{\bar{t}}A_{\bar{s}}(\tau, \vec{\sigma})]]) \approx 0,
\]

(3.9)

where

\[
T_{\tau\tau}(\tau, \vec{\sigma}) = -\frac{1}{2}(\frac{1}{\sqrt{\gamma}} \pi^\tau \eta_{s\bar{s}} - \gamma^s \gamma^\bar{s} F_{s\bar{s}} F_{\bar{s}s})(\tau, \vec{\sigma}),
\]

\[
T_{\bar{t}\bar{s}}(\tau, \vec{\sigma}) = -F_{s\bar{t}}(\tau, \vec{\sigma}) \pi^\tau(\tau, \vec{\sigma}) = -\epsilon_{\bar{t}\bar{a}} \eta_{\bar{a}s} \pi^\tau(\tau, \vec{\sigma}) B_{\bar{a}}(\tau, \vec{\sigma}) = \pi^\tau(\tau, \vec{\sigma}) × B(\tau, \vec{\sigma}),
\]

(3.10)

are the energy density and the Poynting vector respectively. We use the notation \((\vec{\pi} × \vec{B})_s = (\vec{E} × \vec{B})_s\) because it is consistent with \(\epsilon_{\bar{t}\bar{a}} \pi^\tau B_{\bar{a}}\) in the flat metric limit \(g_{\bar{A}\bar{B}} \rightarrow \eta_{\bar{A}\bar{B}}\); in this limit \(T_{\tau\tau} \rightarrow \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\).

This form of the constraint displays both the tangential \(z^\mu_\tau(\tau, \vec{\sigma})\) and normal \(l^\mu(\tau, \vec{\sigma})\) components of the momentum \(\rho^\mu(\tau, \vec{\sigma})\) conjugate to the embedding variables \(z^\mu(\tau, \vec{\sigma})\).

Again, being solved in terms of the momenta \(\rho_\mu(\tau, \vec{\sigma})\), these constraints are first class with exactly zero Poisson brackets \(\{\mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma'})\} = 0\) and their existence implies once again that the description of the system is independent of the choice of foliation.

Moreover, we have the (Lorentz scalar) primary constraints of the electromagnetic field connected with the gauge invariance of the action
\[ \pi^\tau(\tau, \vec{\sigma}) \approx 0. \quad (3.11) \]

From these constraints we construct the Dirac Hamiltonian. But first we must construct the canonical Hamiltonian \( H_C \). The canonical Hamiltonian is

\[
H_c = -\sum_{i=1}^{N} \kappa_{\mu}(\tau) \eta_i^\mu(\tau) + \int d^3\sigma [\pi^A(\tau, \vec{\sigma}) \partial_\tau A^A(\tau, \vec{\sigma}) - \rho_\mu(\tau, \vec{\sigma}) z^\mu(\tau, \vec{\sigma}) - \mathcal{L}(\tau, \vec{\sigma})] =
= \int d^3\sigma \left[ \partial_\tau (\pi^r(\tau, \vec{\sigma}) A_r(\tau, \vec{\sigma})) - A_r(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}) \right] - \int d^3\sigma A_r(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}), \quad (3.12)
\]

after the elimination of a surface term.

Note that because of the \( \tau \) and \( \vec{\sigma} \) reparametrization invariance, \( H_C \) nearly vanishes, except for the portion involving

\[
\Gamma(\tau, \vec{\sigma}) \equiv \partial_\tau \pi^r(\tau, \vec{\sigma}) - \sum_{i=1}^{N} Q_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)). \quad (3.13)
\]

Thus the Dirac Hamiltonian is (\( \lambda^\mu(\tau, \vec{\sigma}) \) and \( \lambda_r(\tau, \vec{\sigma}) \) are Dirac multipliers)

\[
H_D = \int d^3\sigma [\lambda^\mu(\tau, \vec{\sigma}) \mathcal{H}_\mu(\tau, \vec{\sigma}) + \lambda_r(\tau, \vec{\sigma}) \pi^r(\tau, \vec{\sigma}) - A_r(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma})]. \quad (3.14)
\]

The requirement that the primary constraints be \( \tau \) independent (\( \{\pi^\tau(\tau, \vec{\sigma}), H_D\} \approx 0, \{\mathcal{H}^\mu(\tau, \vec{\sigma}), H_D\} \approx 0 \)) leads only to the Gauss’s law secondary constraint

\[
\Gamma(\tau, \vec{\sigma}) \approx 0. \quad (3.15)
\]

Since the embedding variables \( z^\mu(\tau, \vec{\sigma}) \) are the only configuration variables with Lorentz indices, the ten conserved generators of the Poincaré transformations are:

\[
P^\mu = p^\mu_s = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}),
J^{\mu\nu} = J^{\mu\nu}_s = \int d^3\sigma (z^\mu \rho^\nu - z^\nu \rho^\mu)(\tau, \vec{\sigma}), \quad (3.16)
\]

(the subscript \( s \) stands for hypersurface variable). From the first of these we obtain

\[
\{z^\mu(\tau, \vec{\sigma}), p^\nu_s\} = -\eta^{\mu\nu}. \quad (3.17)
\]
We can restrict ourselves to foliations whose leaves are spacelike hyperplanes \( \Sigma^\tau_H \) with constant timelike normal \( b^\mu_\dot{r} = 0 \), by imposing the following gauge fixings \([x^\mu_s \text{ is an arbitrary origin}]
\)

\[
\zeta^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) - x^\mu_s(\tau) - b^\mu_\dot{r}(\tau)\sigma^\dot{r} \approx 0.
\] (3.18)

In this expression \( b^\mu_\dot{r}(\tau), \dot{r} = 1, 2, 3 \) are three orthonormal vectors, such that the constant and future pointing normal to the hyperplane \( \Sigma^\tau_H \) \([b^\mu_A = (b^\mu_\tau, b^\mu_\dot{r}) \text{ are orthonormal tetrads}]\)

\[
l^\mu(\tau, \vec{\sigma}) \approx l^\mu = b^\mu_\tau = \varepsilon^\mu_{\alpha\beta\gamma} b^\alpha_\tau b^\beta_\dot{1}(\tau) b^\gamma_\dot{3}(\tau).
\] (3.19)

Using the definitions of the vierbeins we obtain from the above gauge fixing the simplifications

\[
\begin{align*}
z^\mu_\dot{r}(\tau, \vec{\sigma}) & \approx b^\mu_\dot{r}(\tau), \\
z^\mu_\tau(\tau, \vec{\sigma}) & \approx i^\mu_s(\tau) + \dot{b}^\mu_\tau(\tau)\sigma^\dot{r}, \\
g_{\dot{r}\dot{s}}(\tau, \vec{\sigma}) & \approx -\delta_{\dot{r}\dot{s}}, \quad \gamma^{\dot{r}\dot{s}}(\tau, \vec{\sigma}) \approx -\delta^{\dot{r}\dot{s}}, \quad \gamma(\tau, \vec{\sigma}) \approx 1,
\end{align*}
\] (3.20)

as well as a natural decomposition of the Lorentz generators into orbital and spin portions

\[
\begin{align*}
J^\mu_{\dot{s}s} &= x^\mu_s p^\nu_s - x^\nu_s p^\mu_s + S^\mu_{\dot{s}s}, \\
S^\mu_{\dot{s}s} &= b^\mu_\dot{r}(\tau) \int d^3\sigma \sigma^\dot{r} \rho^\dot{r}(\tau, \vec{\sigma}) - b^\mu_\tau(\tau) \int d^3\sigma \sigma^\dot{r} \rho^\dot{s}(\tau, \vec{\sigma}).
\end{align*}
\] (3.21)

Here \( S^\mu_{\dot{s}s} \) is the spin part of the Lorentz generators.

These gauge fixings have the following Poisson brackets with the primary constraint \( \mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0 \)

\[
\{\zeta^\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma})\} = -\eta^\mu_\nu \delta^3(\vec{\sigma} - \vec{\sigma}').
\] (3.22)

Therefore, we get a continuum set of second class constraints. They can be eliminated by forming the Dirac brackets
For example, one finds that
\[
\{x_\mu^s, p_\nu^s\}^* = -\eta^{\mu\nu}.
\] (3.24)

In this way the infinity of continuum hypersurface degrees of freedom \(z^\mu(\tau, \bar{\sigma})\), \(\rho_\mu(\tau, \bar{\sigma})\), are reduced to 20: i) 8 are \(x_\mu^s(\tau)\), \(p_\nu^s\); ii) 12 are the 6 independent pairs of canonical variables hidden in \(b_\mu^s\) and \(S^\mu_\nu = J^\mu_\nu - (x_\mu^s p_\nu^r - x_\nu^r p_\mu^s)\) [they have the following brackets consistent with the orthonormality of the tetrads \(b_\mu^s\) [1]: \(\{b_\mu^s, b_\nu^r\}_B = 0\), \(\{S^\mu_\nu, b_\rho_\sigma\}_A = \eta^{\mu\nu}b_\rho^s - \eta^\rho\sigma b_\mu^s\), \(\{S^\mu_\nu, S^{\alpha\beta}\} = C^\mu_\gamma_\delta S^{\gamma\delta}_\nu\), where \(C^\mu_\gamma_\delta = \delta_\gamma^\epsilon\delta_\delta^\sigma\eta^{\epsilon\sigma} + \delta_\gamma^\mu\delta_\delta^\nu\eta^{\mu\nu} - \delta_\gamma^\mu\delta_\delta^\nu\eta^{\sigma\sigma} - \delta_\gamma^\nu\delta_\delta^\mu\eta^{\sigma\sigma}\) are the structure constants of the Lorentz algebra].

It can be shown [1] that the following 10 first class constraints survive at the level of the Dirac brackets [they are 10 combinations of the primary constraints whose gauge freedom is not fixed by the gauge fixings (3.18)]

\[
\tilde{\mathcal{H}}^\mu(\tau) = \int d^3\sigma \mathcal{H}^\mu(\tau, \bar{\sigma}) = \\
= p_\mu^s - \bar{\mu}\left[\sum_{i=1}^{N} \sqrt{m_i^2 + [\bar{\kappa}_i(\tau) - Q_i A_i(\tau, \bar{\eta}_i(\tau))]^2}\right] + \\
+ \frac{1}{2} \int d^3\sigma [\bar{\pi}^2 + \bar{B}^2](\tau, \bar{\sigma}) + \\
+ b_\nu^s(\tau) \left[\sum_{i=1}^{N} [\kappa_{ip}(\tau) - Q_i A_p(\tau, \bar{\eta}_i(\tau))]\right] + \int d^3\sigma [\bar{\pi} \times \bar{B}]_p(\tau, \bar{\sigma}) \approx 0, \quad (3.25)
\]

and

\[
\tilde{\mathcal{H}}^\mu(\tau) = b_\nu^s(\tau) \int d^3\sigma \sigma^\mu \mathcal{H}^\nu(\tau, \bar{\sigma}) - b_\nu^s(\tau) \int d^3\sigma \sigma^\mu \mathcal{H}^\nu(\tau, \bar{\sigma}) = \\
= S^{\mu_\nu}_\sigma - [b_\nu^s(\tau)b_\nu^r - b_\nu^r(\tau)b_\nu^s] \left[\sum_{i=1}^{N} \eta_i^s(\tau)\sqrt{m_i^2 + [\bar{\kappa}_i(\tau) - Q_i A_i(\tau, \bar{\eta}_i(\tau))]^2}\right] + \\
+ \frac{1}{2} \int d^3\sigma \sigma^\mu [\bar{\pi}^2 + \bar{B}^2](\tau, \bar{\sigma}) - \\
- [b_\nu^s(\tau)b_\nu^r(\tau) - b_\nu^r(\tau)b_\nu^s(\tau)] \left[\sum_{i=1}^{N} \eta_i^s(\tau)[\kappa_i^s(\tau) - Q_i A_i(\tau, \bar{\eta}_i(\tau))]\right] + \\
+ \int d^3\sigma \sigma^\mu [\bar{\pi} \times \bar{B}]^s_p(\tau, \bar{\sigma}) \approx 0. \quad (3.26)
\]
These constraints say that $p_s^\mu$ coincides with the total 4-momentum of the isolated system and that $S_s^{\mu\nu}$ is determined by its spin tensor.

The Dirac Hamiltonian becomes

$$H_D = \tilde{\lambda}_\mu(\tau)\tilde{H}^\mu(\tau) + \tilde{\lambda}_{\mu\nu}(\tau)\tilde{H}^{\mu\nu}(\tau) + \int d^3\sigma[c_\tau(\tau,\vec{\sigma})\pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\Gamma(\tau,\vec{\sigma})],$$

where, due to the associated Hamilton equations, the new Dirac multipliers have the following interpretation [1]:

$$\tilde{\lambda}^\mu(\tau) = -\dot{x}^\mu(\tau), \quad \tilde{\lambda}^{\mu\nu}(\tau) = -\dot{\lambda}^{\mu\nu}(\tau) \approx \frac{1}{2}[\tilde{b}_\tau^\mu(\tau)\tilde{b}_\tau^\nu(\tau) - \tilde{b}_\tau^\nu(\tau)\tilde{b}_\tau^\mu(\tau)].$$

Restricting our considerations to configurations with $p_s^2 > 0$, we make a further canonical reduction to the special foliation whose hyperplanes are orthogonal to $p_s^\mu$. These hyperplanes are intrinsically determined by the system itself and are called the Wigner hyperplanes $\Sigma_W(\tau)$. They can be identified [1] by requiring the gauge fixings $b_A^\mu(\tau) \approx L_A^\mu(\tau) = 0$, where $L_A^\mu(\tau)$ is the standard boost for timelike Poincaré orbits. This implies $\upsilon^\mu = b_\tau^\mu \approx p_s^\mu / \sqrt{p_s^2}$.

The rest frame form of a timelike fourvector $p^\mu$ is

$$\hat{p}^\mu = \eta_\mu\sqrt{p^2}(1; \vec{0}) = \eta^\mu \eta_\mu \sqrt{p^2}, \quad \hat{p}^2 = p^2,$$

where $\eta = \text{sign } p^0$. Since we restricted ourselves to positive energy particles, $\eta_i = +1$, we shall put $\eta = 1$. The standard Wigner boost transforming $\hat{p}^\mu$ into $p^\mu$ is

$$L_\upsilon^\mu(p, \hat{p}) = \epsilon_\upsilon^\mu(u(p)) =$$

$$= \eta_\upsilon^\mu + \frac{2P_\upsilon^\mu\hat{p}_\upsilon}{p^2} - \frac{(p^\mu + \hat{p}^\mu)(p_\upsilon + \hat{p}_\upsilon)}{p^\mu \hat{p}^\mu + p^2} =$$

$$= \eta_\upsilon^\mu + 2u^\mu(p)u_\mu(\hat{p}) - \frac{(u^\mu(p) + u^\mu(\hat{p}))(u_\upsilon(p) + u_\upsilon(\hat{p}))}{1 + u^0(p)},$$

$$\nu = 0 \quad \epsilon_0^\mu(u(p)) = u^\mu(p) = p^\mu / \sqrt{p^2};$$

$$\nu = \tau \quad \epsilon_\tau^\mu(u(p)) = (-u_\tau(p); \delta^\mu_\tau - \frac{u^\mu(p)u_\tau(p)}{1 + u^0(p)}).$$

The inverse of $L_\nu^\mu(p, \hat{p})$ is $L_\nu^\mu(\hat{p}, p)$, the standard boost to the system rest frame, defined by

$$L_\nu^\mu(\hat{p}, p) = L_\nu^\mu(p, \hat{p}) = L_\nu^\mu(p, \hat{p})|_{p \rightarrow -\hat{p}}.$$  (3.28)

We also use these boosts to define the following vierbeins [the $\epsilon_\upsilon^\mu(u(p))$’s are also called
polarization vectors; the indices \( r, s \) will be used for \( A = 1, 2, 3 \) and \( \bar{\alpha} \) for \( A = 0 \)

\[
\epsilon^\mu_A(u(p)) = L^\mu_A(p, \vec{p}),
\]

\[
\epsilon^\mu_\bar{\alpha}(u(p)) = \eta_{\mu\alpha}\epsilon^\nu_{\bar{\alpha}}(u(p)),
\]

\[
\epsilon^\mu_\bar{\alpha}(u(p)) = \eta_{\mu\alpha}\epsilon^\nu_{\bar{\alpha}}(u(p)) = u_\mu(p),
\]

\[
\epsilon^\mu_r(u(p)) = -\delta^{rs}\eta_{\mu\alpha}\epsilon^\nu_s(u(p)) = (\delta^{rs}u_s(p); \delta^r_j - \delta^{rs}\delta_{jh}u_h(p) + \delta^{rs}u_o(p)),
\]

\[
\epsilon^\mu_o(u(p)) = u_A(p), \tag{3.29}
\]

which satisfy

\[
\epsilon^\mu_A(u(p))\epsilon^\nu_A(u(p)) = \eta^\mu_\nu,
\]

\[
\epsilon^\mu_A(u(p))\epsilon^\nu_B(u(p)) = \eta^A_B,
\]

\[
\eta^{\mu\nu} = \epsilon^\mu_A(u(p))\eta^{AB}\epsilon^\nu_B(u(p)) = u^\mu(p)u^\nu(p) - \sum_{r=1}^{3} \epsilon^\mu_r(u(p))\epsilon^\nu_r(u(p)),
\]

\[
\eta^{AB} = \epsilon^\mu_A(u(p))\eta^{\mu\nu}\epsilon^\nu_B(u(p)),
\]

\[
p_\alpha\frac{\partial}{\partial p_\alpha}\epsilon^\mu_A(u(p)) = p_\alpha\frac{\partial}{\partial p_\alpha}\epsilon^\mu_A(u(p)) = 0. \tag{3.30}
\]

With the Wigner rotation corresponding to the Lorentz transformation \( \Lambda \) being

\[
R^{\mu\nu}(\Lambda, p) = [L(\vec{p}, p)\Lambda^{-1}L(\Lambda p, \vec{p})]^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & R^i_j(\Lambda, p) \end{pmatrix},
\]

\[
R^i_j(\Lambda, p) = (\Lambda^{-1})^i_j - \frac{(\Lambda^{-1})^i_\alpha p_\beta(\Lambda^{-1})^\beta_j}{p_\rho(\Lambda^{-1})^\rho_\alpha + \eta\sqrt{p^2}} - \frac{p^i}{p^\alpha + \eta\sqrt{p^2}}[(\Lambda^{-1})^\alpha_j - \frac{(\Lambda^{-1})^\alpha_\rho p_\beta(\Lambda^{-1})^\beta_j}{p_\rho(\Lambda^{-1})^\rho_\alpha + \eta\sqrt{p^2}}]. \tag{3.31}
\]

we have that the polarization vectors transform under the Poincaré transformations \((a, \Lambda)\) in the following way:

\[
\epsilon^\mu_r(u(\Lambda p)) = (R^{-1})^r_s \Lambda^{\mu}_{\nu} \epsilon^\nu_s(u(p)). \tag{3.32}
\]

These boosts can be used to obtain the further canonical reduction referred to above. This takes place in two steps: i) firstly, one boosts to the rest frame the variables \( b^\mu_A, S^\mu_{\nu}, \)
with the standard Wigner boost $L^\mu \nu(p_s, p_s)$ for timelike Poincaré orbits; ii) then, one adds the gauge-fixings $b^\mu_A - L^\mu_A(p_s, p_s) \approx 0$ and goes to Dirac brackets. It can be shown [1] that after this special gauge fixing the Lorentz scalar 3-indices $\tilde{r}$ become Wigner spin 1 3-indices $r$. Therefore, we get a rest-frame instant form of dynamics with Wigner covariance.

The Lorentz generators become $J_{s}^{\mu \nu} = \tilde{x}_{s}^{\mu} p_{s}^{\nu} - \tilde{p}_{s}^{\nu} \tilde{x}_{s}^{\mu} + \tilde{S}_{s}^{\mu \nu}$ with $\tilde{S}_{s}^{\mu \nu}$ given in Eq.(59) of Ref. [1]. We now get $\hat{H}^{\mu \nu}(\tau) \equiv 0$, i.e. $S_{s}^{\mu \nu}$ is forced to coincide with the spin tensor of the isolated system.

If we define the rest-frame spin tensor

$$S_{s}^{AB} = \epsilon_{\mu}^{A}(u(p_s)) \epsilon_{\nu}^{B}(u(p_s)) S_{s}^{\mu \nu} \equiv \left[ \eta_{\mu}^{A} \eta_{\nu}^{B} - \eta_{\mu}^{B} \eta_{\nu}^{A} \right] \left[ \frac{1}{2} \int d^{3} \sigma \sigma^{\tilde{r}} \left[ \tilde{\pi}^{2} + \tilde{B}^{2} \right](\tau, \tilde{\sigma}) + \right]$$

$$+ \sum_{i=1}^{N} \eta_{\mu}^{i}(\tau) \sqrt{m_{i}^{2} + \left[ \tilde{\kappa}_{i}(\tau) - Q_{i} \tilde{A}(\tau, \tilde{\eta}_{i}(\tau)) \right]^{2}} - \left[ \eta_{\mu}^{A} \eta_{\nu}^{B} - \eta_{\mu}^{B} \eta_{\nu}^{A} \right] \left[ \int d^{3} \sigma \sigma^{\tilde{r}} \left[ \tilde{\pi} \times \tilde{B} \right]_{s}(\tau, \tilde{\sigma}) + \right]$$

$$+ \sum_{i=1}^{N} \eta_{\mu}^{i}(\tau) \left[ \kappa_{i}^{s}(\tau) - Q_{i} A^{s}(\tau, \tilde{\eta}_{i}(\tau)) \right],$$

$$S_{s}^{rs} \equiv \sum_{i=1}^{N} \left( \eta_{r}^{i}(\tau) \left[ \kappa_{i}^{s}(\tau) - Q_{i} A^{s}(\tau, \tilde{\eta}_{i}(\tau)) \right] - \right.$$

$$\left. - \sum_{i=1}^{N} \eta_{r}^{i}(\tau) \left[ \kappa_{i}^{r}(\tau) - Q_{i} A^{r}(\tau, \tilde{\eta}_{i}(\tau)) \right] + \right.$$

$$\left. + \int d^{3} \sigma \left[ \sigma^{\tilde{r}} \left[ \tilde{\pi} \times \tilde{B} \right]_{s}(\tau, \tilde{\sigma}) - \sigma^{s} \left[ \tilde{\pi} \times \tilde{B} \right]_{r}(\tau, \tilde{\sigma}) \right), \right.$$

$$S_{s}^{0r} \equiv - \sum_{i=1}^{N} \eta_{0}^{i}(\tau) \sqrt{m_{i}^{2} + \left[ \tilde{\kappa}_{i}(\tau) - Q_{i} \tilde{A}(\tau, \tilde{\eta}_{i}(\tau)) \right]^{2}} -$$

$$- \frac{1}{2} \int d^{3} \sigma \sigma^{\tilde{r}} \left[ \tilde{\pi}^{2} + \tilde{B}^{2} \right](\tau, \tilde{\sigma}). \right.$$ (3.33)

it can be shown that the form of $S_{s}^{\mu \nu}$ implies that the rest-frame “external” Poincaré generators are [1]

$$J_{s}^{ij} : = \tilde{x}_{s}^{i} p_{s}^{j} - \tilde{p}_{s}^{j} \tilde{x}_{s}^{i} + \delta^{ij} \delta^{rs} S_{s}^{rs},$$

$$J_{s}^{oi} : = \tilde{x}_{s}^{o} p_{s}^{i} - \tilde{p}_{s}^{i} \tilde{x}_{s}^{o} - \frac{\delta^{ij} S_{s}^{rs} p_{s}^{s}}{p_{s}^{i} + \eta_{s} \sqrt{p_{s}^{2}}},$$

(3.34)
Only in this special gauge do we get the separation of a decoupled “external” canonical non-covariant center of mass described by the 4 pairs \( \hat{x}_s^\mu(\tau), p_s^\mu \), of canonical variables \( \{\hat{x}_s^\mu, p_s^\mu\}^{**} = -\eta^{\mu\nu} \) identifying the Wigner hyperplane \( \Sigma_W(\tau) \) [see Eq.(59) of Ref. [1] for the expression of \( \hat{x}_s^\mu(\tau) \) in terms of \( x_s^\mu \) and of the spin tensor] and of the “internal” Wigner-covariant canonical variables \( \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), A_A(\tau, \vec{\sigma}), \pi^A(\tau, \vec{\sigma}) \) living inside the Wigner hyperplane and with the Dirac brackets coinciding with the original Poisson brackets \( \{\eta_i^r(\tau), \kappa_j^s(\tau)\}^{**} = -\delta_{ij}\delta^r_s, \{A_A(\tau, \vec{\sigma}), \pi^B(\tau, \vec{\sigma}')\} = \eta_A^B \delta^3(\vec{\sigma} - \vec{\sigma}') \).

As shown in Ref. [1] one can replace \( \hat{x}_s^\mu, p_s^\mu \) with a new canonical basis \( T_s = p_s \cdot \hat{x}_s/\sqrt{p_s^2} = p_s \cdot x_s/\sqrt{p_s^2} \) (it is the Lorentz-invariant rest frame time), \( \varepsilon_s = \sqrt{p_s^2}, z_s = \sqrt{p_s^2}(\hat{x}_s - \hat{x}_s p_s/p_s^2), k_s = \tilde{u}(p_s) \) with \( z_s \) having the same covariance of the Newton-Wigner position operator under the little group \( O(3) \) of the timelike Poincaré orbits. The 3-position canonical variable \( z_s/\varepsilon_s \) is the classical background of this operator and describes the decoupled “external” 3-center of mass, whose 4-position is \( \hat{x}_s^\mu \).

In this special gauge there is no restriction on \( p_s^\mu \): the four velocity \( u^\mu(p_s) = p_s^\mu/\sqrt{p_s^2} = l^\mu \) describes the orientation of the Wigner hyperplane with respect to an arbitrary Lorentz frame.

We obtain the following form for the constraints \( \hat{H}^\mu(\tau) \approx 0 \):

\[
\hat{H}^\mu(\tau) = \int d^3\sigma \hat{H}^\mu(\tau, \vec{\sigma}) = p_s^\mu - \\
- u^\mu(p) \left( \frac{1}{2} \int d^3\sigma [\vec{p}^2 + \vec{B}_s^2](\tau, \vec{\sigma}) + \\
+ \sum_{i=1}^N \sqrt{m_i^2 + [\vec{k}_i(\tau) - Q_i \vec{A}(\tau, \vec{\eta}_i(\tau))|^2} \right) - \\
- \epsilon^\mu(u(p)) \left( \int d^3\sigma [\vec{\pi} \times \vec{B}](\tau, \vec{\sigma}) + \\
+ \sum_{i=1}^N [\vec{k}_i(\tau) - Q_i \vec{A}(\tau, \vec{\eta}_i(\tau))|^r \right) \right) \approx 0. \tag{3.35}
\]

Their projections along the normal and the tangents to the Wigner hyperplane are

\[
\mathcal{H}(\tau) = u^\mu(p_s) \hat{H}_\mu(\tau) = \\
= \sqrt{p_s^2} - \left( \sum_{i=1}^N \sqrt{m_i^2 + [\vec{k}_i(\tau) - Q_i \vec{A}(\tau, \vec{\eta}_i(\tau))|^2} \right)
\]

33
\[ + \frac{1}{2} \int d^3 \sigma [\vec{\pi}^2 + \vec{B}^2](\tau, \vec{\sigma})] \approx 0, \quad (3.36) \]

\[ \tilde{\mathcal{H}}_p(\tau) = \sum_{i=1}^{N} [\vec{\kappa}_i(\tau) - Q_i \vec{A}(\tau, \vec{\eta}_i(\tau))] + \int d^3 \sigma [\vec{\pi} \times \vec{B}](\tau, \vec{\sigma}) \approx 0. \quad (3.37) \]

The first one gives the mass spectrum of the isolated field plus particle system, while the other three say that the total 3-momentum of the \( N \) charged particles plus fields vanishes inside the Wigner hyperplane \( \Sigma_W(\tau) \). This condition is the rest-frame condition identifying the Wigner hyperplane as the rest frame of the isolated system.

The Dirac Hamiltonian is now
\[
H_D = \tilde{\lambda}(\tau) \tilde{\mathcal{H}}_p(\tau) = \lambda(\tau) \mathcal{H}(\tau) - \tilde{\lambda}(\tau) \cdot \tilde{\mathcal{H}}_p(\tau),
\]
with
\[
\lambda(\tau) \approx -\dot{x}_{s\mu}(\tau) u_{\mu}(ps), \quad \lambda_r(\tau) \approx -\dot{x}_{s\mu}(\tau) \epsilon_{\mu}^r(u(ps)).
\]

The two additional electromagnetic constraints are
\[
\pi^\tau(\tau, \vec{\sigma}) \approx 0, \quad \Gamma(\tau, \vec{\sigma}) \approx 0. \quad (3.38)
\]

In the rest-frame instant form of dynamics on the Wigner hyperplanes they are Lorentz scalar constraints \([A_r(\tau, \vec{\sigma}) \text{ and } \pi^\tau(\tau, \vec{\sigma}) \text{ are Lorentz scalars, while } \vec{A}(\tau, \vec{\sigma}) \text{ and } \vec{\pi}(\tau, \vec{\sigma}) \text{ are spin-1 Wigner 3-vectors}]\).

We now eliminate the electromagnetic gauge degrees of freedom by decomposing the above spin-one Wigner 3-vector canonical field variables into their transverse and longitudinal components \([1]\)
\[
\vec{A}(\tau, \vec{\sigma}) = \vec{A}_\perp(\tau, \vec{\sigma}) - \frac{\vec{\partial}}{\Delta} \vec{\partial} \cdot \vec{A}(\tau, \vec{\sigma}),
\]
\[
\vec{\pi}(\tau, \vec{\sigma}) = \vec{\pi}_\perp(\tau, \vec{\sigma}) - \frac{\vec{\partial}}{\Delta} \vec{\partial} \cdot \vec{\pi}(\tau, \vec{\sigma}) \approx \vec{\pi}_\perp(\tau, \vec{\sigma}) + \frac{\vec{\partial}}{\Delta} \sum_{i=1}^{N} Q_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)), \quad (3.39)
\]
with \( \Delta = -\vec{\partial}^2 \).

We re-express everything in terms of the Dirac observables: i) \( \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma}) \),
\[
\{\vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma})\} = -P^{rs}_\perp(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}') \quad [P^{rs}_\perp(\vec{\sigma}) = \delta^{rs} + \frac{\sigma_r \sigma_s}{\Delta}] \text{ for the electromagnetic...}
field; ii) \( \tilde{n}_i(\tau), \tilde{k}_i(\tau) = \tilde{n}_i(\tau) + Q_i \frac{\tilde{\mathbf{A}}}{\tilde{\mathbf{r}}} \cdot \tilde{\mathbf{A}}(\tau, \tilde{\sigma}) \) for the particles [they now become dressed with a Coulomb cloud]; iii) \( \tilde{\theta}_i(\tau), \tilde{\dot{\theta}}_i(\tau) \), such that \( Q_i = e\tilde{\theta}_i = e\tilde{\theta}_i^* \).

This is known as the Wigner-covariant rest-frame radiation gauge. Note that \( \tilde{n}_i(\tau) - Q_i \tilde{A}(\tau, \tilde{n}_i(\tau)) = \tilde{n}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{n}_i(\tau)) \). Using the Gauss law constraint \( \tilde{\partial} \cdot \tilde{\pi}(\tau, \tilde{\sigma}) \approx \sum Q_i \delta^3(\tilde{\sigma} - \tilde{n}_i) \) with \( \frac{1}{4\pi} \delta^3(\tilde{\sigma} - \tilde{n}_i) = -1/(4\pi |\tilde{\sigma} - \tilde{n}_i|) \) and integrating by parts we separate out the Coulomb portion of the rest frame energy from the field energy integral. A similar procedure on the field momentum integral simplifies the rest frame condition (and the expression for the internal angular momentum in the next section). Thus we find that the reduced form of the 4 constraints is

\[
\mathcal{H}(\tau) = \epsilon_s - \left\{ \sum_{i=1}^{N} \sqrt{m_i^2 + (\tilde{n}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{n}_i(\tau)))^2} + \right.
\sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\tilde{n}_i(\tau) - \tilde{n}_j(\tau)|} + \left. \int d^3\sigma \frac{1}{2} (\tilde{\pi}^2 + \tilde{\mathbf{B}}^2)(\tau, \tilde{\sigma}) \right\} = \\
= \epsilon_s - M \approx 0, \\
\tilde{\mathcal{H}}_\rho(\tau) = \sum_{i} \tilde{k}_i(\tau) + \int d^3\sigma [\tilde{\pi}_\perp \times \tilde{\mathbf{B}}](\tau, \tilde{\sigma}) \approx 0, \\
\tag{3.40}
\]
rest-frame conditions $\mathcal{H}_\mu(\tau) \approx 0$.

**B. The Energy-Momentum Tensor.**

The Euler-Lagrange equations from the action (3.4)) are

\[
\begin{align*}
\left( \frac{\partial L}{\partial z_\mu} - \partial_A \frac{\partial L}{\partial z_\mu^A} \right)(\tau, \vec{\sigma}) &= \eta_{\mu\nu} \partial_{\vec{\nu}} \left[ \sqrt{g} T^{A\vec{B}} \right](\tau, \vec{\sigma}) \overset{\circ}{=} 0, \\
\left( \frac{\partial L}{\partial \eta^i_\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\eta}^i_\mu} \right)(\tau) &= \overset{\circ}{=} 0, \\
\left( \frac{\partial L}{\partial A^A} - \partial_{\vec{B}} \frac{\partial L}{\partial \partial_{\vec{B}} A^A} \right)(\tau, \vec{\sigma}) &= \overset{\circ}{=} 0,
\end{align*}
\]

where we introduced the total energy-momentum tensor $[\dot{\eta}_A^A(\tau) = (1; \dot{\eta}_r^r(\tau))]$

\[
T^{A\vec{B}}(\tau, \vec{\sigma}) = -\left[ \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{A\vec{B}}} \right](\tau, \vec{\sigma}) =
\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{m_i \dot{\eta}_i^A(\tau) \dot{\eta}_i^B(\tau)}{\sqrt{g} \sqrt{g_{CD} \dot{\eta}_i^C(\tau) \dot{\eta}_i^D(\tau)}} + \\
\left[ F^{A\vec{C}} F_{C\vec{B}} + \frac{1}{4} g^{A\vec{B}} F^{C\vec{D}} F_{C\vec{D}} \right](\tau, \vec{\sigma}).
\]

When $\partial_A \left[ \sqrt{g} z_\mu^A \right](\tau, \vec{\sigma}) \neq 0$ as happens on the Wigner hyperplanes in the gauge $T_s - \tau \approx 0$, $\vec{\lambda}(\tau) = 0$, we get the conservation of the energy-momentum tensor $T^{A\vec{B}}(\tau, \vec{\sigma})$, i.e. $\partial_{\vec{A}} T^{A\vec{B}} \overset{\circ}{=} 0$. Otherwise there is compensation coming from the dynamics of the hypersurface.

On the Wigner hyperplanes the energy-momentum tensor becomes

\[
T^{rr}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 + [\vec{k}_i(\tau) - Q_i A_i(\tau, \vec{\eta}_i(\tau))]^2} + \frac{1}{2} [\vec{\pi}^2 + \vec{B}^2](\tau, \vec{\sigma}),
\]

\[
T^{rs}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) [k_i^r(\tau) - Q_i A_i^r(\tau, \vec{\eta}_i(\tau))] + [\vec{\pi} \times \vec{B}](\tau, \vec{\sigma}),
\]

\[
T^{rs}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{[k_i^s(\tau) - Q_i A_i^s(\tau, \vec{\eta}_i(\tau))] [k_i^r(\tau) - Q_i A_i^r(\tau, \vec{\eta}_i(\tau))] - [\vec{k}_i(\tau) - Q_i A_i(\tau, \vec{\eta}_i(\tau))]^2}{\sqrt{m_i^2 + [\vec{k}_i(\tau) - Q_i A_i(\tau, \vec{\eta}_i(\tau))]^2}} - \frac{1}{2} [\vec{\pi}_r \vec{\pi} + \vec{B}^2 - \vec{\pi}_s \vec{\pi} + B^r B^s](\tau, \vec{\sigma}).
\]

Finally, after the canonical reduction, which eliminates the electromagnetic gauge degrees of freedom and the choice of the gauge $T_s \equiv \tau$, we get
\[ T^{\tau\tau}(\tau, \sigma) = \sum_{i=1}^{N} \delta^3(\sigma - \eta_i(\tau)) \sqrt{m_i^2 + [\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))]^2} + \]
\[ + \frac{1}{2} \left[ \left( \tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \tilde{\delta} \delta^3(\sigma - \tilde{\eta}_i(\tau)) \right)^2 + \tilde{B}^2 \right](\tau, \sigma), \]
\[ T^{\tau\tau}(\tau, \sigma) = \sum_{i=1}^{N} \delta^3(\sigma - \tilde{\eta}_i(\tau))[\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))] + \]
\[ + \left[ \left( \tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \tilde{\delta} \delta^3(\sigma - \tilde{\eta}_i(\tau)) \right) \times \tilde{B} \right](\tau, \sigma), \]
\[ T^{\tau\tau}(\tau, \sigma) = \sum_{i=1}^{N} \delta^3(\sigma - \tilde{\eta}_i(\tau)) \frac{[\tilde{k}_i^*(\tau) - Q_i \tilde{A}_\perp^*(\tau, \tilde{\eta}_i(\tau))][\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))] - \]
\[ - \frac{1}{2} \delta^r[r(\tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \tilde{\delta} \delta^3(\sigma - \tilde{\eta}_i(\tau)))]^2 + B^2 \] - \]
\[ - \left[ \left( \tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \tilde{\delta} \delta^3(\sigma - \tilde{\eta}_i(\tau)) \right)^r \left( \tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \tilde{\delta} \delta^3(\sigma - \tilde{\eta}_i(\tau)) \right)^s + \]
\[ + B^r B^s \right](\tau, \sigma). \] (3.44)
IV. INTERNAL POINCARE\'E ALGEBRA AND EQUATIONS OF MOTION FOR
THE ELECTROMAGNETIC FIELD AND THE N CHARGED PARTICLES.

In this Section we first build a realization of the Poincaré algebra inside the Wigner
hyperplane using results of the previous Section. Then, by identifying the Lorentz scalar
rest-frame time $T_s$ with the invariant time $\tau$ labeling the hypersurfaces $\Sigma(\tau)$, we arrive at
the Dirac Hamiltonian $H_D = M - \vec{\lambda}(\tau) \cdot \vec{H}_p(\tau)$, in which the only gauge freedom left is
the one associated with the rest-frame condition. We then obtain the Hamilton and Lagrange
equations for fields and particles. Then we describe how to find the canonical “internal”
center of mass $\vec{q}_+$ for fields and particles on the Wigner hyperplane. The natural gauge
fixing to the rest-frame conditions $\vec{H}_p(\tau) \approx 0$ are $\vec{q}_+ \approx 0$: they imply $\vec{\lambda}(\tau) = 0$ and the
decoupling of the “internal” center of mass from the “internal” relative motions. In this
way only the “external” decoupled 3-center of mass $\vec{z}_s$ remains (the Newton-Wigner-like
3-position which replaces the 4-center of mass $\tilde{x}_\mu_s$ in the gauge $T_s \equiv \tau$). A property of the
particle accelerations of any order, which will be needed in the next Section, is derived.

A. Internal Poincaré Algebra

In the rest-frame instant form of the dynamics there is another realization of the Poincaré
algebra besides the “external” one given in Eq.(3.34). This is the “internal” realization built
in terms of the variables living inside each Wigner hyperplane. The associated generators of
this internal Poincaré group are given by (for positive energies in the Wigner-covariant rest
frame radiation gauge)

$$\mathcal{P}_{(int)}^\tau = M = \sum_{i=1}^N \sqrt{m_i^2 + (\vec{k}_i(\tau) - Q_i \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)))^2 +}
+ \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|}
+ \int d^3\sigma \frac{1}{2} [\vec{\pi}_\perp + \vec{\pi}_\perp(\tau, \vec{\sigma})],$$

$$\mathcal{P}_{(int)} = \vec{H}_p = \vec{\lambda}(\tau) + \int d^3\sigma [\vec{\pi}_\perp \times \vec{B}](\tau, \vec{\sigma}) \approx 0,$$
\begin{align*}
\mathcal{J}_{(\text{int})}^r &= \varepsilon_{r\st} \bar{S}_{\st}^r = \sum_{i=1}^N (\vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau))^r + \int d^3\sigma \left( \vec{\pi}_\perp \times \vec{B} \right)^r (\tau, \vec{\sigma}), \\
\mathcal{K}_{(\text{int})}^r &= \bar{S}_{\st} \bar{S}^\st = -\sum_{i=1}^N \bar{\eta}_i(\tau) \sqrt{m_i^2 + \left[ \vec{\kappa}_i(\tau) - Q_i \vec{A}_\perp(\tau, \vec{\eta}_i) \right]^2} + \\
&+ \sum_{i=1}^N \left[ \sum_{j \neq i} Q_i Q_j \left( \frac{1}{\Delta_{ij}} \frac{\partial}{\partial \eta^r_j} c(\bar{\eta}_i(\tau) - \bar{\eta}_j(\tau)) - \eta^r_j(\tau) c(\bar{\eta}_i(\tau) - \bar{\eta}_j(\tau)) \right) + \\
&+ Q_i \int d^3\sigma \vec{\pi}_\perp(\tau, \vec{\sigma}) c(\vec{\sigma} - \vec{\eta}_i(\tau)) \right] - \frac{1}{2} \int d^3\sigma \sigma^r (\vec{\pi}_\perp^2 + \vec{B}^2)(\tau, \vec{\sigma}), \tag{4.1}
\end{align*}

in which \( \vec{\kappa}_i(\tau) = \Sigma_i \kappa_i(\tau) \) and \( c(\eta_i - \eta_j) := -1/(4\pi|\eta_j - \eta_i|) \). The latter two Lorentz generators are determined as the components of the spin tensor \( \bar{S}_{\st}^{AB} \) defined in Eq.(3.33) inside each Wigner hyperplane.

The Dirac Hamiltonian is

\[ H_D = \lambda(\tau) \mathcal{H}(\tau) - \vec{\lambda}(\tau) \cdot \vec{H}_p(\tau). \tag{4.2} \]

As already said, if we add the gauge-fixing

\[ \chi = T_s - \tau \approx 0, \quad T_s \equiv \frac{p_s \cdot \vec{x}_s}{\sqrt{p_s^2}} = \frac{p_s \cdot x_s}{\sqrt{p_s^2}}, \tag{4.3} \]

implying that the Lorentz scalar parameter \( \tau \) labelling the leaves of the foliation of Minkowski spacetime with Wigner hyperplanes coincides with the rest-frame time \( T_s \) of the decoupled point particle clock (the “external” center of mass) \( \vec{x}_s \). Its conservation in \( \tau \) will imply \( \lambda(\tau) = -1 \) so that, after taking the Dirac brackets associated with the second class constraints \( \epsilon_s - M \approx 0 \) and \( T_s - \tau \approx 0 \) (this eliminates \( T_s \) and \( \epsilon_s \)), the final Dirac Hamiltonian in this gauge would be \( H_D = -\vec{\lambda}(\tau) \cdot \vec{H}_p(\tau) \). However, if we wish to reintroduce the evolution in \( \tau \equiv T_s \) in this frozen phase space [containing the canonical variables \( \vec{z}_s, \vec{k}_s, \eta_i(\tau), \vec{k}_i(\tau), \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\eta}_\perp(\tau, \vec{\sigma}) \)] we must use the Hamiltonian

\[ H_D = M - \vec{\lambda}(\tau) \cdot \vec{H}_p(\tau), \tag{4.4} \]

because \( M = \mathcal{P}_{(\text{int})}^r \) is the invariant mass and the “internal” energy generator of the isolated system [it is like with the frozen Hamilton-Jacobi theory, in which the time evolution can be reintroduced by using the energy generator of the Poincaré group as Hamiltonian].
The only remaining first class constraints are the rest-frame conditions $\mathcal{H}_p(\tau) \approx 0$. The Dirac multipliers $\vec{\lambda}(\tau)$ describe the remaining gauge freedom on the location of the “internal” center of mass on the Wigner hyperplanes. In the next Subsection we will study the natural gauge fixings for these first class constraints. After this final canonical reduction the isolated system will be described by the decoupled “external” 3-center-of-mass canonical variables $\vec{z}_s$, $\vec{k}_s$ and by relative Wigner-covariant degrees of freedom on the Wigner hyperplane, with the invariant mass $M$ as the Hamiltonian for the evolution in $\tau \equiv T_s$.

**B. The equations of motion for particles and fields.**

The Hamilton-Dirac equations associated to the previous Hamiltonian are

$$
\dot{\vec{\eta}}_i(\tau) \overset{\circ}{=} \frac{\tilde{\kappa}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + (\tilde{\kappa}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \vec{\eta}_i(\tau)))^2}} - \tilde{\lambda}(\tau),
$$

$$
\dot{\tilde{\kappa}}_i(\tau) \overset{\circ}{=} -\sum_k Q_i Q_k (\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)) + Q_i (\vec{\eta}_i^a(\tau) + \lambda^a(\tau)) \frac{\partial}{\partial \vec{\eta}_i} \tilde{A}_\perp^a(\tau, \vec{\eta}_i(\tau)) - Q_i (\vec{\eta}_i^a(\tau) + \lambda^a(\tau)) \frac{\partial}{\partial \vec{\eta}_i} \tilde{A}_\perp^a(\tau, \vec{\eta}_i(\tau)) + \int d^3\sigma [\tilde{\pi}_\perp \times \tilde{B}](\tau, \vec{\sigma}) \approx 0.
$$

(4.5)

in which $\overset{\circ}{=} \text{ means evaluated on the equations of motion.}$

The Hamilton-Dirac equations for the fields are

$$
\dot{\tilde{A}}_\perp(\tau, \vec{\sigma}) \overset{\circ}{=} -\tilde{\pi}_\perp(\tau, \vec{\sigma}) - [\tilde{\lambda}(\tau) \cdot \tilde{\partial}] \tilde{A}_\perp(\tau, \vec{\sigma}),
$$

$$
\dot{\tilde{\pi}}_\perp(\tau, \vec{\sigma}) \overset{\circ}{=} \Delta \tilde{A}_\perp^a(\tau, \vec{\sigma}) - [\tilde{\lambda}(\tau) \cdot \tilde{\partial}] \tilde{\pi}_\perp^a(\tau, \vec{\sigma}) + \sum_i Q_i P_{\perp}^a(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)).
$$

(4.6)

The associated Lagrangian, obtained by means of an inverse Legendre transformation [12], is

$$
L_R(\tau) = \dot{\vec{\eta}}_i(\tau) \cdot \dot{\tilde{\kappa}}_i(\tau) - \int d^3\sigma [\tilde{\dot{A}}_\perp(\tau, \vec{\sigma}) \cdot \tilde{\pi}_\perp(\tau, \vec{\sigma}) - H_R(\tau) = \sum_{i=1}^N \left[ -m_i \sqrt{1 - (\dot{\vec{\eta}}_i(\tau) + \tilde{\lambda}(\tau))^2} + Q_i [\dot{\vec{\eta}}_i(\tau) + \tilde{\lambda}(\tau)] \cdot \tilde{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right]
$$

40
Here $\vec{\lambda}(\tau)$ is now interpreted as a non-linear Lagrange multiplier needed to get the rest-frame conditions $\mathcal{H}_p = \mathcal{P}_{(int)} \approx 0$. Its Euler-Lagrange equations $\frac{d}{d\tau} \frac{\partial L}{\partial \dot{\eta}_i} = \frac{\partial L}{\partial \eta_i} ; \frac{\partial L}{\partial \dot{\lambda}} = 0$ yield

$$
\frac{d}{d\tau} \left[ m_i \frac{\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau))^2}} + Q_i \ddot{A}_\perp(\tau, \ddot{\eta}_i(\tau)) \right] = -\sum_{k \neq i} \frac{Q_i Q_k (\ddot{\eta}_i(\tau) - \ddot{\eta}_k(\tau))}{4\pi | \ddot{\eta}_i(\tau) - \ddot{\eta}_k(\tau) |^3} + Q_i (\ddot{\eta}_i^n(\tau) + \dot{\lambda}^n(\tau)) \frac{\partial}{\partial \ddot{\eta}_i} A^u_\perp(\tau, \ddot{\eta}_i(\tau)),
$$

and [these are the Lagrangian rest-frame conditions]

$$
\sum_{i=1}^{N} \left[ m_i \frac{\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau))^2}} + Q_i \ddot{A}_\perp(\tau, \ddot{\eta}_i(\tau)) \right] + \int d^3 \sigma \sum_{r} \left[ (\ddot{A}_\perp^r(\tau, \ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau)) \right](\tau, \ddot{\eta}_i(\tau)) = 0.
$$

The Lagrangian expression for the conserved invariant mass $M = \mathcal{P}_{(int)}$ is

$$
E_{rel} = \sum_{i=1}^{N} \frac{m_i}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau))^2}} + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi | \ddot{\eta}_i(\tau) - \ddot{\eta}_j(\tau) |^3} + \int d^3 \sigma \frac{1}{2} [\vec{E}_\perp + \vec{B}]^2 (\tau, \ddot{\eta}_i(\tau)) = const.
$$

Eq.(4.8) may be rewritten as

$$
\frac{d}{d\tau} \left[ m_i \frac{\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau)}{\sqrt{1 - (\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau))^2}} \right] = -\sum_{k \neq i} \frac{Q_i Q_k (\ddot{\eta}_i(\tau) - \ddot{\eta}_k(\tau))}{4\pi | \ddot{\eta}_i(\tau) - \ddot{\eta}_k(\tau) |^3} + Q_i [\vec{E}_\perp(\tau, \ddot{\eta}_i(\tau)) + (\ddot{\eta}_i(\tau) + \ddot{\lambda}(\tau)) \times \vec{B}(\tau, \ddot{\eta}_i(\tau))],
$$

(4.11)
where the notation $\tilde{E}_\perp = -\tilde{A}_\perp - [\tilde{X}(\tau) \cdot \tilde{\partial}i] \tilde{A}_\perp = \tilde{\pi}_\perp$ has been introduced.

Eqs.(4.11) and (4.8) are the rest-frame analogues of the usual equations for charged particles in an external electromagnetic field and of the electromagnetic field with external particle sources in which both particles and electromagnetic field are dynamical. Eq.(4.9) defines the rest frame by using the total (Wigner spin 1) 3-momentum of the isolated system formed by the particles plus the electromagnetic field. Eq.(4.10) gives the constant invariant mass of the isolated system: the electromagnetic self-energy of the particles has been regularized by the Grassmann-valued electric charges $[Q_i^2 = 0]$ so that the invariant mass is finite.

C. The “internal” center of mass and the last gauge fixing.

The rest-frame conditions $\tilde{H}_p = \tilde{P}_{(int)} \approx 0$ show that there are still 3 gauge degrees of freedom among the reduced canonical variables $\tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau), \tilde{A}_\perp(\tau, \vec{\sigma}), \tilde{\pi}_\perp(\tau, \vec{\sigma})$ on each Wigner hyperplane. They correspond to our freedom in the choice of the point of the Wigner hyperplane that locates the “internal” 3-center of mass $\vec{q}_+$ of the isolated system. After the gauge fixing $\vec{q}_+ \approx 0$ only Wigner-covariant relative variables are left on the Wigner hyperplane and there is no double counting of the center of mass [only the decoupled canonical non-covariant “external” one $\vec{z}_s, \vec{k}_s$ is left].

In Refs. [1,12] there was a naive choice $\tilde{\eta}_+ = \frac{1}{N} \sum_{i=1}^{N} \tilde{\eta}_i$ of the Wigner spin 1 3-vector conjugate to $\tilde{H}_p = \tilde{P}_{(int)}$. Then, after realizing that $\tilde{\eta}_+ \approx 0$ does not imply $\tilde{\lambda}(\tau) = 0$, in Ref. [5] a different choice $\vec{q}_+$ was made by utilizing the group-theoretical results of Ref. [7]: now the time constancy of the gauge fixings $\vec{q}_+ \approx 0$ implies $\tilde{\lambda}(\tau) = 0$. Moreover, the nonrelativistic limit of $\vec{q}_+$ is now the unique nonrelativistic center of mass.

In this gauge we get the simplest description of the dynamics on the Wigner hyperplanes: $\vec{\sigma} = \vec{q}_+ \approx 0$ implies that the “internal” center of mass is put at the origin $x^\mu_+(\tau)$ of the coordinates $[z^\mu(\tau, \vec{\sigma}) = x^\mu_+(\tau) + c^\mu(u(p_s))\sigma^\tau$ with $x^\mu_+(\tau) = x^\mu_+(0) + u^\mu(p_s)\tau$] of the Wigner hyperplane. In this gauge the origin acquires the property $\dot{x}^\mu_+(\tau) = u^\mu(p_s)$ and becomes also
the “external” Fokker-Pryce center of inertia of the isolated system [5] [in a future paper [6] there will be a more detailed analysis of these problems].

In order to find $\vec{q}_+$ one must take advantage of the “internal” realization of the Poincaré algebra inside the Wigner hyperplane. Ref. [7] implies the following definition of the canonical “internal” 3-center of mass

$$
\vec{q}_+ = \frac{-\vec{K}_{(int)}^a}{\sqrt{(\vec{P}_{(int)}^a)^2 - (\vec{P}_{(int)}^a)^2}} + \frac{\vec{J}_{(int)} \times \vec{P}_{(int)}}{\sqrt{(\vec{P}_{(int)}^a)^2 - (\vec{P}_{(int)}^a)^2}[\vec{P}_{(int)}^a + \sqrt{(\vec{P}_{(int)}^a)^2 - (\vec{P}_{(int)}^a)^2}]}
$$

$$
\vec{P}_{(int)} \approx 0 \quad -\frac{\vec{K}_{(int)}^a}{\vec{P}_{(int)}^a} = \vec{R}_. 
$$

Imposing the rest-frame condition, it is seen that $\vec{q}_+$ weakly coincides with the non-canonical “internal” Møller center of energy $\vec{R}_$. In that same limit it is also equal to the “internal” Fokker-Pryce center of inertia defined by

$$
\vec{q}_+ = \vec{q}_+ + \frac{\vec{S}_{(int)} \times \vec{P}_{(int)}}{\sqrt{(\vec{P}_{(int)}^a)^2 - (\vec{P}_{(int)}^a)^2}[\vec{P}_{(int)}^a + \sqrt{(\vec{P}_{(int)}^a)^2 - (\vec{P}_{(int)}^a)^2}]},
$$

where

$$
\vec{S}_{(int)} \equiv \vec{J}_{(int)} - \vec{q}_+ \times \vec{P}_{(int)} \approx \vec{S}_. 
$$

With the gauge fixing condition $\vec{q}_+ \approx 0$ and with $T_S = \tau$ one finds the following expression for the origin $x^\mu_a(\tau)$ of the coordinates on the Wigner hyperplane [$x^\mu_a(0)$ is arbitrary]

$$
x^{(q_+)}(T_S) = x^\mu_a(0) + u^\mu(p_a)T_S.
$$

It can be shown that this coincides with the covariant noncanonical “external” Fokker-Pryce center of inertia $Y^\mu(\tau)$. However, it is different from both the “external” center of mass $\bar{x}^\mu_a(\tau)$ and the “external” center of energy of Møller $R^\mu(\tau)$.
Since \( \frac{d}{d\tau} \vec{q}_+ = \{ \vec{q}_+, M - \vec{\lambda}(\tau) \cdot \vec{H}_p \} = -\vec{\lambda}(\tau) \approx 0 \), there is no gauge freedom left and we could eliminate the variables \( \vec{q}_+ \), \( \vec{P}_{(\text{int})} = \vec{H}_p \) and look for a canonical basis of (Dirac observable) relative variables on the Wigner hyperplane.

Instead of doing that [see Ref. [6]], in this paper we will go on to work with all the variables \( \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma}) \), but we shall restrict their equations of motion to the gauge \( \vec{\lambda}(\tau) = 0 \) without explicitly introducing \( \vec{q}_+ \approx 0 \).

The equations of motion for the particles and for the electromagnetic field then become

\[
\frac{d}{d\tau} \left( m_i \frac{\vec{\eta}_i(\tau)}{\sqrt{1 - \vec{\eta}_i^2(\tau)}} \right) = -\sum_{k \neq i} Q_i Q_k (\vec{\eta}_k(\tau) - \vec{\eta}_i(\tau)) \frac{\delta^3(\vec{\sigma} - \vec{\eta}_k(\tau))}{4\pi |\vec{\eta}_k(\tau) - \vec{\eta}_i(\tau)|^3} + Q_i \left[ \vec{E}_\perp(\tau, \vec{\eta}_i(\tau)) + \vec{\eta}_i(\tau) \times \vec{B}(\tau, \vec{\eta}_i(\tau)) \right],
\]

(4.16)

\[
\square \vec{A}_\perp(\tau, \vec{\sigma}) = \frac{\partial^2 \vec{A}_\perp(\tau, \vec{\sigma})}{\partial \tau^2} + \Delta \vec{A}_\perp(\tau, \vec{\sigma}) \overset{\text{def}}{=} J^\tau_{\perp}(\tau, \vec{\sigma}) =
\]

\[
= \sum_{i=1}^N Q_i P^r_{\perp}(\vec{\sigma}) \vec{\eta}^r(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) =
\]

\[
= \sum_{i=1}^N Q_i \vec{\eta}^s(\tau) (\delta^{rs} + \frac{\partial^r}{\partial^s} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) =
\]

\[
= \sum_{i=1}^N Q_i \vec{\eta}^s(\tau) [\delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) +
\]

\[
+ \int d^3 \sigma' \frac{\pi^{rs}(\vec{\sigma} - \vec{\sigma})}{|\vec{\sigma} - \vec{\sigma}'|^3} \delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau)),
\]

(4.17)

with

\[
\pi^{rs}(\vec{\sigma} - \vec{\sigma}') = \delta^{rs} - 3(\sigma^r - \sigma'^r)(\sigma^s - \sigma'^s)/(\vec{\sigma} - \vec{\sigma}')^2.
\]

(4.18)

We point out that defining \( \vec{\beta}_i(\tau) = \vec{\eta}_i(\tau) = \frac{d\vec{\eta}_i(\tau)}{d\tau} = \frac{1}{c} \frac{d\vec{h}_i(t)}{dt} \) \( [\tau = ct, \vec{\eta}_i(t) = \vec{\eta}_i(\tau) \); even if we use everywhere \( c = 1 \), we have momentarily reintroduced it] and \( \vec{\beta}^{(h)}_i = d^h \vec{\beta}_i/d\tau^h \) and writing the particle equations of motion as (no sum over \( i \))

\[
\frac{d}{d\tau} \left( m_i \frac{\vec{\beta}_i(\tau)}{\sqrt{1 - \vec{\beta}_i^2(\tau)}} \right) = \frac{m_i}{\sqrt{1 - \vec{\beta}_i^2(\tau)}} [\vec{\beta}_i^{(1)} + \vec{\beta}_i \frac{\vec{\beta}_i^{(1)} \cdot \vec{\beta}_i}{1 - \vec{\beta}_i^2(\tau)}] = Q_i \vec{F}_i,
\]

(4.19)

we obtain
\[ m_i \frac{\beta_i^{(1)} \cdot \beta_i}{(1 - \beta_i^2(\tau))^{3/2}} = Q_i \beta_i \cdot \vec{F}_i, \quad (4.20) \]

so that

\[ \beta_i^{(1)} = \frac{\sqrt{1 - \beta_i^2(\tau)}}{m_i} Q_i (\vec{F}_i - \beta_i \beta_i \cdot \vec{F}_i). \quad (4.21) \]

Thus in general we will have for every \( h \geq 1 \)

\[ \beta_i^{(h)} = Q_i G_i, \quad (4.22) \]

so that using the Grassmann property of the charges

\[ Q_i \beta_i^{(h)} = 0, \quad h \geq 1. \quad (4.23) \]

This will lead to important simplifications later allowing us to drop acceleration dependent terms in the force.

Due to the projector \( P^{rs}_\perp(\vec{\sigma}) \) required by the rest-frame radiation gauge, the sources of the transverse (Wigner spin 1) vector potential becomes non-local and one has a system of integrodifferential equations (like with the equations generated by Fokker-Tetrode actions) with the open problem of how to define an initial value problem.

The Lagrangian equations identifying the rest frame become

\[ \sum_{i=1}^{N} (\eta_i m_i \frac{\tilde{\eta}_i(\tau)}{\sqrt{1 - \eta_i^2(\tau)}} + Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))) + \]

\[ + \int d^3 \sigma \sum_r [(\tilde{A}_\perp^r(\tau, \tilde{\sigma}) = 0. \quad (4.24) \]
V. ELECTROMAGNETIC LIENARD-WIECHERT POTENTIALS.

In this Section we will study the Lienard-Wiechert solutions of the previous radiation gauge field equations in the gauge $\tilde{\lambda}(\tau) = 0$ in absence of incoming radiation by using the results of Ref. [12]. We shall study the retarded, advanced and $\frac{1}{2}(\text{retarded} + \text{advanced})$ Lienard-Wiechert potentials. Using the Smart-Winter [46] expansion for retarded and advanced time dependence, we obtain an infinite series form of the retarded and advanced Lienard-Wiechert potentials depending on instantaneous accelerations of every order. It will be shown that the results of the previous Section imply that on the solutions of the particle equations of motion the higher accelerations decouple due to the semiclassical regularization $Q_i^2 = 0$. We show that this implies that the $\frac{1}{2}(\text{retarded} - \text{advanced})$ Lienard-Wiechert potential vanishes at the semiclassical level and that there is only one semiclassical Lienard-Wiechert potential: $\text{retarded} = \text{advanced} = \frac{1}{2}(\text{retarded} + \text{advanced})$. This allows us to re-express the semiclassical Lienard-Wiechert potential in terms of particle canonical coordinates and momenta [the same can be done for the Lienard-Wiechert electric field, as it will be shown in the next Section]. Therefore, we get a Hamiltonian description of the Lienard-Wiechert semiclassical solution: this sector of solutions can be identified as the symplectic submanifold of the space of solutions of the electromagnetic field equations determined by two pairs of second class constraints, which force the electromagnetic field to coincide with the semiclassical Lienard-Wiechert one. After having gone to Dirac brackets with respect to them, we get a reduced phase space with only particles and we find a canonical basis $\tilde{\eta}_i$, $\tilde{\kappa}_i$ for these brackets for arbitrary $N$. In the new variables the rest-frame condition becomes $\sum_{i=1}^{N} \tilde{\kappa}_i \approx 0$, as expected in an instant form of dynamics.
A. Grassmann truncated form of the advanced and retarded Lienard Wiechert Solutions.

Here we develop the Grassmann truncated forms for the Lienard-Wiechert vector potential [see the next Section for the transverse electric and magnetic fields]. The \( \frac{1}{2} \) (retarded + advanced) solutions are given (for \( \tilde{\chi}(\tau) = 0 \)) by [for the sake of notational simplicity we will use the notation \( \kappa_i(\tau), \bar{A}_i(\tau, \bar{\sigma}), \bar{\pi}_i(\tau, \bar{\sigma}) \), instead of \( \hat{\kappa}_i(\tau), \bar{A}_i(\tau, \bar{\sigma}), \bar{\pi}_i(\tau, \bar{\sigma}) \), from now on]

\[
A^\tau_{\perp S}(\tau, \bar{\sigma}) = \frac{1}{2} [A^\tau_{\perp +} + A^\tau_{\perp -}](\tau, \bar{\sigma}) = \\
= \frac{1}{2} P^\tau_{\perp}(\bar{\sigma}) \sum_{i=1}^{N} Q_i \int d\tau_i d^3 \sigma_1 [\theta(\tau - \tau_i) + \theta(\tau - \tau^-_i)] \\
\delta[(\tau - \tau^-_i)^2 - (\bar{\sigma} - \bar{\eta}_i(\tau^-_i))^2] \beta^\tau_i(\tau^-_i) = \\
= P^\tau_{\perp}(\bar{\sigma}) \sum_{i=1}^{N} Q_i \int dt_1 \delta[(t - t^-_1)^2 - \frac{1}{c^2}(\bar{\sigma} - \bar{\eta}_i(ct^-_1))^2] \beta^\tau_i(ct^-_1) = \\
: = \sum_{i=1}^{N} Q_i A^\tau_{\perp S_i}(\tau, \bar{\sigma}), 
\]

in which we have put \( \tau = ct, \bar{\beta}_i(\tau) = \bar{\eta}_i(\tau) = \frac{1}{c} \frac{d\eta(t)}{dt} \) and \( \bar{A}_{\perp +} = \bar{A}_{\perp RET} \) (\( \bar{A}_{\perp -} = \bar{A}_{\perp ADV} \)) for the retarded (advanced) solution. The equation for \( t^-_1 \) is \( c^2(t - t^-_1)^2 = (\bar{\sigma} - \bar{\eta}_i(ct^-_1))^2 \) with the two solutions being

\[
t^+_{i+}(\tau, \bar{\sigma}) = \frac{1}{c} \tau^+_{i+}(\tau, \bar{\sigma}) = t + \frac{1}{c} \tau^+_{i+}(\tau, \bar{\sigma}), \bar{\sigma} = \frac{\tau}{c} - T^+_{i+}(\tau, \bar{\sigma}), \\
t^+_{i-}(\tau, \bar{\sigma}) = \frac{1}{c} \tau^+_{i-}(\tau, \bar{\sigma}) = t + \frac{1}{c} \tau^+_{i-}(\tau, \bar{\sigma}), \bar{\sigma} = \frac{\tau}{c} + T^+_{i-}(\tau, \bar{\sigma}), \tag{5.2}
\]

for the retarded and for the advanced case respectively. The light cone delta function is

\[
\delta[(\tau - \tau^-_1)^2 - (\bar{\sigma} - \bar{\eta}_1(\tau^-_1))^2] = \frac{1}{c^2} \delta[(t - t^-_1)^2 - \frac{1}{c^2}(\bar{\sigma} - \bar{\eta}_i(ct^-_1))^2] = \\
= \delta[\tau^-_1 - \tau^-_{1+}(\tau, \bar{\sigma})] + \delta[\tau^-_1 - \tau^-_{1-}(\tau, \bar{\sigma})] = \frac{1}{2|\tau - \tau^-_1 - \bar{\beta}_i(\tau^-_1)|} + \frac{1}{2|\tau - \tau^-_1 - \bar{\beta}_i(\tau^-_1)|}. \tag{5.3}
\]

The relative space location between the field point and the retarded or advanced particle position is

\[
\bar{\sigma} - \bar{\eta}_1(\tau_{i \pm}(\tau, \bar{\sigma})) = \bar{r}_i(\tau_{i \pm}(\tau, \bar{\sigma}), \bar{\sigma}) = r_{i \pm}(\tau_{i \pm}(\tau, \bar{\sigma}), \bar{\sigma}) \hat{r}_{i \pm}(\tau_{i \pm}(\tau, \bar{\sigma}), \bar{\sigma}), \tag{5.4}
\]
and its length is related to the time interval by

\[ r_{i \pm}(\tau, \bar{\sigma}) = |\bar{\sigma} - \bar{\eta}_i(\tau, \bar{\sigma})| = c T_{i \pm}(\tau, \bar{\sigma}) = |\tau - \tau_{i \pm}(\tau, \bar{\sigma})|, \]

\[ \Rightarrow \quad \tau - \tau_{i \pm}(\tau, \bar{\sigma}) = \pm c T_{i \pm}(\tau, \bar{\sigma}) = \pm r_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma}). \]  

(5.5)

The effective spatial interval is defined by

\[ \rho_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma}) = r_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})[1 \mp \beta_i(\tau, \bar{\sigma}) \cdot \hat{r}_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})]. \]  

(5.6)

In terms of these variables, the retarded, advanced and time symmetric solutions are

\[ A_{i \pm}^R(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \frac{Q_i^{\pm}}{4 \pi} \frac{\beta_i^R(\tau, \bar{\sigma})}{\rho_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})}, \]

\[ A_{i}^S(\tau, \bar{\sigma}) = \sum_{i=1}^{N} Q_i A_{i \mp}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \frac{Q_i^{\pm}}{8 \pi} \beta_i \left[ \frac{\beta_i^R(\tau, \bar{\sigma})}{\rho_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})} \right], \]  

(5.7)

We use the Smart-Wintner expansion [46,18,19]

\[ f(\tau_{i \pm}) = f(\tau \mp c T_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})) = f(\tau - [\pm r_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})]) = \]

\[ = f(\tau) + \sum_{k=1}^{\infty} \frac{(-)^k}{k!} \frac{d^{k-1}}{d\tau^{k-1}} \left[ (\pm r_{i \pm}(\tau, \bar{\sigma})) \frac{df(\tau)}{d\tau} \right] = \]

\[ = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \frac{d^k}{d\tau^k} \left[ (\pm r_{i \pm}(\tau, \bar{\sigma})) \right] \left[ 1 \mp \beta_i(\tau, \bar{\sigma}) \cdot \hat{r}_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma}) \right] f(\tau), \]  

(5.8)

where

\[ \hat{r}_{i}(\tau, \bar{\sigma}) = r_{i}(\tau, \bar{\sigma}) \hat{r}_{i}(\tau, \bar{\sigma}) = \bar{\sigma} - \bar{\eta}_i(\tau) = r_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})|_{\tau_{i \pm}(\tau, \bar{\sigma}) = \tau}, \]

\[ f(\tau_{i \pm}) = \frac{\beta_i^R(\tau_{i \pm})}{\rho_{i \pm}(\tau_{i \pm})} = \frac{\beta_i^R(\tau_{i \pm})}{r_{i \pm}(\tau_{i \pm})[1 \mp \beta_i(\tau_{i \pm}) \cdot \hat{r}_{i \pm}(\tau, \bar{\sigma}, \bar{\sigma})]}, \]  

(5.9)

and where the last line in Eq.(5.8) is identical to the previous one since \( \frac{dr_{i \pm}(\tau, \bar{\sigma})}{d\tau} = -\beta_i(\tau) \cdot \hat{r}_{i}(\tau, \bar{\sigma}) \).

Hence we get

\[ A_{i \pm}^R(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \frac{Q_i^{\pm}}{4 \pi} \beta_i^R(\tau, \bar{\sigma}) \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \frac{d^k}{d\tau^k} \left[ r_{i \pm}(\tau, \bar{\sigma}) \beta_i^R(\tau), \right], \quad A_{i \pm}^S = \frac{1}{2} (A_{i \pm}^R + A_{i \pm}^L). \]  

(5.10)
In order to evaluate the above derivatives we need the Leibnitz formula for the \( k \)th derivative of the product \( f(\tau)g(\tau) \)

\[
\frac{d^k}{d\tau^k}(fg) = \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \frac{d^m f}{d\tau^m} \frac{d^{k-m} g}{d\tau^{k-m}}. \tag{5.11}
\]

Thus we get

\[
\sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{(\mp)^k}{m!(k-m)!} \frac{d^m r^{i_1\ldots i_{k-1}\beta^s}}{d\tau^m} \frac{d^{k-m} \beta^s}{d\tau^{k-m}} = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{(\mp)^k}{m!(k-m)!} \frac{d^m r^{i_1\ldots i_{k-1}h}}{d\tau^m} \frac{d^{k-m} h}{d\tau^{k-m}}. \tag{5.12}
\]

Using the notation \( \beta^{(h)s} = \frac{d^h \beta^s}{d\tau^h} \) we obtain the following expression for the vector potential

\[
A^r_{i\pm}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \mathcal{P}^r_{\pm}(\vec{\sigma}) \sum_{h=0}^{\infty} \frac{(\mp)^h}{h!} \beta^{(h)s}_{i}(\tau) \phi_{i\pm,h}(\tau, \vec{\sigma}), \tag{5.13}
\]

in which

\[
\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \frac{d^m r^{i_1\ldots i_{h+m-1}}(\tau, \vec{\sigma})}{d\tau^m} = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \frac{d^m}{d\tau^m} [\sqrt{(\vec{\sigma} - \vec{n}_h(\tau))^2}]^{m+h-1}. \tag{5.14}
\]

In order to display the result of the evaluation of the derivative we use the formula

\[
\frac{d^m}{d\tau^m} R(f(\tau)) = \sum_{n=0}^{m} \sum_{n_1 n_2 \ldots} \frac{m!}{n_1! n_2! \ldots} \frac{d^n R(f(\tau))}{d\tau^n} |_{f=f(\tau)} \left( \frac{1}{1!} \frac{df(\tau)}{d\tau} \right)^{n_1} \left( \frac{1}{2!} \frac{d^2 f(\tau)}{d\tau^2} \right)^{n_2} \ldots \tag{5.15}
\]

(with the summations restricted so that \( \sum_r n_r = n, \sum_r rn_r = m \)) to obtain

\[
\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \sum_{n=0}^{m} \sum_{n_1 n_2 \ldots} \frac{m!}{n_1! n_2! \ldots} \nabla^{n_1 \ldots n_2} \frac{d^n r^{i_1\ldots i_{h+m-1}}(\tau, \vec{\sigma})}{d\tau^n} \circ \left( \frac{-\beta_i(\tau)}{1!} \right)^{n_1} \left( \frac{-\beta_i(\tau)}{2!} \right)^{n_2} \ldots \tag{5.16}
\]

In this expression the symbol \( \circ \) represents a scalar product between the tensors to the left and to the right with the summation \( \sum_r n_r = n \) indicating how the indices would be matched.

Changing the \( m \) summation index to \( k = m - n \) we obtain

\[
\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n_1 n_2 \ldots} \frac{(\mp)^{k+n}}{n_1! n_2! \ldots} \nabla^{k+n h+1} \frac{d^{k+n h+1} r^{i_1\ldots i_{h+m-1}}(\tau, \vec{\sigma})}{d\tau^{k+n h+1}} \circ \left( \frac{\beta_i(\tau)}{1!} \right)^{n_1} \left( \frac{\beta_i(\tau)}{2!} \right)^{n_2} \ldots \tag{5.17}
\]
(In this latter summation $\sum_r n_r = n$, $\sum_r r n_r = n + k$.)

Now we can take advantage of the Grassmann charges to significantly simplify the above multi-summations. As we have seen above, with a semiclassical $Q_i$ there are no accelerations on shell ($Q_i \vec{\beta}_i^l = 0$) in the equations of motion of the particle ‘i’, since both the Coulomb potential and the Lienard-Wiechert Lorentz force on particle ‘i’ produced by the other particles, i.e. $Q_i [\vec{E}_l (\tau, \vec{\eta}_i (\tau)) + \vec{\beta}_i (\tau) \times \vec{B}(\tau, \vec{\eta}_i (\tau))]$, are proportional to $Q_i$. Therefore, the full set of Hamilton equations (4.5), (4.6) for both fields and particles imply that at the semiclassical level we have a natural “order reduction” of the final particle equation of motion in the Lienard-Wiechert sector [only second order differential equations].

One effect of this truncation is the elimination of multi-particle forces; all the interactions will be pairwise, in both the Lagrangian and Hamiltonian formalisms. This was to be expected since the rest-frame instant form is an equal-time description of the $N$ particle system: (acceleration-independent) 3-body..., N-body forces appear as soon as we go to a description with no concept of equal time, like in the standard approach with $N$ first class constraints [47].

Thus the only contributing indices are $n_2 = n_3 = .. = 0$, $n_1 = n$ and our expression for the transverse vector potentials simplify to

$$A^r_{\perp \pm}(\tau, \vec{\sigma}) = \sum_{i=1}^N \frac{Q_i}{4\pi} P_{\perp}^r(\vec{\sigma}) \beta^s_i(\tau) \phi_{i \pm, 0}(\tau, \vec{\sigma})$$

$$= \sum_{i=1}^N \frac{Q_i}{4\pi} P_{\perp}^r(\vec{\sigma}) \beta^s_i(\tau) \sum_{n=0}^{\infty} \frac{(\pm)^n}{n!} \partial^n r_i^{n-1}(\tau, \vec{\sigma}) \bigg( \frac{\beta_i(\tau)}{1!} \bigg)^n,$$

$$A^r_{\perp S}(\tau, \vec{\sigma}) = \frac{1}{2} (A^r_{\perp +} + A^r_{\perp -})(\tau, \vec{\sigma}). \quad (5.18)$$

Since $r_i = \sqrt{\vec{r}_i^2}$, we see that for odd $n=2m+1$ we get

$$\frac{\partial^{2m+1}}{\partial \vec{r}_i^{2m+1}} (\sqrt{\vec{r}_i^2})^{2m+1} = \frac{\partial^{2m+1}}{\partial \vec{r}_i^{2m+1}} (\vec{r}_i^2)^m = 0, \quad (5.19)$$

and this implies the equality of the retarded, advanced and symmetric Lienard-Wiechert potentials on-shell

$$A^r_{\perp S}(\tau, \vec{\sigma}) = A^r_{\perp \pm}(\tau, \vec{\sigma}) = \sum_{i=1, i \neq u}^N \frac{Q_i}{4\pi} P_{\perp}^r(\vec{\sigma}) \beta^s_i(\tau)$$
\[
\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \vec{\beta}(\tau) \cdot \frac{\partial^{2m}}{\partial \vec{r}_i^{2m}} \right) \vec{\eta}_i^{2m-1}(\tau, \vec{\sigma}). \tag{5.20}
\]

Therefore, at the semiclassical level there is only one Lienard-Wiechert sector with a uniquely determined standard action-at-a-distance interaction.

We use a tensor notation to write the transverse symmetric vector potential above as
\[
\vec{A}_{\perp S}(\tau, \vec{\sigma}) \propto \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{\eta}_i}{\vec{\eta}_i} \sum_{m=0}^{\infty} \frac{\partial^{2m} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m-1}}{\partial \sigma_j \partial \sigma_{j2m}}.	ag{5.21}
\]

Using the definition of the Coulomb projection operator
\[
\mathcal{P}(\vec{\sigma})_{\perp \perp} = \delta_{\perp \perp} \frac{1}{4\pi} \int d^3 \sigma' \frac{\partial^2}{\partial \sigma_i \partial \sigma_k} \frac{1}{|\vec{\sigma} - \vec{\sigma}'|} F(\vec{\sigma}'),	ag{5.22}
\]
and compactifying the notation still further we obtain \( [\vec{\nabla}_\sigma = \partial / \partial \vec{\sigma} ] \)
\[
\vec{A}_{\perp S}(\tau, \vec{\sigma}) \propto \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{\eta}_i}{\vec{\eta}_i} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \vec{\eta}_i(\tau) \cdot \vec{\eta}_i(\tau) \right)^{2m-1} - \frac{1}{4\pi} \int d^3 \sigma' \langle \vec{\nabla}_\sigma (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma) \rangle \frac{1}{|\vec{\sigma} - \vec{\sigma}'|} (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma)^{2m-1}. \tag{5.23}
\]

Integration by parts and changing from \( \frac{\partial}{\partial \sigma} \) to \( \frac{\partial}{\partial \vec{\sigma}} \) and translation gives
\[
\vec{A}_{\perp S}(\tau, \vec{\sigma}) \propto \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{\eta}_i}{\vec{\eta}_i} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma \right)^{2m+1} - \frac{1}{4\pi} \int d^3 \sigma' \langle \vec{\nabla}_\sigma (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma) \rangle^{2m+1} \frac{1}{|\vec{\sigma} - \vec{\sigma}'|} (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma)^{2m-1}. \tag{5.24}
\]

The integral above is finite, and thus we can view it as the \( \Lambda \to \infty \) limit of an integral with a cutoff \( \Lambda \) and take the derivatives out. The integral is thus of the form
\[
-\frac{1}{4\pi} \vec{\nabla}_\sigma (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma)^{2m+1} \int d^3 \sigma' \frac{\sigma^{2m-1}}{|\vec{\sigma} - (\vec{\sigma} - \vec{\eta}_i)|}, \tag{5.25}
\]
and
\[
\frac{1}{4\pi} \int_{\Lambda} d^3 \sigma' \frac{\sigma^{2m-1}}{|\vec{\sigma} - (\vec{\sigma} - \vec{\eta}_i)|} = \frac{1}{2} \int_{0}^{\Lambda} d\sigma' \sigma^{2m+1} \int_{-1}^{1} \frac{dz}{\sqrt{\sigma'^2 + (\vec{\sigma} - \vec{\eta}_i)^2}} = \frac{1}{2} \frac{1}{(2m+1)(2m+2)}
\]
and
\[
= \frac{\Lambda^{2m+1}}{2m+1} - \frac{\sigma^{2m+1}}{(2m+1)(2m+2)}. \tag{5.26}
\]

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Note that the $\Lambda$ cutoff will get killed by the $\sigma$ derivatives. Thus, we obtain

$$\vec{A}_{\perp S}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{1}{(2m)!} \hat{\eta}_i(\tau) \cdot \vec{\nabla}_{\sigma} \right]^{2m} |\vec{\sigma} - \hat{\eta}_i(\tau)|^{2m-1} - \frac{1}{(2m+2)!} \nabla_{\sigma}(\hat{\eta}_i(\tau) \cdot \vec{\nabla}_{\sigma})^{2m+1} |\vec{\sigma} - \hat{\eta}_i(\tau)|^{2m+1} := \sum_{i=1}^{N} Q_i \vec{A}_{\perp S_i}(\vec{\sigma} - \hat{\eta}_i(\tau), \hat{\eta}_i(\tau)). \quad (5.27)$$

Using the first half of particle Hamilton equations (4.5) [with $\vec{\lambda}(\tau) = 0$] in the form $\dot{\vec{\eta}}_i = \vec{\kappa}_i / \sqrt{m_i^2 + \vec{\kappa}_i^2} + O(Q_i)$, we can, as shown in Appendix A, arrive at the following closed form of the vector potential $[\vec{\eta}_i = \vec{\eta}_i(\tau), \vec{\kappa}_i = \vec{\kappa}_i(\tau)]$

$$\vec{A}_{\perp S}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \vec{A}_{\perp S_i}(\vec{\sigma} - \hat{\eta}_i(\tau), \hat{\eta}_i(\tau)),$$

$$\vec{A}_{\perp S_i}(\vec{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) = \frac{1}{4\pi |\vec{\sigma} - \hat{\eta}_i|} \left[ \frac{\hat{\kappa}_i}{\sqrt{m_i^2 + (\hat{\kappa}_i \cdot \frac{\vec{\sigma} - \hat{\eta}_i}{|\vec{\sigma} - \hat{\eta}_i|})^2}} - \hat{\kappa}_i \cdot \left( \frac{I - (\vec{\sigma} - \hat{\eta}_i)(\vec{\sigma} - \hat{\eta}_i)}{|\vec{\sigma} - \hat{\eta}_i|^2} \right) \left( \frac{\sqrt{m_i^2 + \hat{\kappa}_i^2}}{\sqrt{m_i^2 + (\hat{\kappa}_i \cdot \frac{\vec{\sigma} - \hat{\eta}_i}{|\vec{\sigma} - \hat{\eta}_i|})^2}} - 1 \right) \right] \times$$

$$\frac{\sqrt{m_i^2 + \hat{\kappa}_i^2}}{\hat{\kappa}_i^2 - (\hat{\kappa}_i \cdot \frac{\vec{\sigma} - \hat{\eta}_i}{|\vec{\sigma} - \hat{\eta}_i|})^2} \right]. \quad (5.28)$$

**B. Lienard-Wiechert Second-Class Constraints, their Dirac Brackets and the New Canonical Variables.**

Thus far we have the reduced phase space of $N$ charged particles plus the transverse electromagnetic field. This is a well defined isolated system with a global Darboux basis $[\vec{\eta}_i, \vec{\kappa}_i, \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma})]$ and a well defined physical Hamiltonian, the invariant mass $M = \mathcal{P}_\text{(int)}$. All possible configurations of motion take place in this reduced phase space. The space of solutions of Hamilton’s equations is a symplectic space in that there is a definition of Poisson brackets on the space of solutions. The question arises whether one can select a subset of solutions of the equations of motion which is still a symplectic manifold: an
arbitrarily chosen set of solutions will not form a symplectic manifold. The method we propose here is to add by hand a set of second class constraints “compatible with the equations of motion” which amounts to the selection of a symplectic submanifold of the symplectic manifold of solutions.

The above Grassmann truncated semiclassical Lienard Wiechert solution \( \vec{A}_{\perp S} \) for the vector potential with \( \vec{\pi}_{\perp S} = \vec{E}_{\perp S} = -\frac{\partial}{\partial \tau} \vec{A}_{\perp S} \) for the canonical conjugate field momentum [see Eq.(6.2) in the next Section] and provide us such a set of second class constraints by way of

\[
\begin{align*}
\chi_1(\tau, \vec{\sigma}) &= \vec{A}_{\perp}(\tau, \vec{\sigma}) - \sum_{i=1}^{N} Q_i \vec{A}_{\perp S_i}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) \approx 0, \\
\chi_2(\tau, \vec{\sigma}) &= \vec{\pi}_{\perp}(\tau, \vec{\sigma}) - \sum_{i=1}^{N} Q_i \vec{\pi}_{\perp S_i}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) \approx 0.
\end{align*}
\]

These constraints allow us to eliminate the canonical degrees of freedom of the radiation field and to get the symmetric Lienard-Wiechert reduced phase space, in which there are only particle degrees of freedom. This has an immediate and important consequence: the independent variables \( \vec{\eta}_i, \vec{\kappa}_i \) will no longer be canonical when one imposes these constraints by way of modified Dirac brackets.

Now in order to compute the effects of these constraints we must use them in the construction of Dirac brackets. This requires that we compute the 6x6 matrix of brackets

\[
\begin{pmatrix}
\{ \chi_1, \chi_1 \} & \{ \chi_1, \chi_2 \} \\
\{ \chi_2, \chi_1 \} & \{ \chi_2, \chi_2 \}
\end{pmatrix}.
\]

(5.30)

It turns out that this matrix bracket is relatively simple, due to the Grassmann charges. Consider, for example the case of two particles. The particle or Lienard-Wiechert parts of the matrix bracket vanish since \( Q_1^2 = Q_2^2 = 0 \) and cross terms vanish because they involve Poisson brackets of particle one variables with particle two variables. Thus the only part of the 6x6 matrix bracket that contributes is from the field variables. It has the form

\[
\{ \chi_1(\tau, \vec{\sigma}_1), \chi_2(\tau, \vec{\sigma}_2) \} = (\mathbf{I} - \frac{\vec{\nabla} \vec{\nabla}}{\vec{\nabla}^2}) \delta^3(\vec{\sigma}_1 - \vec{\sigma}_2),
\]

(5.31)
and since
\[ \{ \chi_1, \chi_1 \} = 0 = \{ \chi_2, \chi_2 \}, \] (5.32)
only the 3x3 off diagonal portion contributes.

In order to have a well defined Dirac bracket we need to use a modified form of the Dirac bracket in which the inverse of the matrix of constraint Poisson brackets is used. Calling this matrix \( C \), we define \( C^{-1} \) so that \( C C^{-1} = (I - \frac{\nabla \nabla}{\nabla^2}) \delta^3(\sigma_1 - \sigma_2) \). But the transverse form of the delta function allows us to use the idempotent property of the projector to show that the inverse of \( C \) in this sense is just \( C \) itself. In that case for two functions \( f(\tilde{\kappa}_i, \tilde{\eta}_i), g(\tilde{\kappa}_i, \tilde{\eta}_i) \) of the particle variables the Dirac bracket becomes
\[ \{ f, g \}^* = \{ f, g \} - \left\{ \int d^3 \sigma \{ f, - \sum_i Q_i \bar{A}_{\perp S_i} (\tilde{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) \} \cdot \left\{ - \sum_j Q_j \bar{\pi}_{\perp S_j} (\tilde{\sigma} - \tilde{\eta}_j(\tau), \tilde{\kappa}_j(\tau)) , g \right\} - \{ f, - \sum_j Q_j \bar{A}_{\perp S_j} (\tilde{\sigma} - \tilde{\eta}_j(\tau), \tilde{\kappa}_j(\tau)) \} \cdot \left\{ - \sum_i Q_i \bar{\pi}_{\perp S_i} (\tilde{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) , g \right\} \right\}. \] (5.33)
This bracket will lead to a new symplectic manifold by altering the basic commutation relations and providing us with new canonical variables. Toward this end we define the following scalar function.
\[ K = \sum_{i=1}^{N-1} \sum_{j=i+1}^N Q_i Q_j K_{ij}(\tilde{\kappa}_i, \tilde{\kappa}_j; \tilde{\eta}_i - \tilde{\eta}_j), \] (5.34)
in which
\[ K_{ij} = \int d^3 \sigma [ \bar{A}_{\perp S_i} (\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \bar{\pi}_{\perp S_j} (\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) - \bar{A}_{\perp S_j} (\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) \cdot \bar{\pi}_{\perp S_i} (\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) ] = K_{ij}(\tilde{\kappa}_i, \tilde{\kappa}_j; \tilde{\eta}_i - \tilde{\eta}_j) = -K_{ji}. \] (5.35)
Let \( \tilde{\eta}_i = \tilde{\eta}_i + \tilde{\alpha}_i, \tilde{\kappa}_i = \tilde{\kappa}_i + \tilde{\beta}_i, i = 1, 2, \ldots N \) where
\[ \tilde{\alpha}_i = a_i \sum_{j=i+1}^N Q_i Q_j \nabla_{\kappa_i} K_{ij} + \tilde{a}_i \sum_{j=1}^{i-1} Q_i Q_j \nabla_{\kappa_i} K_{ji}, \]
\[ \tilde{\beta}_i = b_i \sum_{j=i+1}^N Q_i Q_j \nabla_{\eta_i} K_{ij} + \tilde{b}_i \sum_{j=1}^{i-1} Q_i Q_j \nabla_{\eta_i} K_{ji}. \] (5.36)
Since they do not appear in these equations, we may choose \( \tilde{a}_1 = \bar{b}_1 = a_N = b_N = 0 \).

We determine relations between the unknown coefficients by requiring that \( \tilde{\eta}_i, \tilde{\kappa}_j \) be independent canonical variables. So, for example, (for \( k < l \))

\[
\{ \tilde{\eta}_k, \tilde{\eta}_l \}^{*} = \{ \bar{\eta}_k, \bar{\eta}_l \} - \left[ \int d^3 \sigma \left\{ \tilde{\eta}_k, - \sum_i Q_i \tilde{A}_i(S_i(\tilde{\sigma} - \bar{\eta}_i, \bar{\kappa}_i)) \right\} \cdot \left\{ - \sum_j Q_j \tilde{A}_j(S_j(\tilde{\sigma} - \bar{\eta}_j, \bar{\kappa}_j), \bar{\eta}_k) \right\} - \left\{ \tilde{\eta}_k, - \sum_j Q_j \tilde{A}_j(S_j(\tilde{\sigma} - \bar{\eta}_j, \bar{\kappa}_j)) \right\} \cdot \left\{ - \sum_i Q_i \tilde{A}_i(S_i(\tilde{\sigma} - \bar{\eta}_i, \bar{\kappa}_i), \bar{\eta}_l) \right\} = \{ \tilde{\eta}_k, \alpha_l \} + \{ \alpha_k, \tilde{\eta}_l \} + Q_k Q_l \nabla_{\kappa_k} \nabla_{\kappa_l} K_{kl} = 0. \tag{5.37} \]

Then using the expressions for \( \tilde{\alpha}_i \) leads to

\[
a_l - \bar{a}_k = 1, \quad k > l; \quad \bar{a}_l - a_k = 1, \quad l > k. \tag{5.38} \]

Solving this gives

\[
\bar{a}_2 = \bar{a}_3 = \ldots = \bar{a}_N := \bar{a},
\]

\[
a_1 = a_2 = \ldots a_{N-1} := a. \tag{5.39} \]

Similarly, requiring that \( \{ \tilde{\kappa}_k, \tilde{\kappa}_l \}^{*} = 0 \) leads to

\[
\bar{b}_2 = \bar{b}_3 = \ldots = \bar{b}_N := \bar{b},
\]

\[
b_1 = b_2 = \ldots = b_{N-1} := b. \tag{5.40} \]

Requiring that \( \{ \tilde{\eta}_i, \tilde{\kappa}_i \}^{*} = \bar{1} \) leads to

\[
a_i + b_i = 0, \quad i = 1, \ldots, N - 1; \quad \bar{a}_i + \bar{b}_i = 0, \quad i = 2, \ldots, N. \tag{5.41} \]

This condition implies that the first two conditions are equivalent to one another. The requirement that \( \{ \tilde{\eta}_k, \tilde{\kappa}_j \}^{*} = 0 \) for \( k < l \) leads to

\[
ak + b_l = 1; \quad k < l, \tag{5.42} \]

or

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\[ a_k - \bar{a}_l = 1; \quad k < l, \quad (5.43) \]

which is also the same as the first condition. While for \( l < k \) it leads to

\[ \bar{a}_k + b_l = -1; \quad l < k, \quad (5.44) \]

or

\[ \bar{a}_k - a_l = -1; \quad l < k, \quad (5.45) \]

which again is the same as the first condition. This leaves us with just two unknowns \( a \) and \( \bar{a} \).

So in summary we have

\[
\tilde{\eta}_i = \eta_i + a \sum_{j=i+1}^{N} Q_i Q_j \bar{\nabla}_{\eta_i} K_{ij} + \bar{a} \sum_{j=1}^{i-1} Q_i Q_j \bar{\nabla}_{\eta_i} K_{ji},
\]

\[
\tilde{\kappa}_i = \kappa_i - a \sum_{j=i+1}^{N} Q_i Q_j \bar{\nabla}_{\eta_i} K_{ij} - \bar{a} \sum_{j=1}^{i-1} Q_i Q_j \bar{\nabla}_{\eta_i} K_{ji}. \quad (5.46)
\]

Let us rewrite the rest frame condition Eq.\((3.40)\)

\[
\vec{H}_p = \vec{P}_{(int)} = \sum_{i=1}^{N} \vec{\kappa}_i + \int d^3 \sigma [\vec{p}_\perp \times \vec{B}](\tau, \vec{\sigma}) =
\]

\[
= \sum_{i=1}^{N} \bar{\kappa}_i + \sum_{i<j} Q_i Q_j \int d^3 \sigma \left[ \vec{p}_\perp S_i (\vec{\eta} - \bar{\eta}_i, \bar{\kappa}_i) \times (\bar{\nabla}_\sigma \times \vec{A}_\perp S_j (\vec{\eta} - \bar{\eta}_j, \bar{\kappa}_j)) \right] +
\]

\[
+ \vec{p}_\perp S_j (\vec{\eta} - \bar{\eta}_j, \bar{\kappa}_j) \times (\bar{\nabla}_\sigma \times \vec{A}_\perp S_i (\vec{\eta} - \bar{\eta}_i, \bar{\kappa}_i)) \right] \approx 0. \quad (5.47)
\]

in these new canonical variables. Let us expand the cross products, integrate by parts and use the transverse gauge condition to get

\[
\vec{H}_p = \vec{P}_{(int)} = \sum_{i=1}^{N} \bar{\kappa}_i + \sum_{i<j} Q_i Q_j \bar{\nabla}_{\eta_i} K_{ij} = 0. \quad (5.48)
\]

If we choose \( a = -\bar{a} = \frac{1}{2} \) this becomes

\[
\vec{H}_p = \vec{P}_{(int)} = \sum_{i=1}^{N} \bar{\kappa}_i = \sum_{i=1}^{N} \bar{\kappa}_i = 0, \quad (5.49)
\]

like in the case of \( N \) either free or interacting particles on the Wigner hyperplane [1] [in an instant form of dynamics the Poincaré generators \( \vec{P}_{(int)} \) do not depend on the interaction].
In other words the rest frame condition is simply that the sum of the \( N \) (new) canonical momentum is zero. Note that this same choice gives
\[
\sum_{i=1}^{N} \tilde{\eta}_i = \sum_{i-1}^{N} \tilde{\eta}_i - \sum_{i<j} Q_i Q_j \tilde{\eta}_i \eta_j K_{ij}.
\] (5.50)

Therefore, with the choice \( a = -\bar{a} = \frac{1}{2} \), Eq.(5.46) defining the final canonical variables becomes
\[
\tilde{\eta}_i = \eta_i + \frac{1}{2} \sum_{j \neq i} Q_i Q_j \tilde{\eta}_j K_{ij},
\]
\[
\tilde{\kappa}_i = \kappa_i - \frac{1}{2} \sum_{j \neq i} Q_i Q_j \tilde{\eta}_i \kappa_{ij},
\] (5.51)
with \( K_{ij} \) given by Eq.(5.35).

In the next section we will re-express the other internal Poincaré generators \( M = P^\tau_{(int)}, \) \( \tilde{J}_{(int)}, \tilde{K}_{(int)} \) of Eqs.(4.1) and the internal center-of-mass coordinate \( \tilde{q}_+ \approx -\tilde{K}_{(int)}/M \) of Eq.(4.12) in these final canonical variables.
VI. THE EXACT DARWIN HAMILTONIAN FROM THE INARIANT MASS.

In this Section our aim is to use the explicit semiclassical Lienard Wiechert solution of Eqs.(5.27), (5.28) for the transverse vector potential to obtain an explicit form of the instantaneous action-at-a-distance potentials present in the invariant mass $M = \mathcal{P}_{(int)}^\tau$ of Eq.(4.1) after the elimination of the electromagnetic degrees of freedom. In its phase space form this Hamiltonian for the $\tau \equiv T_{s}$-evolution will contain:

i) a vector potential $\vec{V}_i(\tau) = Q_i \sum_{i \neq j}^{N} Q_j \vec{A}_j(\vec{r}_i(\tau) - \vec{r}_j(\tau), \vec{\kappa}_j(\tau))$, minimally coupled to $\vec{\kappa}_i(\tau)$, under the square root kinetic energy term of each particle ‘i’;

ii) a scalar potential $U(\tau) = \frac{1}{2} \int d^3\sigma [\vec{\pi}^2_{\perp S} + \vec{B}^2_S](\tau, \vec{\sigma})$, coming from the field energy, which adds to the Coulomb potentials.

Due to $Q_i^2 = 0$ we can extract the vector potentials from the square roots: the semiclassical contribution from the vector potentials is a new effective scalar potential $U_1 = - \sum_{i=1}^{N} Q_i \vec{\kappa}_i \vec{V}_i \sqrt{m_i^2 + \kappa_i^2}$ and we get the complete Darwin potential $V_{DAR} = U + U_1$ added to the Coulomb one. In this form the invariant mass becomes the exact semiclassical Darwin Hamiltonian and $V_{DAR}$ is the Darwin potential to all orders of $1/c^2$ for every $N$. If we call $V_{LOD}$ the lowest $1/c^2$ order historical Darwin potential [see Eq.(6.11)], we have $U_1 = 2V_{LOD} + U_{1HOD}$ [see Eq.(6.10)], $U = -V_{LOD} - U_{1HOD} + U_{HOD}$ [see Eq.(6.13)], $V_{DAR} = V_{LOD} + U_{HOD}$. [“LOD” and “HOD” mean lowest and higher order in $1/c^2$ respectively]. When we re-express the invariant mass in terms of the final canonical variables, there is an extra contribution $U'_{HOD}$ coming from the square roots, so that at the end the final Darwin potential is

$$\tilde{V}_{DAR} = V_{DAR} + U'_{HOD} = V_{LOD} + V_{HOD}; \quad V_{HOD} = U_{HOD} + U'_{HOD}. \quad (6.1)$$

In this Section we shall evaluate the complete (to all orders in $1/c^2$) Darwin potential in the old (no longer canonical) variables and then we shall re-express it in terms of the new final canonical variables. We begin by obtaining the contribution of the field energy integrals, expressed in terms of the canonical particle variables. Using the truncation properties of
the Grassmann charges we extract from the kinetic piece the vector potential portion and combine it with the field energy integral. In addition to the naive kinetic part (expressed in terms of the old canonical momentum) we obtain the rest frame Coulomb part, a part that generalizes the standard Darwin interaction and a double infinite series containing all higher order corrections. As an extra check on these generalized Darwin interactions in \( M \) we obtain an independent derivation of this series using the Lagrangian expression for the invariant mass. Then we express the kinetic portion in terms of the new canonical momentum to obtain the final form of the complete Darwin Hamiltonian. This interaction Hamiltonian contains no \( N \)-body forces and is a sum of two-body portions. In the center-of-mass rest frame the double infinite series can be summed exactly to obtain a closed form expression for the special case of two particles. Then all the generators of the “internal” Poincaré algebra and the energy-momentum tensor are expressed in the new variables in the \( N \)-particle case. As with the three-momentum we find that the internal angular momentum does not depend on the interaction. Also we obtain the “internal” center of mass \( \vec{q}_+ \) and there are some comments on how to find a collective variable (replacing the center of mass) for a cluster of particles interacting with the remaining ones of the isolated system.

A. Field Energy and Momentum Integrals.

Although we have summed exactly to a closed form (5.28) the semiclassical Lienard-Wiechert solution, we use the series form (5.27) for finding the expression for the invariant mass \( M \), since the closed form provides no simplification in obtaining that expression. From the above we can find expressions for the semiclassical Lienard-Wiechert electric and magnetic fields. For the electric field we find \( \hat{\vec{E}}_i(\tau) \) does not contribute due to Eq.(4.23), and the same is true of \( \hat{\vec{k}}_i(\tau) \); \( \frac{\partial}{\partial \tau} |\vec{\sigma} - \vec{\eta}_i(\tau)| = -\hat{\vec{E}}_i(\tau) \cdot \vec{\nabla}_\sigma |\vec{\sigma} - \vec{\eta}_i(\tau)| \) 

\[
\vec{E}_{\perp S}(\tau, \vec{\sigma}) = \vec{\pi}_{\perp S}(\tau, \vec{\sigma}) = -\left( \frac{1}{2m} \right) \hat{\vec{E}}_i(\tau) \cdot \vec{A}_\perp(\tau) \]

\[
= \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \hat{\vec{E}}_i(\tau) \cdot \vec{A}_\perp(\tau) \right)^{2m+1} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m-1} -
\]
\[-\frac{1}{(2m+2)!}\vec{\nabla}_\sigma (\vec{\eta}_i(\tau) \cdot \vec{\nabla}_\sigma)^{2m+2} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m+1} = \]

\[= \sum_{i=1}^{N} Q_i \vec{E}_{\perp S_i}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \]

\[= \sum_{i=1}^{N} Q_i \cdot \vec{\kappa}_i(\tau) \cdot \vec{\nabla}_\sigma \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)} \vec{A}_{\perp S_i}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \]

\[= -\sum_{i=1}^{N} Q_i \times \frac{1}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \left[ \vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau) \right] \frac{\sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \frac{\sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}}{[m_i^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}^2)]^{3/2} + \]

\[+ \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \left( \vec{\kappa}_i^2(\tau) + (\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau)) \right) \left( \frac{\vec{\kappa}_i^2(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \right) \left( \frac{m_i^2 + \vec{\kappa}_i^2(\tau)}{m_i^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}^2)^2} \right) - 1 \right) + \]

\[+ \left( \vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau) \right) \left( \frac{m_i^2 + \vec{\kappa}_i^2(\tau)}{[m_i^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}^2)^2]^{3/2}} \right). \] (6.2)

The magnetic field is \([B^S = e^{\sigma u} \frac{\partial}{\partial \sigma} A^u\)]

\[\vec{B}_S(\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \vec{B}_{S_i}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \]

\[= -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \hat{\vec{\eta}}_i(\tau) \times \vec{\nabla}_\sigma (\hat{\vec{\eta}}_i(\tau) \cdot \vec{\nabla}_\sigma)^{2m} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m-1} = \]

\[= \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{m_i^2 \vec{\kappa}_i(\tau) \times \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}}{[m_i^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}^2)^2]^{3/2}}. \] (6.3)

In these expressions \(\hat{\vec{\eta}}_i(\tau)\) may be replaced with \(\vec{\kappa}_i(\tau)/\sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}\). We have also given the closed form of the fields.

Let us remark that the Feynman-Wheeler complete absorber assumption is violated in the Maxwell theory, since it would imply (see for instance Ref. [30], where there is the definition of radiation in the Feynman-Wheeler theory) \(\vec{E}_{\perp S}(\tau, \vec{\sigma}) = \vec{E}_{\perp \pm}(\tau, \vec{\sigma}) = 0\) and \(\vec{B}_S(\tau, \vec{\sigma}) = \vec{B}_{\pm}(\tau, \vec{\sigma}) = 0\) everywhere (inside and outside the absorbers).

For the energy we need \(\vec{E}_{\perp S}^2 + \vec{B}_S^2\) and for the momentum we need \(\vec{E}_{\perp S} \times \vec{B}_S\). In Appendix B we evaluate these and show that \(\vec{\eta}_{ij}(\tau) = \eta_{ij}(\tau) \vec{\eta}_{ij}(\tau) = \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau); \vec{\nabla}_{ij} = \partial/\partial \eta_{ij}\)

\[U(\tau) = \frac{1}{2} \int d^3\sigma (\vec{E}_{\perp S}^2 + \vec{B}_S^2)(\tau, \vec{\sigma}) = \sum_{i<j}^{N} \frac{Q_i Q_j}{4\pi} h_1(\hat{\vec{\kappa}}, \hat{\vec{\eta}}, \hat{\eta}_{ij}) = \]

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\[ M = \mathcal{P}^\tau_{(int)} = \sum_{i=1}^{N} \sqrt{m_i^2 + (\vec{k}_i(\tau) - Q_i \vec{A}_{\perp S}(\tau, \vec{\eta}_i(\tau)))^2} + \\
+ \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + \int d^3\sigma \frac{1}{2} [\vec{\sigma}^2 S(\tau, \vec{\sigma}) + \vec{B}_S^2(\tau, \vec{\sigma}) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + [\vec{k}_i(\tau) - \vec{V}_i(\tau)]^2} + \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + U(\tau) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2} + \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + V_{DAR}(\tau), \]

\[ (6.5) \]
\[ V_{DAR}(\tau) := V_{LOD}(\tau) + U_{HOD}(\tau). \]

\[ \bar{\mathcal{P}}_{(\text{int})} = \mathcal{H}_p(\tau) = \bar{\kappa}_+(\tau) + \int d^3\sigma [\bar{\kappa}_{\perp S} \times \vec{B}_S](\tau, \vec{\sigma}) \approx 0, \quad (6.6) \]

The first line of \( M \) is

\[
\sum_{i=1}^{N} \sqrt{m_i^2 + (\bar{\kappa}_i(\tau) - Q_i\bar{A}_{\perp S}(\tau, \vec{\eta}_i(\tau)))^2} = \sum_{i=1}^{N} \left( \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} - \frac{\bar{\kappa}_i(\tau) \cdot Q_i\bar{A}_{\perp S}(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)}} \right) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + \bar{\kappa}_i^2} + U_1,
\]

\[ U_1 = -\sum_{i=1}^{N} \frac{\bar{\kappa}_i \cdot \vec{V}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}}, \quad \Rightarrow \quad V_{DAR}(\tau) = U_1(\tau) + U(\tau) = V_{LOD}(\tau) + U_{HOD}(\tau), \quad (6.7) \]

in which the vector potential is given by the semiclassical Lienard-Wiechert transverse potential Eq.(5.27). By re-expressing this transverse vector potential in terms of the momenta [one may use in that expression either the new or old canonical variables because of the Grassmann truncation] we obtain

\[
\bar{A}_{\perp S}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{1}{(2m)!} \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \left( \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \cdot \vec{\nabla}_\sigma \right)^{2m} \right] - \frac{1}{(2m+2)!} \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \cdot \left( \vec{\nabla}_\sigma \right)^{2m+1} [\vec{\sigma} - \vec{\eta}_i]^{2m+1}, \quad (6.8)
\]

so that for the scalar potential \( U_1 \) we get

\[
U_1 = -\sum_{i=1}^{N} \frac{\bar{\kappa}_i \cdot \vec{V}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} = \sum_{i=1}^{N} \left( -\frac{\bar{\kappa}_i \cdot Q_i\bar{A}_{\perp S}(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \right) = \\
= -\sum_{i<j}^{1,N} \frac{Q_iQ_j}{4\pi} \sum_{m=0}^{\infty} \left[ \left( \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m} + \left( \frac{\bar{\kappa}_j}{\sqrt{m_j^2 + \bar{\kappa}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2m} \right] \frac{\eta_{ij}^{2m-1}}{(2m)!} - \\
-\left( \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \cdot \vec{\nabla}_{ij} \right) \left( \frac{\bar{\kappa}_j}{\sqrt{m_j^2 + \bar{\kappa}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+1} + \\
+\left( \frac{\bar{\kappa}_i}{\sqrt{m_i^2 + \bar{\kappa}_i^2}} \cdot \vec{\nabla}_{ij} \right) \left( \frac{\bar{\kappa}_j}{\sqrt{m_j^2 + \bar{\kappa}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+1} \frac{\eta_{ij}^{2m+1}}{(2m+2)!} \right]. \quad (6.9)
\]
The lowest order part of this kinetic contribution (the $m = 0$ term) is twice the familiar Darwin interaction (but with $m_i \to \sqrt{m_i^2 + \kappa_i^2}$; strictly speaking this is a higher order correction). Thus we have

\[ U_1 = -\sum_{i=1}^{N} \frac{\overrightarrow{\kappa}_i \cdot \overrightarrow{V}_i}{\sqrt{m_i^2 + \kappa_i^2}^2} = 2V_{LOD}(\tau) + U_{1HOD}, \tag{6.10} \]

with

\[ V_{LOD} = -\sum_{i<j}^{N} \frac{Q_i Q_j}{4\pi} \left( \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}} \cdot \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}} \frac{1}{\eta_{ij}} - \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}} \cdot \nabla_{ij} \right) \left( \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}} \right) \frac{\eta_{ij}}{2} \]

\[ = -\sum_{i<j}^{N} \frac{Q_i Q_j}{8\pi \eta_{ij}} \left( \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}} \cdot \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}} \right) + \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}} \cdot \eta_{ij} \left( \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}} \cdot \eta_{ij} \right). \tag{6.11} \]

The standard form for the historical Darwin term is the above but with $\sqrt{m_i^2 + \kappa_i^2} \to m_i$.

The remaining compensating part of the familiar Darwin interaction plus all higher order parts come from the field energy $U(\tau)$. In terms of momentum variables, using Grassmann truncations, the expression Eq.(6.4) for the field energy integral simply) becomes

\[ U(\tau) = \frac{1}{2} \int d^4\sigma (\overrightarrow{E}_{1S}^2 + B_{1S}^2)(\tau, \sigma) = \sum_{i<j}^{N} \frac{Q_i Q_j}{4\pi} h_1 \left( \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}}, \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}}, \overrightarrow{\eta}_{ij} \right) = \]

\[ = \sum_{i<j}^{N} \frac{Q_i Q_j}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\overrightarrow{\kappa}_i}{\sqrt{m_i^2 + \kappa_i^2}} \cdot \frac{\overrightarrow{\kappa}_j}{\sqrt{m_j^2 + \kappa_j^2}} \right) \times \]

\[ \frac{(\overrightarrow{\kappa}_i \cdot \nabla_{ij})^{2m+1} (\overrightarrow{\kappa}_j \cdot \nabla_{ij})^{2n+2m+1} \eta_{ij} \cdot \eta_{ij}^{2n+2m+1}}{(2n+2m+2)!} - \]

\[ - \frac{(\overrightarrow{\kappa}_i \cdot \nabla_{ij})^{2m+2} (\overrightarrow{\kappa}_j \cdot \nabla_{ij})^{2n+2m+3} \eta_{ij} \cdot \eta_{ij}}{(2n+2m+4)!} + \]

\[ + \frac{\overrightarrow{\kappa}_i \cdot \overrightarrow{\kappa}_j}{\sqrt{m_i^2 + \kappa_i^2} \sqrt{m_j^2 + \kappa_j^2}} \frac{(\overrightarrow{\kappa}_i \cdot \nabla_{ij})^{2m+2} (\overrightarrow{\kappa}_j \cdot \nabla_{ij})^{2n+2m+1} \eta_{ij} \cdot \eta_{ij}^{2n+2m+1}}{(2n+2m)!} \]

\[ - \frac{(\overrightarrow{\kappa}_i \cdot \nabla_{ij})^{2m+1} (\overrightarrow{\kappa}_j \cdot \nabla_{ij})^{2n+2m+1} \eta_{ij} \cdot \eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}. \tag{6.12} \]

Single infinite sum pieces can be split off from the double infinite sum in the last two lines of the above expression for the field energy integral. This naturally separates out the
In this form we see that the first summation is the compensating portion of the familiar lowest order Darwin parts plus all remaining higher order Darwin parts, including a piece that cancels exactly $U_{1HOD}$.

$$U(\tau) = \sum_{i<j}^{1,N} \frac{Q_i Q_j}{4\pi} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \eta_{ij} \right) =$$

$$= \sum_{i<j}^{1,N} \frac{Q_i Q_j}{4\pi} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \eta_{ij} \right) +$$

$$+ \sum_{i<j}^{N} \frac{Q_i Q_j}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2m)!} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \eta_{ij} \right)^{2m+1} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \eta_{ij} \right)^{2n+2m+1} -$$

$$- \frac{1}{(2m+2)!} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \eta_{ij} \right)^{2m+1} +$$

$$+ \frac{1}{(2m+4)!} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \eta_{ij} \right)^{2m+3} = -V_{LOD} - U_{1HOD} + U_{HOD},$$

$$\Rightarrow \quad V_{DAR}(\tau) = U_1(\tau) + U(\tau) = V_{LOD}(\tau) + U_{HOD}(\tau). \quad \text{(6.13)}$$

In this form we see that the first summation is $-V_{LOD}$ and combines with the gauge part of the kinetic piece $[2V_{LOD}(\tau)]$ to give the familiar lowest order Darwin piece $V_{LOD}(\tau)$. The
third set of summations is \(-U_{1HOD}\) and exactly cancels with the corresponding term in the kinetic piece. The second and fourth set of summations (the double sums which begin at higher order in \((1/c^2)\)) we define as \(U_{HOD}\), coming only from \(U(\tau)\). Altogether we obtain

\[
[\eta_{ij} = |\bar{\eta}_{ij}| = |\eta_i - \eta_j|]
\]

\[
M = \sum_{i=1}^{N} \sqrt{m_i^2 + (\vec{k}_i(\tau) - Q_i\vec{A}_{1S}(\tau, \bar{\eta}_i(\tau)))^2} + \\
+ \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{4\pi |\bar{\eta}_{ij}(\tau) - \bar{\eta}_{ij}(\tau)|} + \int d^3\sigma \frac{1}{2} [\vec{a}_{1S}^2 + \vec{B}_S^2](\tau, \vec{\sigma}) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + [\vec{k}_i(\tau) - \vec{V}_i(\tau)]^2} + \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{4\pi |\bar{\eta}_{ij}(\tau) - \bar{\eta}_{ij}(\tau)|} + U(\tau) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2(\tau)} + \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{|\bar{\eta}_{ij}(\tau) - \bar{\eta}_{ij}(\tau)|} + V_{LOD}(\tau) + V_{HOD}(\tau) = \\
= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2(\tau)} + \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{4\pi |\bar{\eta}_{ij}(\tau) - \bar{\eta}_{ij}(\tau)|} - \\
- \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{4\pi} \frac{\vec{k}_i}{\sqrt{m_i^2 + \vec{k}_i^2}} \cdot \frac{\vec{k}_j}{\sqrt{m_j^2 + \vec{k}_j^2}} \cdot 1 \cdot \vec{\eta}_i(\vec{\eta}_j) \cdot \vec{\eta}_j(\tau)(\vec{k}_i \cdot \vec{\eta}_j(\vec{\eta}_j) - \vec{k}_j \cdot \vec{\eta}_j(\vec{\eta}_j)) + \\
+ \sum_{i \neq j}^{1, N} \frac{Q_i Q_j}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{\vec{k}_i}{\sqrt{m_i^2 + \vec{k}_i^2}} \cdot \frac{\vec{k}_j}{\sqrt{m_j^2 + \vec{k}_j^2}} \times \right. \\
\left. \frac{\vec{\eta}_i(\vec{\eta}_j)^{2m+1}}{2n+2m+2)! \right. \\
- \frac{\vec{\eta}_i(\vec{\eta}_j)^{2m+2}}{2n+2m+4)!} + \\
+ \frac{\vec{\eta}_i(\vec{\eta}_j)^{2m+3}}{2n+2m+6)!} \\
- \frac{\vec{\eta}_i(\vec{\eta}_j)^{2m+4}}{2n+2m+8)!} - \\
\left. \right] \right].
\]

(6.14)

In Appendix C we show how the multiple directional derivatives in the generalized higher order Darwin interactions can be evaluated in general, thereby obtaining a more readily usable form. (The expression for \(M\) and for \(V_{DAR}\) for arbitrary \(N\) is an immediate generalization.
of the case for $N = 2$ and for simplicity of notation the details in the appendices will be limited to the two-body case.) However, because of the significant complexity of the above form it will be of value to first obtain an alternative derivation of this series. For $N=2$ we have the following Lagrangian expression for the invariant mass [see Eq.(4.10) with $\tilde{\lambda}(\tau) = 0$; $h_1$ is defined in Eq.(6.4)]

$$E_{\text{rel}} = h(\dot{\vec{\eta}}_1, \dot{\vec{\eta}}_2, \vec{\eta}) = \frac{m_1}{\sqrt{1 - \dot{\vec{\eta}}_1^2}} + \frac{m_2}{\sqrt{1 - \dot{\vec{\eta}}_2^2}} + \frac{Q_1 Q_2}{4\pi} h_1(\dot{\vec{\eta}}_1, \dot{\vec{\eta}}_2, \vec{\eta}) := h_0(\dot{\vec{\eta}}_1, \dot{\vec{\eta}}_2, \vec{\eta}) + \frac{Q_1 Q_2}{4\pi} h_1(\dot{\vec{\eta}}_1, \dot{\vec{\eta}}_2, \vec{\eta}).$$

(6.15)

In order to find the Hamiltonian $H(\vec{\kappa}_1, \vec{\kappa}_2; \vec{\eta})$ from $h_1$ we must demand that Hamilton’s equation be satisfied. We use the Dirac bracket since we have used the constraint as a strong condition on the dynamical variables. This will lead to a set of differential equations for $H(\vec{\kappa}_1, \vec{\kappa}_2; \vec{\eta})$ and to a result that agrees exactly with the above expression for $M$. The details of this analysis are given in Appendix D.

Here we mention a comparison of these cross checked results (valid to all order of $1/c^2$) with approximate results obtained elsewhere. In [48] one obtains a single time Lagrangian by expanding the symmetric Green function in the Fokker action, used by Wheeler and Feynman, to all orders in $1/c^2$. From this Lagrangian one obtains the Legendre Hamiltonian $\tilde{h}(\dot{\vec{\eta}}_1, \dot{\vec{\eta}}_2, \vec{\eta}) + \text{terms involving the acceleration and all higher order derivatives}$. Ignoring those higher order accelerations one finds the same Legendre Hamiltonian as above. From that Hamiltonian the authors obtain a final Hamiltonian that although agreeing through order $1/c^2$ with the results above (including the standard Darwin interaction to that order) they differ from our common results (above and in Appendix D) at order $1/c^4$ (no terms of higher order are computed in [48]). The failure to obtain results that agree with our result here is the neglect there of using the proper Dirac brackets in the Kerner reduction. The reason that those Dirac brackets were not used is that the authors took as a starting point the Fokker action, not taking into account that this action itself is a result of imposing constraints on the solutions of the electromagnetic field equations.
By using Eqs.(5.51) we get
\[
\sqrt{m_i^2 + \vec{\kappa}_i^2} = \sqrt{m_i^2 + \vec{\kappa}_i^2 + \vec{\kappa}_i \cdot \sum_{j \neq i} Q_i Q_j \vec{\nabla}_{\vec{\eta}_j} \vec{K}_{ij}} = \\
= \sqrt{m_i^2 + \vec{\kappa}_i^2} + \frac{\vec{\kappa}_i \cdot \sum_{j \neq i} Q_i Q_j \vec{\nabla}_{\vec{\eta}_j} \vec{K}_{ij}}{2\sqrt{m_i^2 + \vec{\kappa}_i^2}},
\]
with
\[
Q_i Q_j \vec{K}_{ij}(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) = Q_i Q_j \vec{K}_{ij}(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j),
\]
\[
\vec{\kappa}_i \cdot \vec{\nabla}_{\vec{\eta}_j} \vec{K}_{ij} = \int d^3 \sigma [\vec{\kappa}_i \cdot \vec{\nabla}_{\vec{\eta}_j} \vec{A}_\perp S_i(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j)] \cdot \vec{A}_\perp S_i(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j)
- \vec{A}_\perp S_i(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) \cdot (\vec{\kappa}_i \cdot \vec{\nabla}_{\vec{\eta}_j} \vec{A}_\perp S_i(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j)]],
\]
(6.16)
so that \( M = P_{(int)}^r \) becomes (due to Grassmann truncation we can replace the old variables by the new variables in the interaction terms)
\[
M = P_{(int)}^r = \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2} + \sum_{i=1}^N \vec{\kappa}_i \cdot \sum_{j \neq i} \frac{Q_i Q_j \vec{\nabla}_{\vec{\eta}_j} \vec{K}_{ij}}{2\sqrt{m_i^2 + \vec{\kappa}_i^2}} + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i - \vec{\eta}_j|} - \\
- \sum_{i<j}^{1..N} \frac{Q_i Q_j}{4\pi} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \frac{1}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} (\frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_{\vec{\eta}_j}) \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \frac{\vec{\nabla}_{\vec{\eta}_j}}{2} \right) + \\
+ \sum_{i<j}^{1..N} \frac{Q_i Q_j}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \times \\
\left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2m+1} \left( \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2n+1} \vec{\eta}_{ij}^{2n+2m+1} - \\
(2n + 2m + 2)! \\
(2n + 2m + 4)! \\
(2n + 2m + 6)! \\
+ \\
\left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2m+2} \left( \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2n+2m+3} - \\
(2n + 2m + 4)! \\
(2n + 2m + 6)! \\
+ \\
\left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2m+3} \left( \frac{\vec{\kappa}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \vec{\nabla}_{\vec{\eta}_j} \right)^{2n+2m+5} - \\
(2n + 2m + 6)! \\
\right],
\]
(6.17)
where \( \vec{\nabla}_{ij} = \partial / \partial \vec{\eta}_{ij} \). We emphasize at this point that the interaction terms are all two-body forces; there are no \( N \) body forces in our semiclassical treatment.
In this subsection we will present the final expression for \( M = \mathcal{P}_{(m)}^\tau \) in the final canonical variables for arbitrary \( N \). Now from Eq.(5.48) above we can replace \( \sum_{i<j} \nabla \eta_i Q_i \mathcal{K}_{ij} \) by \(-\int d^3 \sigma [\bar{\pi}_{\perp S} \times \bar{B}_S](\tau, \sigma)\). We have already performed that integral (see Eq.(D14)) for the two-body problem and the result for the \( N \) body problem is an immediate generalization.

Using that complete expression we obtain
\[
\sum_{i=1}^N \sqrt{m_i^2 + \kappa_i(\tau)^2} = \sqrt{m_i^2 + \tilde{\kappa}_i(\tau)^2} + U'_{HOD}(\tau)
\]

\[
U'_{HOD}(\tau) = -\sum_{i<j} \frac{Q_i Q_j}{8\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \right] \frac{1}{(2n + 2m + 2)!} \times \left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m+2} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n} + \\
+ 2 \left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m+1} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n+1} + \\
\left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n+2} \eta_{ij}^{2n+2m+1} - \\
\frac{1}{(2n + 2m + 4)!} \left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m+3} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n+1} + \\
+ 2 \left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m+2} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n+2} + \\
\left( \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 + \tilde{\kappa}_i^2}} \cdot \nabla_{ij} \right)^{2m+1} \left( \frac{\tilde{\kappa}_j}{\sqrt{m_j^2 + \tilde{\kappa}_j^2}} \cdot \nabla_{ij} \right)^{2n+3} \eta_{ij}^{2n+2m+3}. \quad (6.18)
\]

We now combine this portion of the complete Darwin Hamiltonian coming from the vector potentials \( \tilde{V}_i(\tau) \) in the kinetic terms with the potential \( U(\tau) \) arising from the field energy integral [which contains \( V_{LOH}(\tau) \)] to obtain the final form of the invariant mass in the rest frame.

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\[ M = P_{(int)} = \sum_{i=1}^{M} \sqrt{m_i^2 + \left( \vec{r}_i(\tau) - Q_i \vec{A}_{\perp S}(\tau, \vec{n}_i(\tau)) \right)^2} +
\]
\[
+ \sum_{i<j} \frac{Q_i Q_j}{4 \pi \eta_{ij}} \int d^3 \sigma \frac{1}{2} \left[ \vec{r}_{ij}(\tau) + \vec{B}_{S}^2 \right](\tau, \vec{\sigma}) =
\]
\[
= \sum_{i=1}^{N} \sqrt{m_i^2 + [\vec{r}_i^2(\tau) - \vec{V}_i(\tau)]^2} + \sum_{i<j} \frac{Q_i Q_j}{4 \pi \eta_{ij}} + U(\tau) =
\]
\[
= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{r}_i^2(\tau)} + \sum_{i<j} \frac{Q_i Q_j}{4 \pi \eta_{ij}} + V_{DAR}(\tau) = \quad \left[ V_{DAR} = V_{LOD} + V_{HOD} \right]
\]
\[
= \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{r}_i^2} + \sum_{i<j} \frac{Q_i Q_j}{4 \pi \eta_{ij}} + \hat{V}_{DAR}(\tau), \quad \left[ \hat{V}_{DAR} = V_{DAR} + U'_{HOD} \right]
\]
\[
\hat{V}_{DAR} = - \sum_{i<j} \frac{Q_i Q_j}{4 \pi} \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \right) - \frac{1}{2} \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right) \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right) \frac{\vec{n}_{ij}}{2} - \frac{1}{8 \pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right]^{2m+2} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n} + \]
\[
+ \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n+2} \eta_{ij}^{2n+2m+1} - \frac{1}{(2n+2m+4)!} \left( \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+3} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n+1} + \right. \]
\[
+ \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+1} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n+3} \eta_{ij}^{2n+2m+3} - \right. \]
\[
- 2 \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+2} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n+2} \eta_{ij}^{2n+2m+3} + \]
\[
+ 2 \left( \frac{\vec{r}_i}{\sqrt{m_i^2 + \vec{r}_i^2}} \cdot \vec{\nabla}_{ij} \right)^{2m+3} \left( \frac{\vec{r}_j}{\sqrt{m_j^2 + \vec{r}_j^2}} \cdot \vec{\nabla}_{ij} \right)^{2n+3} \eta_{ij}^{2n+2m+5} \right] \equiv (6.19)
\]

Notice that the \( N \) body interaction Hamiltonian is a sum of individual 2 body Hamiltonians. (Grassmann truncation eliminates \( N \)-body forces). In this sense these interactions

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correspond to a sum of disconnected and spectator Feynman diagrams.

D. Further Reductions of the N-Body Darwin Hamiltonian and Closed Form Solutions for the Two-Body Problem.

It is of interest to see if we can make further simplifications by deriving an expression for the multiple derivatives. We show in Appendix C that the problematic expression \((\vec{\kappa}_i \cdot \vec{\nabla}_{ij})^a(\vec{\kappa}_j \cdot \vec{\nabla}_{ij})^b\hat{\eta}^{a+b-1}\) is obtained in terms of powers of \(\hat{\eta}_{ij} = |\vec{\eta}_i - \vec{\eta}_j|; \hat{\eta}_{ij} = (\vec{\eta}_i - \vec{\eta}_j)/\hat{\eta}_{ij}\)

\[
\cos^2 \phi_{ij} := \frac{(\vec{\kappa}_i \cdot \vec{\kappa}_j - \vec{\kappa}_i \cdot \hat{\eta}_{ij}\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2}{(\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \hat{\eta}_{ij})^2)(\vec{\kappa}_j^2 - (\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2)}. \tag{6.20}
\]

For \(N = 2\) this expression reduces to 1 if \(\vec{\kappa}_1 = -\vec{\kappa}_2\) (the center of mass rest frame condition for the two body problem). The result for \(a = 2m + 1, b = 2n + 1\) is

\[
(\vec{\kappa}_i \cdot \vec{\nabla}_{ij})^{2m+1}(\vec{\kappa}_j \cdot \vec{\nabla}_{ij})^{2n+1}\hat{\eta}_{ij}^{2(m+n)+1} =
\]

\[
\begin{align*}
&= 2\left[\frac{(2m + 2n + 1)!!}{(2m + n + 1)!}\right]^2\left[\frac{(2m + n + 1)!!}{(n + m + 1)!}\right]^2 \\
&\quad \times \sum_{l=0}^{n} \sum_{k=0}^{n-l} \sum_{h=0}^{k} \left(\begin{array}{c}2n + 1 \\l \end{array}\right)\left(\begin{array}{c}2m + 1 \\l + m - n \end{array}\right)\left(\begin{array}{c}2(n-l) + 1 \\2k \end{array}\right)\left(\begin{array}{c}k \\h \end{array}\right)(-1)^{k+h}(\cos^2 \phi_{ij})^{n-l+h-k}. \tag{6.21}
\end{align*}
\]

For the two body case in the center-of-mass rest frame \(\vec{\kappa}_1 = -\vec{\kappa}_2 := \vec{\kappa}\) this expression reduces to

\[
-\frac{[(2m + 2n + 1)!!]^2}{\hat{\eta}_{ij}}(\vec{\kappa}^2 - (\vec{\kappa} \cdot \hat{\eta})^2)^{m+n+1}. \tag{6.22}
\]

In the general case this expression can be more simply written in terms of \(\cos(2k + 1)\phi_{ij}\) [see Eq.(C55)]. Using the identity

\[
\frac{[(2m + 2n + 1)!!][(n + m + 1)!]}{(2m + n + 1)!} = \frac{1}{2^{m+n+1}}, \tag{6.23}
\]

one obtains for \(m \geq n\) the expression

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\[(\vec{\kappa}_i \cdot \nabla_{ij})^{2m+1}(\vec{\kappa}_j \cdot \nabla_{ij})^{2n+1} \eta_{ij}^{-2(m+n)+1} = \]

\[= \frac{2}{\eta_{ij}} \frac{(2(m + n + 1))!(\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \hat{\eta}_{ij})^2)^{m+1/2}(\vec{\kappa}_j^2 - (\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2)^{n+1/2}}{2^{m+n+2}} \times \sum_{k=0}^{n} \cos(2k + 1) \phi_{ij} \binom{2n + 1}{n - k} \binom{2m + 1}{m - k} \] (6.24)

Hence

\[(\vec{\kappa}_i \cdot \nabla_{ij})^{2m+3}(\vec{\kappa}_j \cdot \nabla_{ij})^{2n+1} \eta_{ij}^{-2(m+n)+1} = \]

\[= \frac{2}{\eta_{ij}} \frac{(2(m + n + 2))!(\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \hat{\eta}_{ij}^2)^2)^{m+3/2}(\vec{\kappa}_j^2 - (\vec{\kappa}_j \cdot \hat{\eta}_{ij}^2)^2)^{n+1/2}}{2^{m+n+3}} \times \sum_{k=0}^{n} \cos(2k + 1) \phi_{ij} \binom{2n + 1}{n - k} \binom{2m + 3}{m + 1 - k} \] (6.25)

By similar methods one finds that

\[(\vec{\kappa}_i \cdot \nabla_{ij})^{2m+2}(\vec{\kappa}_j \cdot \nabla_{ij})^{2n+2} \eta_{ij}^{-2(m+n)+3} = \]

\[= \frac{2}{\eta_{ij}} \frac{(2(m + n + 2))!(\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \hat{\eta}_{ij}^2)^2)^{m+1}(\vec{\kappa}_j^2 - (\vec{\kappa}_j \cdot \hat{\eta}_{ij}^2)^2)^{n+1}}{2^{m+n+4}} \times \left( \sum_{k=0}^{n} \cos(2k + 1) \phi_{ij} \binom{2n + 1}{n - k} \binom{2m + 1}{m - k} + \binom{2m + 2}{m + 1} \binom{2n + 2}{n + 1} \right) \] (6.26)

To determine the other combination note

\[(\vec{\kappa}_i \cdot \nabla_{ij})^{2m}(\vec{\kappa}_j \cdot \nabla_{ij})^{2n} \eta_{ij}^{-2(m+n)-1} = \]

\[= \frac{2}{\eta_{ij}} \frac{(2(m + n))!(\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \hat{\eta}_{ij}^2)^2)^{m}(\vec{\kappa}_j^2 - (\vec{\kappa}_j \cdot \hat{\eta}_{ij}^2)^2)^{n}}{2^{m+n+2}} \times \left( \sum_{k=0}^{n-1} \cos(2k + 1) \phi_{ij} \binom{2n - 1}{n - 1 - k} \binom{2m - 1}{m - 1 - k} + \binom{2m}{m} \binom{2n}{n} \right) \] (6.27)

Thus

\[(\vec{\kappa}_i \cdot \nabla_{ij})^{2m+2}(\vec{\kappa}_j \cdot \nabla_{ij})^{2n} \eta_{ij}^{-2(m+n)+1} = \]

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\[
\sum_{k=0}^{n-1} \cos(2k+1)\phi_{ij}\left(\frac{2n-1}{n-1-k}\right)\left(\frac{2m+1}{m+1-k}\right) + \left(\frac{2m+2}{m+1}\right)^2 \frac{n}{n} (6.28)
\]

Unfortunately, the \( k \) sums cannot be performed analytically to known closed forms for the \( N \)-body problem.

In the special case of the two body system we can obtain a closed form if we use the rest frame condition \( \tilde{\kappa}_1 + \tilde{\kappa}_2 = 0 \). The expression we get in this way may be used with the Dirac brackets associated with \( \tilde{\kappa}_1 + \tilde{\kappa}_2 \approx 0 \), \( \vec{q}_+ \approx 0 \), so that the final reduced phase contains only \( \vec{\eta} = |\vec{\eta}_1 - \vec{\eta}_2| \) and \( \tilde{\kappa} := \tilde{\kappa}_1 = -\tilde{\kappa}_2 \). Using the identity in Eq.(6.22) the higher order Darwin part \( V_{HOD} = U_{HOD} + U'_{HOD} \) becomes

\[
V_{HOD} = -\frac{Q_1Q_2}{8\pi\vec{\eta}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ -\tilde{\kappa}^2 \left[ (2m + 2n + 1)! \right] \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \vec{\eta})^2 \right]^{n+m+1} \right.
\]
\[
\times \left( \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} \right)^{2m+1} \left( \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right)^{2n+1} \left( \frac{1}{m_1^2 + \tilde{\kappa}^2} + \frac{1}{m_2^2 + \tilde{\kappa}^2} \right) +
\]
\[
\left. + \frac{[(2m + 2n + 3)!]^2}{(2n + 2m + 4)!} \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \vec{\eta})^2 \right]^{n+m+2} \left( \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} \right)^{2m+3} \left( \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right)^{2n+3} \right]
\]
\[
- \frac{2[(2m + 2n + 5)!]^2}{(2n + 2m + 6)!} \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \vec{\eta})^2 \right]^{n+m+3} \left( \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} \right)^{2m+3} \left( \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right)^{2n+3} \right].
\]

(6.29)

We use

\[
\frac{[(2m + 2n + 1)!]^2}{(2n + 2m + 2)!} = \frac{(-)^{n+m}}{2(n+m+1)} \left( \frac{-3/2}{n+m} \right),
\]

(6.30)

and let \( m + n = l \) so that \( 0 \leq m \leq l \) and \( 0 \leq l < \infty \). Then we perform the \( m \) sum using

\[
\sum_{m=0}^{l} \left( \frac{x}{y} \right)^m = \frac{y^{l+1} - x^{l+1}}{y^l(y-x)},
\]

(6.31)

and obtain
\[ V_{HOD} = -\frac{Q_1 Q_2}{8\pi \bar{\eta}(m_1^2 - m_2^2)} \times \]
\[ \left( \tilde{\kappa}^2 \left[ \sqrt{\frac{m_2^2 + \tilde{\kappa}^2}{m_2^4 + (\tilde{\kappa} \cdot \hat{n})^2}} - \sqrt{\frac{m_1^2 + \tilde{\kappa}^2}{m_1^4 + (\tilde{\kappa} \cdot \hat{n})^2}} \right] + \left[ \sqrt{\frac{m_2^2 + \tilde{\kappa}^2}{m_2^4 + (\tilde{\kappa} \cdot \hat{n})^2}} \right]^2 \left[ \frac{m_2^2 + \tilde{\kappa}^2}{m_2^4 + (\tilde{\kappa} \cdot \hat{n})^2} - 3 \right] - \left[ \sqrt{\frac{m_1^2 + \tilde{\kappa}^2}{m_1^4 + (\tilde{\kappa} \cdot \hat{n})^2}} \right]^2 \right) \times \]
\[ \begin{align*}
&\sum_{l=0}^{\infty} \frac{(-1)^l}{2(l + 1)^2} (l + 1) \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \hat{n})^2 \right]^{l+2} \left[ \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} - \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right] + \\
&\sum_{l=0}^{\infty} \frac{(-1)^l}{2(l + 2)^2} (l + 1) \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \hat{n})^2 \right]^{l+2} \left[ \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} - \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right] + \\
&\sum_{l=0}^{\infty} \frac{(-1)^l}{2(l + 3)^2} (l + 2) \left[ \tilde{\kappa}^2 - (\tilde{\kappa} \cdot \hat{n})^2 \right]^{l+3} \left[ \frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}} - \frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}} \right]. \end{align*} \]
\[
\left(\frac{m_1^2 + \tilde{\kappa}^2}{m_1^2 + \tilde{\kappa}^2} + \frac{m_2^2 + \tilde{\kappa}^2}{m_2^2 + \tilde{\kappa}^2}\right) + 
+ 2\kappa^2 \left[\frac{(m_2^2 + \tilde{\kappa}^2)}{2\sqrt{m_2^2 + (\tilde{\kappa} \cdot \hat{n})^2}} + \frac{m_2^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_2^2 + \tilde{\kappa}^2} - 3\right] - 
-(m_1^2 + \tilde{\kappa}^2) \left[\frac{m_1^2 + \tilde{\kappa}^2}{m_1^2 + (\tilde{\kappa} \cdot \hat{n})^2} + \frac{m_1^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_1^2 + \tilde{\kappa}^2} - 3\right] \times 
\left(\frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}}\right) \left(\frac{1}{m_2^2 + \tilde{\kappa}^2}\right) - 
-2 \left[(m_2^2 + \tilde{\kappa}^2) \left(\frac{m_2^2 + \tilde{\kappa}^2}{m_2^2 + (\tilde{\kappa} \cdot \hat{n})^2} - \frac{3}{8} \left(\frac{m_2^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_2^2 + \tilde{\kappa}^2}\right)^2 + 
\right. 
+ \left. \frac{5}{4} \left(\frac{m_2^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_2^2 + \tilde{\kappa}^2}\right) - \frac{15}{8}\right) - 
-(m_1^2 + \tilde{\kappa}^2) \left(\frac{m_1^2 + \tilde{\kappa}^2}{m_1^2 + (\tilde{\kappa} \cdot \hat{n})^2} - \frac{3}{8} \left(\frac{m_1^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_1^2 + \tilde{\kappa}^2}\right)^2 + 
\right. 
+ \left. \frac{5}{4} \left(\frac{m_1^2 + (\tilde{\kappa} \cdot \hat{n})^2}{m_1^2 + \tilde{\kappa}^2}\right) - \frac{15}{8}\right) \times 
\left(\frac{1}{\sqrt{m_1^2 + \tilde{\kappa}^2}}\right) \left(\frac{1}{\sqrt{m_2^2 + \tilde{\kappa}^2}}\right). \tag{6.34}
\]

So our final two-body expression is

\[
M = \sqrt{m_1^2 + \tilde{\kappa}^2} + \sqrt{m_2^2 + \tilde{\kappa}^2} + \frac{Q_1Q_2}{4\pi\tilde{\eta}} + \tilde{V}_{DAR},
\]

\[
\tilde{V}_{DAR} = V_{LOD} + V_{HOD}, \quad V_{HOD} = U_{HOD} + U'_{HOD}
\]

\[
V_{LOD} = \frac{Q_1Q_2}{8\pi\tilde{\eta}} \left[\tilde{\kappa}^2 - (\tilde{\kappa} \cdot \hat{n})^2\right] \left(\frac{1}{m_1^2 + \tilde{\kappa}^2}\right) \left(\frac{1}{m_2^2 + \tilde{\kappa}^2}\right) \tag{6.35}
\]

in which we have used the rest-frame condition \(\tilde{\kappa}_1 = -\tilde{\kappa}_2 := \tilde{\kappa}\). Note that \(Q_1Q_2\tilde{\kappa} = Q_1Q_2\tilde{\kappa}\).

In the equal mass limit \(m_1 \to m_2 := m\)

\[
V_{HOD} = \frac{Q_1Q_2}{8\pi\tilde{\eta}} \times 
\frac{m^2[3\tilde{\kappa}^2 + (\tilde{\kappa} \cdot \hat{n})^2] - 2\tilde{\kappa}^2[\tilde{\kappa}^2 - 3(\tilde{\kappa} \cdot \hat{n})^2] \sqrt{\frac{m^2 + \tilde{\kappa}^2}{m^2 + (\tilde{\kappa} \cdot \hat{n})^2}} - [3\tilde{\kappa}^2 + (\tilde{\kappa} \cdot \hat{n})^2][m^2 + (\tilde{\kappa} \cdot \hat{n})^2]}{(m^2 + \tilde{\kappa}^2)[m^2 + (\tilde{\kappa} \cdot \hat{n})^2]},
\]

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so that

\[ M = \mathcal{P}^r_{(\text{int})} = 2\sqrt{m^2 + \tilde{\kappa}^2} + \frac{Q_1Q_2}{4\pi\tilde{\eta}} + \frac{Q_1Q_2}{8\pi\tilde{\eta}} \times \]

\[
\frac{m^2\left[3\tilde{\kappa}^2 + (\tilde{\kappa} \cdot \hat{\eta})^2\right] - 2\tilde{\kappa}^2[\tilde{\kappa}^2 - 3(\tilde{\kappa} \cdot \hat{\eta})^2]\sqrt{\frac{m^2 + \tilde{\kappa}^2}{m^2 + (\tilde{\kappa} \cdot \hat{\eta})^2}} - 2[\tilde{\kappa}^2 + (\tilde{\kappa} \cdot \hat{\eta})^2][m^2 + (\tilde{\kappa} \cdot \hat{\eta})^2]}{(m^2 + \tilde{\kappa}^2)[m^2 + (\tilde{\kappa} \cdot \hat{\eta})^2]}.
\]

(6.37)

Our result differs from the appropriate terms at order \(1/c^4\) with those in [48], [49] and [50] although it does agree with their Darwin interaction at order \(1/c^2\). Since each of these latter three sources took as a starting point the Fokker particle Lagrangian they have not used, as we have done here, the canonical variables which would come from using the pair of secondary constraints arising from the reduction of the field plus particle Lagrangian to the Fokker action. We point out that Molina et al in [49], like earlier work by Golubenkov and Smorodinski [51] obtain order \(1/c^4\) corrections to the Hamiltonian. However, unlike them these authors not only use the Coulomb force law to replace acceleration dependent terms on the Lagrangian but also include the effects of that substitution on the choice of canonical variables by viewing it as a constraint (thus including Dirac brackets), in the reduction to Hamiltonian forms. (Note that since these three approaches, unlike ours, do not use Grassmann charges, they contain acceleration driven terms not contained in our approach. The comparisons we are talking about here with our approach refer to the terms not driven by acceleration dependent Lagrangian potentials).

It is of interest that Hamilton’s equations has a solution for circular orbits just as does the Schild solution for Feynman-Wheeler electrodynamics [42]. In Appendix E we show how this comes about for equal masses. The case of unequal masses is similar.
E. $J_{(int)}^r$, $K_{(int)}^r$, $q_+$ and the Energy-Momentum Tensor in the Final Canonical Variables.

Eqs. (5.49) and (Eq. (6.19) [(Eq. (6.35))] for $N = 2$ give $\vec{P}_{(int)}^r$ and $M = P_{(int)}^r$ in terms of the final canonical variables. For the internal angular momentum we get

$$J_{(int)}^r = \vec{e}^r \vec{e}^r = \sum_{i=1}^{N} (\vec{\eta}_i \times \vec{\kappa}_i)^r + \int d^3 \sigma \left( \vec{\pi}_\perp \times \vec{B} \right)(\tau, \vec{\sigma})^r =$$

$$= \sum_{i=1}^{N} \left[ (\vec{\eta}_i - \vec{\alpha}_i) \times (\vec{\kappa}_i - \vec{\beta}_i) \right]^r +$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} Q_i Q_j \left[ \int d^3 \sigma \left( \vec{\pi}_\perp \times (\vec{\nabla}_\sigma \times \vec{A}_\perp) \right) + i \leftrightarrow j \right]. \quad (6.38)$$

Using the forms for $\vec{\alpha}_i$ and $\vec{\beta}_i$ given in Eqs. (5.51) together with the expression for $K_{ij}$ and expanding the cross products in the integral we obtain

$$J_{(int)}^r = \sum_{i=1}^{N} (\vec{\eta}_i \times \vec{\kappa}_i)^r +$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} Q_i Q_j \left[ (\vec{\eta}_i \times \vec{\nabla}_\eta_i + \vec{\kappa}_i \times \vec{\nabla}_\kappa_i)^r \int d^3 \sigma \left( \vec{A}_\perp \cdot \vec{\pi}_\perp - \vec{A}_\perp \cdot \vec{\pi}_\perp \right) \right]$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} Q_i Q_j \left[ \int d^3 \sigma \left( \vec{\pi}_\perp \cdot \vec{\nabla}_\tau \vec{A}_\perp - \vec{A}_\perp \cdot \vec{\nabla}_\tau \vec{\pi}_\perp \right) \right]. \quad (6.39)$$

Using the transverse nature of the field together with vanishing surface terms we find that

$$\vec{J}_{(int)} = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{\kappa}_i, \quad (6.40)$$

if

$$\int d^3 \sigma \left( \vec{A}_\perp \cdot \vec{\pi}_\perp - \vec{A}_\perp \cdot \vec{\pi}_\perp \right) \left[ \vec{\eta}_i \times \vec{\nabla}_\eta_i + \vec{\kappa}_i \times \vec{\nabla}_\kappa_i \right] \int d^3 \sigma \left( \vec{A}_\perp \cdot \vec{\pi}_\perp - \vec{A}_\perp \cdot \vec{\pi}_\perp \right)$$

$$= \int d^3 \sigma \left[ \vec{A}_\perp \cdot \vec{\pi}_\perp - \vec{A}_\perp \cdot \vec{\pi}_\perp \right] \left[ \vec{A}_\perp \cdot \vec{\pi}_\perp + \vec{A}_\perp \cdot \vec{\pi}_\perp \right] - \vec{\alpha}_i \times \vec{\beta}_i. \quad (6.41)$$

We have given explicit forms for $\vec{A}_\perp$ and $\vec{\pi}_\perp$ in Eqs. (5.28) and (6.2) respectively. Their general forms are
\[ \vec{A}_{\perp Si} = \frac{1}{4\pi|\vec{\sigma} - \vec{\eta}|} \left[ \vec{K}_i f_i (\kappa_i^2, \vec{\kappa}_i \cdot \vec{\rho}_i) + \vec{\rho}_i g_i (\kappa_i^2, \vec{\kappa}_i \cdot \vec{\rho}_i) \right], \]

\[ \vec{\pi}_{\perp Si} = \frac{1}{4\pi|\vec{\sigma} - \vec{\eta}|^2} \left[ \vec{K}_i h_i (\kappa_i^2, \vec{\kappa}_i \cdot \vec{\rho}_i) + \vec{\rho}_i c_i (\kappa_i^2, \vec{\kappa}_i \cdot \vec{\rho}_i) \right]. \tag{6.42} \]

in which \( \vec{\rho}_i \equiv \frac{(\vec{\sigma} - \vec{\eta})}{|\vec{\sigma} - \vec{\eta}|} \). Using

\[ \nabla_\sigma \frac{1}{|\vec{\sigma} - \vec{\eta}|^n} = - \frac{n \hat{\vec{\rho}}_i}{|\vec{\sigma} - \vec{\eta}|^{n+1}}, \quad \nabla_\sigma \hat{\vec{\rho}}_i = - \frac{1}{|\vec{\sigma} - \vec{\eta}|} (I - \hat{\vec{\rho}}_i \hat{\vec{\rho}}_i), \tag{6.43} \]

we find that

\[ [(\vec{\eta}_i \times \nabla_\sigma + \vec{\kappa}_i \times \nabla_\kappa_i) \vec{A}_{\perp Si}] \cdot \vec{\pi}_{\perp Sj} = - \vec{\sigma} \times (\nabla_\sigma A^k_{\perp Si}) \pi^k_{\perp Sj} + \vec{A}_{\perp Si} \times \vec{\pi}_{\perp Sj}, \tag{6.44} \]

while

\[ -A^k_{\perp Sj} [(\vec{\eta}_i \times \nabla_\sigma + \vec{\kappa}_i \times \nabla_\kappa_i) \pi^k_{\perp Sj} = A^k_{\perp Sj} \vec{\sigma} \times \nabla_\sigma \pi^k_{\perp Sj} + \vec{A}_{\perp Sj} \times \vec{\pi}_{\perp Sj}, \tag{6.45} \]

thus verifying Eq.(6.40).

Thus as expected, in an instant form of dynamics, the internal angular momentum does not depend upon the interaction. On the other hand, the interaction-dependent internal boosts have the form of Eq.(4.1). Using the above forms for the final dynamical variables we obtain

\[ \vec{K}_{\text{(int)}} = - \sum_{i=1}^N \vec{\eta}_i \sqrt{m_i^2 + \vec{\kappa}_i^2} + \]

\[ + \vec{\kappa}_i \cdot \sum_{j \neq i} Q_i Q_j \sqrt{m_j^2 + \vec{\kappa}_j^2} \mathcal{K}_{ij} (\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) - 2 \vec{A}_{\perp Sj} (\vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) \]

\[ - \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} Q_i Q_j \sqrt{m_i^2 + \vec{\kappa}_i^2} \nabla_\kappa_i \mathcal{K}_{ij} (\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) + \]

\[ + \sum_{i=1}^N \sum_{j \neq i} \frac{Q_i Q_j}{8\pi} \frac{\vec{\eta}_i - \vec{\eta}_j}{|\vec{\eta}_i - \vec{\eta}_j|} - \sum_{i=1}^N \sum_{j \neq i} \frac{Q_i Q_j}{4\pi} \int d^3\sigma \frac{\vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_i) \cdot \vec{\pi}_{\perp Sj} (\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j)}{|\vec{\sigma} - \vec{\eta}_i|} - \]

\[ - \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} Q_i Q_j \int d^3\sigma \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_i) \cdot \vec{\pi}_{\perp Sj} (\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) + \]

\[ + \vec{B}_{Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) \cdot \vec{B}_{Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) \]

\[ = -P^\ast_{\text{(int)}} \vec{R}_+. \tag{6.46} \]
In the last line the internal Møller center of energy (4.12) is shown explicitly. This equation allows us to express the internal canonical center of mass $\vec{q}_+ \approx \vec{R}_+$ defined in Eq.(4.12) in terms of the final canonical variables. The natural gauge fixing to the rest frame constraints $\vec{P}_{(int)} = \sum_{i=1}^{N} \vec{K}_i \approx 0$ is $\vec{q}_+ \approx 0$.

From Eq.(3.44) we get the following expression for the conserved energy-momentum tensor:

$$
T^{tr}(T_s, \vec{\sigma}) = \sum_{i=1}^{N} \frac{m_i^2}{\sqrt{\{m_i^2 + [\vec{K}_i(T_s) - Q_i \sum_{j \neq i} Q_j A_{ls}^i(T_s, \vec{\eta}_i(T_s))]^2}} \\
+ \frac{1}{2} \sum_{i \neq j} Q_i Q_j [(\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(T_s))) \cdot (\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_j(T_s))) + (\vec{B}_{si} \cdot \vec{B}_{sj})](T_s, \vec{\sigma}),
$$

$$
T^{ts}(T_s, \vec{\sigma}) = \sum_{i=1}^{N} \frac{m_i^2}{\sqrt{\{m_i^2 + [\vec{K}_i(T_s) - Q_i \sum_{j \neq i} Q_j A_{ls}^i(T_s, \vec{\eta}_i(T_s))]^2}} \\
+ \sum_{i \neq j} Q_i Q_j [(\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(T_s))) \cdot (\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_j(T_s))) + (\vec{B}_{si} \cdot \vec{B}_{sj})] \sum_{i \neq j} Q_i Q_j [(\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(T_s))) \cdot (\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_j(T_s))) + (\vec{B}_{si} \cdot \vec{B}_{sj})](T_s, \vec{\sigma}).
$$

(6.47)

By using Eqs.(5.51) we get $[\vec{\eta}_i = \vec{\eta}_i(T_s), \vec{K}_i = \vec{K}_i(T_s); A_{ls}(\vec{\sigma} - \vec{\eta}_i, \vec{\eta}_i) \text{ is given in Eq.(5.28); \vec{K}_{ij} = \vec{K}_{ij}(\vec{K}_i, \vec{K}_j, \vec{\eta}_i - \vec{\eta}_j) \text{ is given in Eqs.(5.34), (5.35)}]$

$$
T^{tr}(T_s, \vec{\sigma}) = \sum_{i=1}^{N} \frac{m_i^2}{\sqrt{\{m_i^2 + [\vec{K}_i - Q_i \sum_{j \neq i} Q_j A_{ls}^i(\vec{\eta}_i - \vec{\eta}_j, \vec{K}_j) + \frac{1}{2} \vec{\nabla} \vec{K}_i] \vec{\nabla} \vec{K}_j]^2}} \\
+ \frac{1}{2} \sum_{i \neq j} Q_i Q_j [(\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(T_s))) \cdot (\vec{\pi}_{ls} + \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_j(T_s))) + (\vec{B}_{si} \cdot \vec{B}_{sj})](T_s, \vec{\sigma}).
$$
\[ T^{\tau\tau}(T_s, \sigma) = \sum_{i=1}^{N} \delta^3(\sigma - \tilde{\eta}_i) + \frac{1}{2} \sum_{j \neq i}^{1..N} Q_i Q_j \tilde{\kappa}_{ij} \tilde{\kappa}_{ij} \]
\[ + \frac{1}{2} Q_i \sum_{j \neq i}^{1..N} \left[ \left( E_{\perp S_i}(\sigma - \tilde{\eta}_i, \tilde{\kappa}_i) + \frac{\bar{\kappa}}{\Delta} \delta^3(\sigma - \tilde{\eta}_i) \right) \cdot \left( E_{\perp S_j}(\sigma - \tilde{\eta}_j, \tilde{\kappa}_j) + \frac{\bar{\kappa}}{\Delta} \delta^3(\sigma - \tilde{\eta}_j) \right) + \right. \]
\[ + \bar{B}_{S_i}(\sigma - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \bar{B}_{S_j}(\sigma - \tilde{\eta}_j, \tilde{\kappa}_j) \right] = \]
\[ \sum_{i=1}^{N} \sqrt{m_i^2 + \bar{\kappa}_i^2} \left[ \delta^3(\sigma - \tilde{\eta}_i) \]
\[ + \frac{1}{2} \bar{\kappa}_i \delta^3(\sigma - \tilde{\eta}_i) \cdot Q_i \sum_{j \neq i} Q_j \tilde{\kappa}_{ij} \tilde{\kappa}_{ij} + \]
\[ + \frac{1}{2} Q_i \sum_{j \neq i}^{1..N} \left[ \left( E_{\perp S_i}(\sigma - \tilde{\eta}_i, \tilde{\kappa}_i) + \frac{\bar{\kappa}}{\Delta} \delta^3(\sigma - \tilde{\eta}_i) \right) \cdot \left( E_{\perp S_j}(\sigma - \tilde{\eta}_j, \tilde{\kappa}_j) + \frac{\bar{\kappa}}{\Delta} \delta^3(\sigma - \tilde{\eta}_j) \right) + \right. \]
\[ + \bar{B}_{S_i}(\sigma - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \bar{B}_{S_j}(\sigma - \tilde{\eta}_j, \tilde{\kappa}_j) \right] = \]
In the final canonical variables there is a dipole term [gradient of delta function] for
each particle like it happens with spinning particles [2]: the role of the spin is taken by
\[ \frac{1}{2} \sum_{j \neq i} Q_j \hat{\partial}_{ij} \hat{\kappa}_{ij}. \]

Let us remark that following [5,6] we can define Dixon’s multipolar expansion [52] of
the energy momentum tensor about an arbitrary point on the Wigner hyperplane. There,
the requirement of having the mass dipole vanishing could be shown to be equivalent to the
identification of the point with \( \vec{R}_+ \approx \vec{q}_+ \), namely the internal center of mass.

For a cluster of \( n \) particles inside the isolated \( N \)-body system, we can now define
a non-conserved energy-momentum tensor \( T_{(n)}^{AB}(T_s, \vec{\sigma}) \) by collecting in the previous equation
all the terms depending on the canonical coordinates \( \vec{\eta}_i, \vec{\kappa}_i \) of the \( n \) particles of the cluster.
This cluster energy-momentum tensor depends also on the canonical coordinates of the other
\( N - n \) particles: for the cluster these are “external fields”. If we make a Dixon multipole
expansion of this cluster energy-momentum tensor with respect to an arbitrary point on the
Wigner hyperplane and we require that the cluster mass dipole vanish, then we can identify a
Møller center of energy \( \vec{R}_+^{(n)} \) for the cluster. This is the only collective configuration variable
which can be defined for a non-isolated cluster, which has no internal conserved Poincaré
algebra associated with it, besides the nonconserved cluster 3-momentum \( \vec{p}_{(int)}^{(n)} = \sum_{i \in \{n\}} \vec{\kappa}_i \).
Moreover, while \( T_{(n)}^{r\tau}(T_s, \vec{\sigma}) \) is the (non-conserved) energy density of the cluster, by analogy
with the theory of dissipative fluids [53] we can say that: i) \( q^\mu(T_s, \vec{\sigma}) = \epsilon^\mu_r(u_p) T_{(n)}^{r\tau}(T_s, \vec{\sigma}) \) is
the heat flow; ii) \( P(T_s, \vec{\sigma}) = \frac{1}{2} \sum_r T_{(n)}^{r\tau}(T_s, \vec{\sigma}) \) is the pressure; iii) \( T_{(n)}^{rs}(T_s, \vec{\sigma}) = T_{(n)}^{rs}(T_s, \vec{\sigma}) - \frac{1}{3} \sum_u T_{(n)}^{uu}(T_s, \vec{\sigma}) \) is the shear (or anisotropic) stress tensor.
VII. CONCLUSIONS.

In this paper we analyzed how to extract the action-at-a-distance interparticle potential hidden in the semiclassical Lienard-Wiechert solution of the electromagnetic field equations, a subset of the solutions of the equations of motion for the isolated system formed by \( N \) scalar charged particles plus the electromagnetic field. The problem is formulated in the Wigner-covariant rest-frame instant form of dynamics, which is defined on the Wigner hyperplanes orthogonal to the total time-like four-momentum of the isolated system and which requires the choice of the sign of the energy of the particles (in this paper we considered only positive energies).

This was possible due to the semiclassical approximation of using Grassmann-valued electric charges \((Q_i^2 = 0, Q_i Q_j \neq 0 \text{ for } i \neq j)\) as an alternative to the extended electron models used for the regularization of the Coulomb self energies. How this happens was shown in Ref. [1], where the Coulomb potential was extracted from the electromagnetic potential by making the canonical reduction of the electromagnetic gauge freedom via the Shanmugadhasan canonical transformation. This is equivalent to the use of a Wigner-covariant radiation (or Coulomb) gauge in the rest-frame instant form.

Ref. [12] presented the retarded Lienard-Wiechert solution for the transverse electromagnetic field in the rest frame instant radiation gauge: in this gauge, due to the transversality, the retarded Lienard-Wiechert potential associated with each charged particle depends on the whole past history of the other particles. At the semiclassical level a single charged particle with Grassmann-valued electric charge does not radiate even if it has a non-trivial Lienard-Wiechert potential, avoiding therefore the acausal features of the Abraham-Lorentz-Dirac equations, and has no mass renormalization. However, a system of \( N \) charged particles produces, by virtue of the interference terms from the various retarded Lienard-Wiechert potentials of the particles, a radiation which reproduces the standard Larmor expression for radiation in the wave zone, when the particles are considered as external sources of the electromagnetic field and their equations of motion are not used.
If instead the particles are considered dynamical, the use of their equations of motion and of the semiclassical approximation lead to a drastic simplification of the Lienard-Wiechert potentials and fields. Indeed, if we make an equal time expansion of the delay by expressing these potentials and fields in terms of particle coordinates, velocities and accelerations of every order, it turns out that all the accelerations decouple at the semiclassical level due to the particle equations of motion.

Therefore, at the semiclassical level the retarded, advanced and symmetric Lienard-Wiechert potentials and the electric and magnetic fields coincide and depend only on the positions and velocities of the particles, so that we can find their phase space expression in terms of particle positions and momenta.

In this way the semiclassical Lienard-Wiechert potential and fields can be reinterpreted as scalar and vector interparticle instantaneous action-at-a-distance potentials. It is then possible to identify a semiclassical reduced phase space containing only particles by eliminating the electromagnetic field by adding by hand second class contraints which force the transverse potential and electric field canonical variables to coincide with the semiclassical Lienard-Wiechert ones in the absence of incoming radiation: \( \mathbf{\tilde{A}}_\perp(\tau, \mathbf{\sigma}) - \mathbf{\tilde{A}}_{\perp, \text{LW}}(\tau, \mathbf{\sigma}) \approx 0, \mathbf{\tilde{\pi}}_\perp(\tau, \mathbf{\sigma}) - \mathbf{\tilde{\pi}}_{\perp, \text{LW}}(\tau, \mathbf{\sigma}) \approx 0. \) Let us remark that this could be done also in presence of an arbitrary incoming radiation \( \mathbf{\tilde{A}}_{\perp, \text{(rad)}}(\tau, \mathbf{\sigma}), \mathbf{\tilde{\pi}}_{\perp, \text{(rad)}}(\tau, \mathbf{\sigma}) = -\frac{\partial}{\partial \tau} \mathbf{\tilde{A}}_{\perp, \text{(rad)}}(\tau, \mathbf{\sigma}) \) [it is an arbitrary solution of the homogeneous wave equation and must not be interpreted as a pair of canonical variables] by modifying the constraints to the form \( \mathbf{\tilde{A}}_\perp(\tau, \mathbf{\sigma}) - \mathbf{\tilde{A}}_{\perp, \text{LW}}(\tau, \mathbf{\sigma}) - \mathbf{\tilde{A}}_{\perp, \text{(rad)}}(\tau, \mathbf{\sigma}) \approx 0, \mathbf{\tilde{\pi}}_\perp(\tau, \mathbf{\sigma}) - \mathbf{\tilde{\pi}}_{\perp, \text{LW}}(\tau, \mathbf{\sigma}) - \mathbf{\tilde{\pi}}_{\perp, \text{(rad)}}(\tau, \mathbf{\sigma}) \approx 0. \)

The reduced phase space is obtained by means of the introduction of the Dirac brackets associated with these second class constraints. Since the old particle positions and momenta are no longer canonical in this reduced phase space, we had to find the new (Darboux) basis of particle canonical variables. The generators of the “internal” Poincaire’ group inside the Wigner hyperplanes in the rest-frame instant form of dynamics can be reexpressed in terms of these new variables: the 3-momentum \( \mathbf{\tilde{P}}_{\text{(int)}} \) and the angular momentum \( \mathbf{\tilde{J}}_{\text{(int)}} \) become equal to those for \( N \) free scalar particles (as expected in an instant form). The interaction
dependent boosts $\mathbf{K}_{\text{init}}$ are proportional to the “internal” canonical center of mass $\mathbf{q}_+^i$ inside the Wigner hyperplane: $\mathbf{q}_+^i \approx 0$ are the gauge-fixings to be be added to the rest-frame conditions $\mathbf{P}_{\text{init}}^r \approx 0$, if one wishes to re-express the dynamics only in terms of particle “internal” relative variables. Also the energy-momentum tensor has been evaluated in the new canonical variables and there is a suggestion on how to find the Möller center of energy of a cluster of $n$ particles contained in the $N$ particle isolated system.

The Hamiltonian in the rest frame frame instant form, generating the evolution in the rest-frame time of the decoupled “external” canonical center of mass, is the “internal” energy generator $M = \mathbf{P}_\text{init}^r$ (the invariant mass of the isolated $N$ particle system). The semiclassical Lienard-Wiechert solution implies the existence of interparticle action-at-a-distance potentials of two types: vector potentials minimally coupled to the Wigner spin 1 particle three-momentum under the square root associated with the kinetic energies; ii) a scalar potential (including the Coulomb potential) outside the square roots. In the semiclassical approximation all these potentials can be replaced by a unique scalar potential, which is the sum of the Coulomb potential and of a generalized Darwin one for arbitrary $N$. It is the (semiclassical) static and non-static complete potential corresponding to the one photon exchange tree Feynman diagrams of scalar electrodynamics and is a completely new result. The expression we find contains no $N$-body forces, being simply a sum of two particle interactions. This is a consequence of our use of Grassmann charges.

In the $N = 2$ case we obtain a closed form of the solution by evaluating it in the rest frame after the gauge fixing $\mathbf{q}_+^i \approx 0$: the lowest order in $1/c^2$ contribution of the generalized Darwin potential agrees with the expression of the standard Darwin potential. We then show that in a semiclassical sense a special solution of the Hamilton equations is the Schild solution [42] in which the two particles move in concentric circular orbits. We evaluate the frequency for equal masses.

Future work will proceed along three parallel courses.

The first is the extention of this work to include semiclassical spinning particles. This
will not only build upon the work here but also that of Ref. [12]. Of particular concern there will be the issue of reproducing the correct spin-orbit and Darwin terms of the appropriate order (the two-body extentions of such terms in the one-body Dirac equation).

The second line is that of the quantization of our general Hamiltonian. Let us remark that quantization of the closed form Hamiltonian for \( N = 2 \) in configuration space would involve not only the usual nonlocal operators for the kinetic energy but also non-local operators for the Darwin portions of the potential. For free scalar particles the positive energy wave equation
\[
\partial_\tau \phi(\tau, \vec{\sigma}) = \sqrt{m^2 + \triangle} \phi(\tau, \vec{\sigma})
\]
has been studied in Ref. [10] by using pseudo-differential operators. Instead see Refs. [54] for the difficulties in quantizing the equal mass two-body problem \((H = 2\sqrt{m^2 + \vec{\kappa}^2} + \frac{\alpha}{\eta})\) with only the Coulomb potential outside the square root. Note that the so-called spinless Salpeter equation would correspond to the quantization of our Hamiltonian with just the Coulomb interaction and at most the lowest order Darwin interaction. For the quantization when there are only scalar potentials inside the square root \((\sqrt{m^2 + V(|\eta|) + \vec{\kappa}^2})\) see Ref. [37].

In our semiclassical approximation we have two options for the action-at-a-distance potentials: i) the vector ones \( \vec{V_i} \) under the square roots and a scalar one \( U \) outside; ii) a unique effective scalar potential sum of the Coulomb and Darwin \((V_{DAR} = V_{LOD} + V_{HOD})\) ones outside the square roots. It can be expected that the results of the quantization of the two options would produce inequivalent theories.

The third line is the development of the quantization of scalar electrodynamics on the Wigner hyperplanes in the rest-frame instant form. This would be a special Wigner-covariant instance of Tomonaga-Schwinger quantum field theory with a well defined covariant concept of “equal times”. To introduce a particle concept in such a quantum formulation, one will have to define Tomonaga-Schwinger asymptotic states and a reduction formalism. The natural candidates for the \( N \)-particle wave functions in such asymptotic states in the case of the Klein-Gordon field would be the wave functions corresponding to the quantization of \( N \) positive energy scalar particles on the Wigner hyperplanes in the rest-frame instant form.

Moreover, in the rest-frame quantum field theory there will be new covariant “equal time”
Green functions. This should allow the definition of a relativistic Schrödinger equation for bound states (replacing the Bethe-Salpeter equation for $N=2$ and avoiding by construction its problems with the spurious solutions, which are a byproduct of the use of asymptotic Fock states, since in a tensor product one cannot eliminate the possibility that an “in” particle be in the absolute future of another one). In the rest-frame quantum field theory it should also be possible to include bound states among the asymptotic Tomonaga-Schwinger states: they should be described by the quantization of isolated $N$ particle systems like the one studied in this paper.

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APPENDIX A: EXACT SUMMATION OF THE TRANSVERSE VECTOR

POTENTIAL SERIES

In this Appendix we perform explicitly the summation of the vector potential below [see Eq.(5.27) with $\hat{\eta}_i = \vec{\beta}_i$ replaced by $\vec{\kappa}_i / \sqrt{m_i^2 + \vec{\kappa}_i^2}$]

$$
\vec{A}_{11}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{1}{(2m)!} \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_\sigma \right)^{2m} |\vec{\sigma} - \hat{\eta}_i|^{2m-1} - \frac{1}{(2m+2)!} \vec{\nabla}_\sigma \left( \frac{\vec{\kappa}_i}{\sqrt{m_i^2 + \vec{\kappa}_i^2}} \cdot \vec{\nabla}_\sigma \right)^{2m+1} |\vec{\sigma} - \hat{\eta}_i|^{2m+1} \right] := \vec{A}_{11}(\vec{\sigma}, \tau) + \vec{A}_{12}(\vec{\sigma}, \tau). \tag{A1}
$$

Using the result of Appendix C that $(\vec{\kappa}_i \cdot \vec{\nabla}_\eta)^{2m} |\hat{\eta}|^{2m-1} = [(2m - 1)!!]^{2} \frac{1}{|\eta|} [\vec{\kappa}^2 - (\vec{\kappa}_i \cdot \frac{\vec{\eta}}{|\vec{\eta}|})^2]^m$, we get $[\vec{\nabla}_\eta = \partial / \partial \hat{\eta}]

$$
\vec{A}_{11}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{(2m - 1)!!}{(2m)!} \frac{\vec{\kappa}_i}{|\vec{\sigma} - \hat{\eta}_i|} \left( \frac{\vec{\kappa}_i}{|\vec{\sigma} - \hat{\eta}_i|} \cdot \vec{\nabla}_\sigma \right)^{m+1} \left( \vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \frac{\vec{\sigma}}{|\vec{\sigma} - \hat{\eta}_i|})^2 \right)^m \right]. \tag{A2}
$$

By using

$$
\frac{(2m - 1)!!}{(2m)!} = \frac{(m - 1/2)!}{(m)!2^{2m}} = \frac{\sqrt{\pi}(-)^{m-1}m^{-1/2}}{(1/2 - m)!m!} = \frac{\sqrt{\pi}(-)^{m}}{(-1/2 - m)!m!} = (-)^{m}\left(\frac{1}{m}\right) \tag{A3}
$$

we find that

$$
\vec{A}_{11}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{\kappa}_i}{|\vec{\sigma} - \hat{\eta}_i|} \sqrt{m_i^2 + \vec{\kappa}_i^2} \left( \frac{\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \frac{\vec{\sigma}}{|\vec{\sigma} - \hat{\eta}_i|})^2}{m_i^2 + \vec{\kappa}_i^2} \right) \sum_{m=0}^{\infty} (-)^{m} \left( \frac{\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \frac{\vec{\sigma}}{|\vec{\sigma} - \hat{\eta}_i|})^2}{m_i^2 + \vec{\kappa}_i^2} \right)^m = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{\kappa}_i}{|\vec{\sigma} - \hat{\eta}_i|} \sqrt{m_i^2 + \vec{\kappa}_i^2} \frac{1}{(\vec{\kappa}_i \cdot \frac{\vec{\sigma}}{|\vec{\sigma} - \hat{\eta}_i|})^2}. \tag{A4}
$$

For $\vec{A}_{12}(\tau, \vec{\sigma})$ we need an expression for $(\vec{\kappa}_i \cdot \vec{\nabla}_\sigma)^{2m+1} |\hat{\eta}|^{2m+1}$. One can show by an induction procedure that

$$
(\vec{\kappa}_i \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_i|^{2m+1} = [(2m + 1)!!]^{2} \sum_{l=0}^{m} (-)^{l} \left( \frac{m}{l} \right) \left( \frac{\vec{\kappa}_i^2 - (\vec{\kappa}_i \cdot \frac{\vec{\sigma}}{|\vec{\sigma} - \hat{\eta}_i|})^2}{2l + 1} \right)^{2l+1}, \tag{A5}
$$

and hence
\( \vec{A}_{\perp 2}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \nabla_{\sigma} V_{\sigma} \sum_{m=0}^{\infty} (-)^{m+1} \left( \frac{\kappa_i^2}{m_i^2 + \kappa_i^2} \right)^m \times \right) \]

Now

\[
\frac{(2m+1)!!}{(2m+2)!} = (-)^{m+1} \left( \frac{-1/2}{m+1} \right)
\]

so that

\[
\vec{A}_{\perp 2}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \nabla_{\sigma} V_{\sigma} \sum_{m=0}^{\infty} (-)^{m+1} \left( \frac{\kappa_i^2}{m_i^2 + \kappa_i^2} \right)^m \times \right) \]

\[
\int_0^{(\vec{\alpha} \cdot \nabla_{\sigma})} \sum_{l=0}^{m} (-)^l \left( \frac{m}{\kappa_i^2} \right) dw = \right)

\[
= -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \nabla_{\sigma} V_{\sigma} \int_0^{(\vec{\alpha} \cdot \nabla_{\sigma})} \left( \frac{\kappa_i^2 - w^2}{m_i^2 + \kappa_i^2} \right)^{-1} \times \right) \]

\[
\sum_{m=0}^{\infty} (-)^{m+1} \left( \frac{\kappa_i^2 - w^2}{m_i^2 + \kappa_i^2} \right)^{m+1} dw = \right)

\[
= -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \nabla_{\sigma} \int_0^{(\vec{\alpha} \cdot \nabla_{\sigma})} \left( \frac{\kappa_i^2 - w^2}{m_i^2 + \kappa_i^2} \right)^{-1} \left[ \frac{1}{\sqrt{m_i^2 + w^2}} - \frac{1}{\sqrt{m_i^2 + \kappa_i^2}} \right] dw \right.
\]

Now

\[
\nabla_{\sigma} \left( \vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|} \right) = \frac{\vec{k}_i}{|\vec{\sigma} - \vec{\eta}_i|} \cdot \left( I - \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|} \right) \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|} .
\]

So we obtain

\[
\vec{A}_{\perp 2}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \frac{Q_i}{4\pi} \frac{\vec{k}_i}{|\vec{\sigma} - \vec{\eta}_i|} \cdot \left( I - \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|} \right) \times \right) \]

\[
\left( \sqrt{m_i^2 + \kappa_i^2} \right) \frac{\sqrt{m_i^2 + \kappa_i^2}}{\sqrt{m_i^2 + \kappa_i^2}} - \frac{1}{\kappa_i^2 - (\vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|})^2} ,
\]

which when combined with the expression for \( \vec{A}_{\perp 1}(\tau, \vec{\sigma}) \) yields the results given by Eq.(5.28).
APPENDIX B: COMPUTATION OF FIELD ENERGY AND MOMENTUM INTEGRALS

Here we carry out the details in the computation of the field energy and momentum for the case \( N = 2 \). The general \( N \) results obtained in the text are an immediate generalization.

From Eq.(6.2) and Eq.(6.3) we find that

\[
\vec{E}_{\perp S}(\tau, \sigma) = \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m)!} \left[ (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} \right] - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n+2)! (2m)!} \times \\
\left[ (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} \right] - (\hat{\eta_2} \cdot \vec{\nabla}_\sigma)^{2n+2} |\vec{\sigma} - \vec{\eta}_2|^{2n+1} - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n+2)! (2m)!} \times \\
\left[ (\hat{\eta_2} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_2|^{2m-1} \right] + \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n+2)! (2m+2)!} \times \\
\left[ \vec{\nabla}_\sigma (\hat{\eta_2} \cdot \vec{\nabla}_\sigma)^{2n+2} |\vec{\sigma} - \vec{\eta}_2|^{2n+1} \right] \times \\
\vec{B}_{S}(\tau, \sigma) = \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m)!} \left[ (\vec{\nabla}_\sigma \hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2n} |\vec{\sigma} - \vec{\eta}_1|^{2n-1} \right] - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m)!} \times \\
\left[ (\vec{\nabla}_\sigma \hat{\eta_2} \cdot \vec{\nabla}_\sigma)^{2n} |\vec{\sigma} - \vec{\eta}_2|^{2n-1} \right] - (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m)!} \times \\
\left[ (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} \right] - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m+2)!} \times \\
\left[ (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} \right] - \frac{Q_1 Q_2}{8\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! (2m+2)!} \times \\
\left[ (\hat{\eta_1} \cdot \vec{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \vec{\eta}_1|^{2m-1} \right]
\]
Our aim here is to compute

\[
\int d^3\sigma (\hat{\nabla}_\sigma \hat{\eta}_1 \cdot \vec{\sigma})^2 m |\vec{\sigma} - \hat{\eta}_2 |^{2n + 1} [(\hat{\nabla}_\sigma \hat{\eta}_1 \cdot \vec{\sigma})^2 m + 2 |\vec{\sigma} - \hat{\eta}_1 |^{2m + 1}]
\]

\[
+ \frac{Q_1 Q_2}{16 \pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)!(2m)!} \hat{\eta}_2 
\]

\[
\left[ \nabla_\sigma (\hat{\nabla}_\sigma \hat{\eta}_1 \cdot \vec{\sigma})^2 m + 2 |\vec{\sigma} - \hat{\eta}_1 |^{2m + 1} \right] \cdot \left[ \nabla_\sigma (\hat{\nabla}_\sigma \hat{\eta}_2 \cdot \vec{\sigma})^2 m |\vec{\sigma} - \hat{\eta}_2 |^{2n + 1} \right]
\]

\[
+ (1 \rightarrow 2).
\]

Our aim here is to compute

\[
\frac{1}{2} \int d^3\sigma (\vec{E}_{\perp S} + \vec{B}_S)(\tau, \vec{\sigma}) := \frac{Q_1 Q_2}{16 \pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)!(2m)!} \left[ (\hat{\eta}_1 \cdot \hat{\eta}_2) I_{1mn} - \frac{1}{(2n + 1)(2n + 2) I_{2mn} - \frac{1}{(2m + 1)(2m + 2) I_{3mn} + \frac{1}{(2n + 1)(2n + 2)(2m + 1)(2m + 2) I_{4mn} + \hat{\eta}_1 \cdot \hat{\eta}_2 I_{5mn} - I_{6mn} \right],
\]

\[
\int d^4\sigma (\vec{E}_{\perp S} \times \vec{B}_S)(\tau, \vec{\sigma}) := \frac{Q_1 Q_2}{16 \pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)!(2m)!} \left[ (\hat{\eta}_1 \cdot \hat{\eta}_2) I_{7mn} - \frac{1}{(2m + 1)(2m + 2) I_{8mn} + \hat{\eta}_1 \cdot \hat{\eta}_2 I_{9mn} + \frac{1}{(2m + 1)(2m + 2) I_{10mn} \right] + \frac{1}{(2n + 1)(2n + 2)},
\]

which involve ten different integrals defined by

\[
I_{1mn} = \int d^4\sigma \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right] \left[ (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2m+1} \right],
\]

\[
I_{2mn} = \int d^4\sigma \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2n+1} \right] \left[ (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2m+1} \right],
\]

\[
I_{3mn} = \int d^4\sigma \left[ (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right] \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right],
\]

\[
I_{4mn} = \int d^4\sigma \left[ \hat{\nabla}_\sigma (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right] \cdot \left[ \hat{\nabla}_\sigma (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right],
\]

\[
I_{5mn} = \int d^4\sigma \left[ \hat{\nabla}_\sigma (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_1 |^{2n+1} \right] \cdot \left[ \hat{\nabla}_\sigma (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_2 |^{2m+1} \right],
\]

\[
I_{6mn} = \int d^4\sigma \left[ (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)(\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2n+1} \right] \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)(\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2m+1} \right],
\]

\[
I_{7mn} = \int d^4\sigma \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right] \left[ \hat{\nabla}_\sigma (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right],
\]

\[
I_{8mn} = \int d^4\sigma \left[ (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right] \left[ \hat{\nabla}_\sigma (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right],
\]

\[
I_{9mn} = \int d^4\sigma \left[ \hat{\nabla}_\sigma (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right] \left[ \hat{\nabla}_\sigma (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right],
\]

\[
I_{10mn} = \int d^4\sigma \left[ \hat{\nabla}_\sigma (\hat{\eta}_1 \cdot \hat{\nabla}_\sigma)^{2m+1} |\vec{\sigma} - \hat{\eta}_1 |^{2m+1} \right] \cdot \left[ \hat{\nabla}_\sigma (\hat{\eta}_2 \cdot \hat{\nabla}_\sigma)^{2n+1} |\vec{\sigma} - \hat{\eta}_2 |^{2n+1} \right].
\]
The powers of $\sigma$ in each of the ten integrands is $\sigma^{2m-1-2m-1}\sigma^{2n-1-2n-1} \sim \sigma^{-4}$ Therefore the integrals converge. Thus

\[
I_i = \lim_{\Lambda \to \infty} \int_0^\Lambda d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_2mn = -\left(\hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_3mn = -\left(\hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_4mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_5mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_6mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_7mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_8mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m-1}\sigma^{2n-1},
\]

\[
I_9mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m+1}\sigma^{2n-1},
\]

\[
I_{10}mn = -\left(\hat{\nabla}_\eta \hat{\eta}_1 \cdot \hat{\nabla}_\eta\right)^{2m+2} \int d\bar{\sigma} \int d\Omega_\sigma(\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)^{2m+1}\sigma^{2n-1}.
\]

The integrals that remain to be evaluated are each of the form

\[
\int d\bar{\sigma} \sigma^{2n-1} |\bar{\sigma} - (\bar{\eta}_1 - \bar{\eta}_2)|^{2m-1} = 2\pi \int_{-1}^{1} dz (\sigma^2 + \eta^2 - 2\eta \sigma z)^{m-1/2} =
\]

\[
= -\frac{2\pi}{\eta(2m + 1)} \int_0^\Lambda d\sigma \sigma^{2n} \int_{-1}^{1} dz (\sigma^2 + \eta^2 - 2\eta \sigma z)^{m-1/2} =
\]

\[
= -\frac{2\pi \eta^{2n+1} + 2m+1}{(2m + 1)} \sum_{k=0}^{2m+1} \left[ \frac{(\Lambda/\eta)^{2n+k+1}}{2n + k + 1} \left(2m+1\right) \left(-1\right)^{k-1} - 2 \frac{(-1)^{k+1}}{2n+k+1} \left(\frac{2m+1}{k}\right) \right].
\]

It is of interest to show how the $\Lambda$ dependent terms vanish after the $\eta$ derivatives have acted.

First note that only the even $k$ terms of the $\Lambda$ dependent sum survive. Of the ones that remain, the power of $\eta$ is $\eta^{2n+1+2m-2n-2k+1} = \eta^{2m-2k} = \eta^{\text{even power}}$. But since the power is

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and since the number of $\eta$ derivatives always exceeds that even power, the derivative vanishes, e.g.

$$\frac{\partial^3}{\partial \eta \partial \eta \partial \eta} \eta^2 = \frac{\partial^2}{\partial \eta \partial \eta} 2\eta_k = 0,$$
$$\frac{\partial^5}{\partial \eta \partial \eta \partial \eta \partial \eta \partial \eta} \eta^4 = 4 \frac{\partial^4}{\partial \eta \partial \eta \partial \eta \partial \eta} 2\eta_m = 4 \frac{\partial^3}{\partial \eta \partial \eta \partial \eta} (2\eta_m + \eta^2 \delta_{ml}) = 0. \tag{B10}$$

Note further that the last sum is

$$2 \sum_{k=0}^{2m+1} (-1)^k \binom{2m + 1}{k} \int_0^1 dx \ x^{2n+k} = 2 \int_0^1 dx \ x^{2n} (1-x)^{2m+1} = 2B(2n+1, 2m+2). \tag{B11}$$

Thus the portion of the integral that “survives” is

$$\int_{\Lambda} d^3 \sigma \sigma^{2n-1} |\mathbf{\bar{\sigma}} - \mathbf{\bar{\eta}}|^{2m-1} n = -\frac{4\pi (2n)! (2m)!}{(2n + 2m + 2)!} \eta^{2n+2m+1}, \tag{B12}$$

and our ten integrals appear as (using $\nabla^2 \eta^l = l (l - 1) \eta^{-2}$)

$$\frac{I_{1mn}}{16\pi^2 (2n)! (2m)!} = \frac{1}{4\pi (2n + 2m + 2)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+1},$$
$$\frac{I_{2mn}}{16\pi^2 (2n)! (2m)!} = -\frac{1}{4\pi (2n + 2m + 4)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+3},$$
$$\frac{I_{3mn}}{16\pi^2 (2n)! (2m)!} = -\frac{1}{4\pi (2n + 2m + 4)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+3},$$
$$\frac{I_{4mn}}{16\pi^2 (2n)! (2m)!} = \frac{1}{4\pi (2n + 2m + 6)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+5} =$$
$$\frac{I_{5mn}}{16\pi^2 (2n)! (2m)!} = \frac{1}{4\pi (2n + 2m + 2)!} \nabla^2 (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+1},$$
$$\frac{I_{6mn}}{16\pi^2 (2n)! (2m)!} = -\frac{1}{4\pi (2n + 2m + 2)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+1},$$
$$\frac{I_{7mn}}{16\pi^2 (2n)! (2m)!} = \frac{1}{4\pi (2n + 2m + 2)!} \nabla^2 (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+1},$$
$$\frac{I_{8mn}}{16\pi^2 (2n)! (2m)!} = -\frac{1}{4\pi (2n + 2m + 4)!} (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+3},$$
$$\frac{I_{9mn}}{16\pi^2 (2n)! (2m)!} = \frac{1}{4\pi (2n + 2m + 4)!} \nabla^2 (\hat{\eta}_1 \cdot \nabla \eta) (\hat{\eta}_2 \cdot \nabla \eta) \eta^{2n+2m+3} =$$

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\[ I_{8mn} = \frac{I_{8mn}}{16\pi^2(2n)!(2m)!}. \] (B13)

Substitution into Eq.(B4) and Eq.(B5) leads to Eq.(6.4) and Eq.(6.5)
APPENDIX C: THE EVALUATION OF $(\vec{\kappa}_1 \cdot \vec{\partial})^A(\vec{\kappa}_2 \cdot \vec{\partial})^B \eta^{A+B-1}$

Consider the case with $a = 2m + 1, \ b = 2n + 1$,

$$(\vec{\kappa}_1 \cdot \vec{\partial})^{2m+1}(\vec{\kappa}_2 \cdot \vec{\partial})^{2n+1}\eta^{2m+2n+1}. \tag{C1}$$

Let

$$\vec{\kappa}_1 = \kappa_1 \hat{\kappa}_1; \quad \vec{\kappa}_2 = \kappa_2 \hat{\kappa}_2; \quad \hat{\kappa}_1 = 1 = \hat{\kappa}_2, \tag{C2}$$

and

$$\hat{\kappa}_2 = \alpha \hat{\kappa}_1 + \beta \hat{\kappa}_1 \perp; \quad \hat{\kappa}_1 \cdot \hat{\kappa}_1 \perp = 0; \quad \hat{\kappa}_1 \cdot \hat{\kappa}_2 = \alpha; \quad \alpha^2 + \beta^2 = 1; \quad \beta = (1 - (\hat{\kappa}_1 \cdot \hat{\kappa}_2)^2)^{1/2}; \tag{C3}$$

so

$${\nu}.(C1) = \kappa_1^{2m+1} \kappa_2^{2n+1} \sum_{j=0}^{2n+1} \binom{2n+1}{j} \alpha^j \beta^{2n+1-j} (\hat{\kappa}_1 \cdot \vec{\partial})^{2m+1+j} (\hat{\kappa}_1 \perp \cdot \vec{\partial})^{2n+1-j} \eta^{2m+n+1}. \tag{C4}$$

Orient our axes so that

$$\vec{\eta} = x \hat{x} + y \hat{y} + Z \hat{z}, \tag{C5}$$

with

$$Z = \vec{\eta} \cdot (\hat{\kappa}_1 \times \hat{\kappa}_2), \tag{C6}$$

so that, the $Z$ direction is perpendicular to the plane containing $\vec{\kappa}_1$ and $\vec{\kappa}_2$. Further orient axes so that

$$\vec{\eta} = x \hat{\kappa}_1 + y \hat{\kappa}_2 + Z \hat{\kappa}_1 \times \hat{\kappa}_2. \tag{C7}$$

Thus

$${\nu}.(C4) = \kappa_1^{2m+1} \kappa_2^{2m+1} \sum_{j=0}^{2n+1} \binom{2n+1}{j} \alpha^j \beta^{2n+1-j} (\partial_x)^{2m+1+j} (\partial_y)^{2n+1-j} \eta^{2m+n+1}. \tag{C8}$$
Consider just the portion
\[ \partial_x^{2m+1+j} \partial_y^{2n+1-j} \eta^{2m+2n+1}. \] (C9)

Let
\[ \eta = (\rho^2 + Z^2)^{\frac{1}{2}} ; \quad \rho^2 = x^2 + y^2 = zz^* ; \quad z = x + iy, z^* = x - iy. \] (C10)

Thus
\[ \partial_x = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} = \partial_z \frac{\partial}{\partial x} \partial_z^* + \partial_{z^*} \frac{\partial}{\partial x} \partial_{z^*}^* ; \quad \partial_y = i(\partial_z - \partial_{z^*}), \] (C11)

and
\[ \text{Eq. (C9)} = i^{2n+1-j} \sum_{h=0}^{2m+1+j} \binom{2m+1+j}{h} \sum_{k=0}^{2n+1-j} \binom{2n+1-j}{k} \times \partial_z^{h+k} (-)^{2n+1-y-k} \partial_{z^*}^{2m+2n+2-h-k} (zz^* + Z^2)^{m+n+\frac{1}{2}}. \] (C12)

Since \( \partial_z z^* = 0 = \partial_{z^*} z \), the derivatives become relatively simple
\[ \partial_z^{h+k} (zz^* + Z^2)^{m+n+\frac{1}{2}} = \frac{(m+n + \frac{1}{2})!}{(m+n + \frac{1}{2} - h - k)!} z^{h+k}(zz^* + Z^2)^{m+n+\frac{1}{2} - h - k}, \] (C13)

and
\[ \partial_{z^*}^{2m+2n+2-h-k} z^{h+k}(zz^* + Z^2)^{m+n+\frac{1}{2} - h - k} = \sum_{i=0}^{2m+2n+2-h-k} \binom{2m+2n+2-h-k}{i} \partial_z^{i} z^{h+k} \partial_{z^*}^{2m+2n+2-h-k-i} (zz^* + Z^2)^{l+\frac{1}{2} - h - k}. \] (C14)

Now
\[ \partial_{z^*}^{i} z^{h+k} = \frac{(h+k)!}{(h+k-i)!} z^{h+k-i}, \] (C15)

Note that the factorial in denominator takes care of cutoff, so we can avoid having to worry about upper limit on sum.
\[ \partial_{z^*}^{2(m+n+1)-h-k-i} (zz^* + Z^2)^{l+\frac{1}{2} - h - k} = \frac{(m+n + \frac{1}{2} - h - k)!}{(i - m - n - \frac{3}{2})!} z^{2(m+n+1)-h-k-i} (zz^* + Z^2)^{i-m-n-\frac{3}{2}}. \] (C16)
Thus combining factors

\[ Eq.(C8) = \kappa_1^{2m+1} \kappa_2^{2n+1} (m + n + \frac{1}{2})! \eta^{-2m+2n+1} z^{2m+2n+1} \beta^{2n+1} (-)^{n+1} \times \]

\[
\sum_{j=0}^{2n+1} \binom{2n+1}{2n+1-j} \left( \frac{i \alpha}{\beta} \right)^j \sum_{h=0}^{(m+n+1)+j} \sum_{k=0}^{2n+1-j} \sum_{i=0}^{2(2m+n+1)-h-k} \left( \frac{n^2 z^i}{z} \right)^h \frac{1}{(i-m-n-3/2) (h+k-i)}.
\]

(C17)

Consider second two lines and let upper limits be \( \infty \) and let the factorials set the limits.

That becomes of form

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} f(j, h, k, i).
\]

(C18)

Let \( q = 2n + 1 - j \), \( p = h + k \). Then

\[
Eq.(C18) = (2n+1)! \left( \frac{i \alpha}{\beta} \right)^{2n+1} \sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{-i \beta}{\alpha} \right)^q \left( \frac{n^2 z^i}{z} \right)^p (-)^k \times \frac{1}{q! (2n+1-q)! k!(q-k)! (p-k)! (2(m+n+1)-q-p+k)!} \times \frac{1}{i!(2(m+n+1)-p-i)! (p-i)! (i-m-n-3/2)!}.
\]

(C19)

Use factorials to place upper bound on summation limits. Consider \( q \) first; \( q \geq 0, q \geq k, q \leq 2(m+n+1)-p+k; q \geq 2n+1 \). Let \( s = q-k \geq 0 \). Upper bound appears ambiguous.

One can make the replacement

\[
\frac{(2(m+n+1)-q)!}{(2n+1-q)!} = \left( \frac{d}{dw} \right)^{2n+1} w^{2(n+m+1)-q} \bigg|_{w=1} = \left( \frac{d}{dw} \right)^{2m+1} \bigg|_{w=1},
\]

(C20)

and so our \( s \) sum is restricted to

\[
0 \leq s \leq 2(m+n+1)-p .
\]

(C21)

Note that the right hand side must be positive. So we can perform the \( q \) (or \( s \)) sum involved in (C19). It is (using \( 1 = \frac{(2(m+n+1)-p)!}{(2(m+n+1)-p)!} \))
We can perform the sum now has well defined limits since $2(m + n + 1) − p − 1 \geq 0$ and therefore $i \leq p \leq 2(m + n + 1) − i$.

Thus

$$
\text{Eq. (C23)} = (2n + 1)\left(\frac{i\alpha}{\beta}\right)^2n+1\left(\frac{d}{dw}\right)^{2n+1}\sum_{i=0}^{\infty} \sum_{p=0}^{2(m+n+1)-i} w^{p-i} \left(1 + \frac{i\beta}{\alpha}\right)^p
\times \left(\frac{\eta^2}{zz^*}\right)^i \left(\frac{z^*}{z}\right)^p \frac{1}{(2(m + n + 1) - p - i)! (i - m - n - \frac{3}{2})!}.
$$

Next, noting that $w^p \left(1 + \frac{i\beta}{\alpha}\right)^p = (w + i\beta)^p$ and that the $p$ sum produces $\frac{1}{(2(m+n+1)-2i)!}$ we see that $i$ is restricted to $0 \leq i \leq m + n + 1$.

The $p$ sum is best performed by changing variables to $r = p - i$. Then $0 \leq r \leq 2(m + n + 1 - i)$. Thus we obtain

$$
\text{Eq. (C24)} = (2n + 1)\left(\frac{i\alpha}{\beta}\right)^2n+1\left(\frac{d}{dw}\right)^{2n+1}\sum_{i=0}^{\infty} \sum_{r=0}^{2(m+n+1)-i} w^r \left(1 + \frac{i\beta}{\alpha}\right)^r
\times \left(\frac{\eta^2}{zz^*}\right)^i \left(\frac{z^*}{z}\right)^r \left[1 + \frac{z^*(w + i\beta)}{z(w - i\beta/\alpha)}\right]^{2(m+n+1)-i} \left(\frac{w - i\beta}{\alpha}\right)^{2(m+n+1)}.
$$

To perform the $i$ sum, notice that

$$
\frac{1}{(i - m - n - 3/2)! (2(m + n + 1) - i)!} = \frac{\pi^{1/2}}{2^{(m+n+1-i)}} (m + n + 1 - i)! (m + n + \frac{1}{2} - i)! (1 - m - n - 3/2)! = \frac{\pi^{1/2} \sin(\pi (i - m - n - \frac{1}{2}))}{2^{(m+n+1-i)}} (m + n + 1 - i)! = \frac{(-)^{i-m-n+1}}{\pi^{1/2}2^{(m+n+1-i)}} (m + n + 1 - i)!.
$$
Further notice that the 2nd line of (C25) can be simplified to

\[
(w^2 + \beta^2/\alpha^2)^i \left[ z(w - i\beta/\alpha) + z^*(w + i\beta/\alpha) \right]^{2(m+n+1)-1} / \eta^i \eta^2_i (m+n+1)!
\].

(C27)

So

\[
\text{Eq.(C25)} = (2n+1)! \left( \frac{i\alpha}{\beta} \right)^{2n+1} (\frac{d}{dw})^{2m+1} \text{ } \frac{(-)^{m+n+1}}{\pi^{\frac{2}{2}(m+n+1)}} (z(w - \frac{1}{2}\beta) + z^*(w + i\beta/\alpha))^{2(m+n+1)} \times \sum_{i=0}^{m+n+1} \frac{1}{i!} \left( \frac{2^{2i}}{(m+n+1-i)!} \right) \eta^2_i (w^2 + \beta^2/\alpha^2)^i (-)^i \]

(C28)

The sum produces

\[
\frac{1}{(m+n+1)!} \left[ 1 - \frac{4\eta^2(w^2 + \beta^2/\alpha^2)}{\eta^2 \eta^2_i} \right]^{m+n+1} \]

(C29)

Combine this with (C28) and then (C17) gives

\[
\text{Eq.(C17)} = \kappa_2^{2m+1} \kappa_2^{2n+1} (m+n+\frac{1}{2})! \eta^{-2m-2n-3} (-)^i \alpha^{2n+1} (2n+1)! (-)^{m+n+1} \times
\]

\[
\left( \frac{d}{dw} \right)^{2m+1} \left[ (z(w - i\beta/\alpha) + z^*(w + i\beta/\alpha))^{2} - 4\eta^2(w^2 + \beta^2/\alpha^2) \right]^{m+n+1} / (m+n+1)! \]

(C30)

Before expanding or simplifying consider case when

\[
\kappa_2 = -\kappa_1 = -\kappa \Rightarrow \beta = 0.
\]

Then

\[
\text{Eq.(C30)} = (-)^{m+n} \frac{\kappa^{2m+2n+2}}{\pi^{\frac{2}{2}(m+n+1)}} (m+n+\frac{1}{2})! \eta^{-2m-2n-3} (2n+1)! \times
\]

\[
\left( \frac{d}{dw} \right)^{2m+1} (w^2(4x^2 - \eta^2))^{m+n+1} / (m+n+1)! \]

(C31)

Use

\[
\frac{d^{2m+1}}{dw^{2m+1}} w^{2m+2n+1} \bigg|_{w=1} = \frac{(2(m+n+1))!}{(2n+1)!}.
\]

(C32)

Also \((m+n+\frac{1}{2})! = \frac{(2(m+n+1))!}{(m+n+1)!} \pi^{\frac{2}{2}(m+n+1)} \). Thus

\[
\text{Eq.(C30)} = -\kappa^{2m+2n+2} \left[ (2(m+n+1))! \right]^2 / (m+n+1)! \frac{1}{\eta^2} \left( 1 - \frac{\eta^2}{\eta^2} \right)^{m+n+1} \frac{1}{\eta} =
\]

\[
= -[(2m+2n+1)!]^{\frac{1}{\eta^2}} \eta^2 (\kappa^2 - (\kappa \cdot \eta^2))^{m+n+1}.
\]

(C33)
Return to (C30) and consider $z(w - i\beta/\alpha) + z^*(w + i\beta/\alpha) = 2(xw + y\beta/\alpha)$. In (C30) this factor becomes

$$[\ ]^{m+n+1} = 2^{(m+n+1)}(-)^{m+n+1} \left[ \eta^2(w^2 + \beta^2/\alpha^2) - x^2w^2 - 2xyw\beta/\alpha - y^2\beta^2/\alpha^2 \right]^{m+n+1} = 2^{(m+n+1)}(-)^{m+n+1}[w^2(\eta^2 - x^2) + \beta^2/\alpha^2(\eta^2 - y^2) - 2xyw\beta/\alpha]^{m+n+1} = 2^{(m+n+1)}(-)^{m+n+1}[\eta^2(w^2 + \beta^2/\alpha^2) - (xw + y\beta/\alpha)^2]^{m+n+1}. \quad (C34)$$

Thus

$$\text{Eq. (C30)} = \kappa_1^{2m+1}\kappa_2^{2n+1} \left[ \frac{2(m+n+1)!}{2^{2m+2n+2}[(n + m + 1)!]^2} \frac{\alpha^{2n+1}(2n+1)!}{\eta} \right] \times \left( \frac{d}{dw} \right)^{2m+1} \left[ (w^2 + \beta^2/\alpha^2) - \left( \frac{xw}{\eta} + \frac{y\beta}{\eta\alpha} \right)^2 \right]^{m+n+1} \bigg|_{w=1}. \quad (C35)$$

The derivative is of the form

$$\left( \frac{d}{dw} \right)^{2m+1} \left( aw^2 + bw + c \right)^{m+n+1} = \left( \frac{d}{dw} \right)^{2m+1} F(Q(w)) = \left( \frac{d}{dw} \right)^{2m+1} f(w). \quad (C36)$$

This radical is

$$\omega^2\left(1 - \frac{x^2}{\eta^2}\right) - \frac{2xy\beta w}{\eta^2\alpha} + \frac{\beta^2}{d^2} \left(1 - \frac{y^2}{\eta^2}\right), \quad (C37)$$

with roots

$$w = \frac{xy\beta}{\eta^2\alpha} \pm \sqrt{\frac{x^2y^2\beta^2}{\eta^2\alpha^2} - \frac{\beta^2}{d^2} \left(1 - \frac{x^2}{\eta^2}\right) \left(1 - \frac{y^2}{\eta^2}\right)} \left(1 - \frac{x^2}{\eta^2}\right)^{-1}. \quad (C38)$$

Since the argument of the root is

$$-\frac{\beta^2}{\alpha^2} \left(1 - \frac{x^2}{\eta^2} - \frac{z^2}{\eta^2}\right) < 0, \quad (C39)$$

the roots are complex. Thus

$$aw^2 + bw + c = a(w - \gamma - i\delta)(w - \gamma + i\delta),$$

where

$$\gamma = \frac{xy\beta}{\alpha(\eta^2 - x^2)}; \quad \delta = \frac{\beta}{\alpha(\eta^2 - x^2)} \cdot Z\eta. \quad (C40)$$

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Consider the derivative:

\[
\frac{d^{2m+1}}{dw^{2m+1}}(w - \gamma - i\delta)^{m+n+1}(w - \gamma + i\delta)^{m+n+1} =
\]

\[
= \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left( \frac{d}{dw} \right)^j (w - \gamma - i\delta)^{m+n+1} \left( \frac{d}{dw} \right)^{2m+1-j} (w - \gamma + i\delta)^{m+n+1} =
\]

\[
= \sum_{j=0}^{2m+1} \binom{2m+1}{j} \frac{[(m+n+1)!]^2}{(m+n+1-j)! (n+j-m)!} (w - \gamma - i\delta)^{m+n+1-j} (w - \gamma + i\delta)^{n+j-m}.
\]

(C41)

The factorial further restricts the sum to

\[
\sum_{j=\max(0,n-m)}^{\min(2m+1,m+n+1)}.
\]

Consider the case when \( \beta = 0 = \delta = \gamma \). Then as \( w = 1 \)

\[
Eq. (C41) = \sum_{j=\max(0,n-m)}^{\min(2m+1,n+m+1)} \binom{2m+1}{j} \frac{(m+n+1)!}{(m+n+1-j)! (n+j-m)!} =
\]

\[
= \frac{[(m+n+1)!]^2}{(2n+1)!} \sum_{j=\max(0,n-m)}^{\min(2m+1,n+m+1)} \binom{2m+1}{j} \binom{2n+1}{n+j-m}.
\]

(C42)

In order to perform this sum, consider the related product (for \( s = 1 \))

\[
\sum_{j=0}^{2m+1} \binom{2m+1}{j} s^j \sum_{i=m-n}^{n+m+1} \binom{2n+1}{n+j-m} s^i =
\]

\[
= \text{(let } l = i - m - n = \sum_{j=0}^{2m+1} \binom{2m+1}{j} s^j \sum_{i=0}^{2n+1} \binom{2n+1}{l} s^l)^{2n+m} =
\]

\[
= s^{(n+m)} (1 + s)^{2(m+n+1)} = \sum_{h=0}^{2(m+n+1)} \binom{2(m+n+1)}{h} s^h s^{n+m}.
\]

(C43)

But consider the product of the two sums. Let \( h = j + \ell \) so that \( 0 \leq h \leq 2(m + n + 1) \). Let \( l = h - j \). Thus, since \( l \geq 0 \) \( \Rightarrow \) \( h \geq j \) and since \( 2m + 1 \geq j \), we have \( j \leq \min(2m + 1, h) \).

Now we also have \( 2n + 1 \geq l \) or \( j \geq h - 2n - 1 \); thus \( j \geq \max(0, h - 2n - 1) \) so

\[
\sum_{h=0}^{2(m+n+1)} \binom{2(m+n+1)}{h} s^h = \sum_{h=0}^{\min(2m+1,h)} \sum_{j=\max(0,h-2n-1)}^{\min(2m+1,j)} \binom{2m+1}{j} \binom{2n+1}{h-j}.
\]

(C44)

Consider the term \( h = m + n + 1 \) in the sum. Then

\[
\frac{(2(m+n+1)!}{[(m+n+1)!]^2} = \sum_{j=\max(0,m-n)}^{\min(2m+1,m+n+1)} \binom{2m+1}{j} \binom{2n+1}{n+j-m},
\]

(C45)
where we have used \( \binom{2n+1}{h-j} = \binom{2n+1}{n+m+1-j} = \binom{2n+1}{n+j-m} \). Thus

\[
\text{Eq. (C42)} = \frac{(2(m + n + 1))!}{(2n + 1)!}.
\]

Note that

\[
\text{Eq. (C35)}|_{\beta=0} = -\kappa^{2(m+n+1)} \left\{ \frac{(2(m + n + 1))!^2}{2^{2(m+n+1)}[(m + n + 1)!]^2} \frac{1}{\eta^2} \left( 1 - \frac{x^2}{\eta^2} \right)^{m+n+1} \right. =
\]

\[
= -([2m + 2n + 1]!!) \frac{1}{\eta^2} (\kappa^2 - (\kappa \cdot \eta)^2)^{m+n+1},
\]

which checks again with Eq. (C33). Thus, with this we write again

\[
\text{Eq. (C35)} = \kappa_1^{2m+1} \kappa_2^{2n+1} \frac{(2(m + n + 1))!}{2^{2m+2+1}[(m + n + 1)!]^2} \frac{\alpha^{2n+1}}{\eta^2} \left( 1 - \frac{x^2}{\eta^2} \right)^{m+n+1} \times
\]

\[
\min(2m+1,n+m+1) \sum_{j=\max(0,m-n)}^{2m+1} \binom{2m+1}{j} \binom{2n+1}{n+j-m} (w - \gamma - i\delta)^{m+n+1-j} (w - \gamma + i\delta)^{n+j-m}|_{w=1}.
\]

To simplify, let \( j = k - n \). Also note

\[
(w - \gamma - i\delta)^{m+n+1-j} (w - \gamma + i\delta)^{n+j-m}|_{w=1} =
\]

\[
= \left[ \frac{1}{\alpha(\eta^2 - x^2)} \right]^{2n+1} [\alpha(\eta^2 - x^2) - \beta(xy + i\eta Z)]^{m+2n+1-k}
\]

\[
[\alpha(\eta^2 - x^2) - \beta(xy - i\eta Z)]^{k-m}.
\]

Thus Eq. (C47) becomes

\[
\frac{[(2m + 2n + 1)!!]^2}{(2(m + n + 1))!} \kappa_1^{2m+1} \kappa_2^{2n+1} \left( \frac{1 - \frac{x^2}{\eta^2}}{\eta^2} \right)^{min(2m+n+1,2n+m+1)} \sum_{k=\max(m,n)}^{m+n+1} \left( \frac{2m+1}{k-n} \right) \left( \frac{2n+1}{k-m} \right) (\sigma^2 \alpha - \beta(xy + i\eta Z))^{m+2n+1-k} (\sigma^2 \alpha - \beta(xy - i\eta Z))^{k-m}.
\]

where

\[
\sigma^2 = \eta^2 - x^2.
\]

If \( m \geq n \) then sum becomes \( (l := k - m) \)

\[
\sum = \sum_{l=0}^{2m+1} \left( \frac{2n+1}{l} \right) \left( \frac{2m+1}{\ell + m + n} \right) (\sigma^2 \alpha - (xy + i\eta Z)\beta)^{2n+1-l} (\sigma^2 \alpha - (xy - i\eta Z)\beta)^l.
\]

(C51)
Because of the symmetry we have
\[
\sum = \sum_{l=0}^{n} \binom{2n+1}{l} \binom{2m+1}{\ell + m - n} \left[ (\sigma^2\alpha - (xy + i\eta Z)\beta)^{2n+1-l} (\sigma^2\alpha - (xy - i\eta Z)\beta)^l + c.c. \right].
\]
\[(C52)\]

The bracket \([ \ ]\) in (C52) above is of the form
\[
[ \ ] = (u - iv)^{2n+1-l}(u + iv)^l + c.c. = (let \ u + iv = re^{i\phi}) =
\]
\[
r^{2n+1}(e^{i(2(l-n)-1)} + c.c.) = 2\gamma^{2n+1}\cos(2(n - l) + 1)\phi,
\]
\[(C53)\]
in which
\[
r = \sqrt{(\sigma^2\alpha - xy\beta)^2 + \eta^2 Z^2 \beta^2},
\]
\[
\phi = \tan^{-1} \frac{\beta \eta z}{\sigma^2\alpha - xy\beta}. \quad (C54)
\]

We further need
\[
\cos((2(n - l) + 1)\phi) = \sum_{k=0}^{n-l} \binom{2(n - l) + 1}{2k} (-)^k (\sin\phi)^{2k} (\cos\phi)^{2(n-l-k) + 1} =
\]
\[
= \sum_{k=0}^{n-l} \binom{2(n - l) + 1}{2k} (-)^k \sum_{j=0}^k \binom{k}{j} (-)^j (\cos\phi)^{2(n-l-k+j) + 1}, \quad (C55)
\]
where
\[
\cos\phi = \frac{\sigma^2\alpha - xy\beta}{r}. \quad (C56)
\]

Recall the original variables
\[
\eta = |\vec{\eta}|; \ \eta^2 = \rho^2 + Z^2 = \sigma^2 + x^2 = x^2 + y^2 + Z^2;
\]
\[
\alpha = \hat{\kappa}_1 \cdot \hat{\kappa}_2; \ \beta = \sqrt{1 - (\hat{\kappa}_1 \cdot \hat{\kappa}_2)^2},
\]
\[
x = \hat{\kappa}_1 \cdot \vec{\eta}; \ \beta y = \hat{\kappa}_1 \perp \cdot \vec{\eta} = (\hat{\kappa}_2 - \alpha \hat{\kappa}_1) \cdot \vec{\eta}. \quad (C57)
\]

Thus
\[
\sigma^2\alpha - xy\beta = \alpha (\eta^2 - (\vec{\eta} \cdot \hat{\kappa}_1)^2) - \eta^2 \hat{\kappa}_1 (\hat{\kappa}_2 - \alpha \hat{\kappa}_1) \cdot \vec{\eta} =
\]
\[
= \eta^2 (\hat{\kappa}_1 \cdot \hat{\kappa}_2 - \hat{\kappa}_1 \cdot \hat{\eta} \hat{\kappa}_2 \cdot \vec{\eta}),
\]
\[
\eta^2 Z^2 \beta^2 = \eta^4 (1 - (\vec{\eta} \cdot \hat{\kappa}_1)^2 - (\vec{\eta} \cdot \hat{\kappa}_2)^2 - (\hat{\kappa}_1 \cdot \hat{\kappa}_2)^2 - 2 \hat{\kappa}_1 \cdot \hat{\kappa}_2 \hat{\eta} \hat{\kappa}_2 \vec{\eta})), \quad (C58)
\]
and
\[ r^2 = \eta^4(1 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)(1 - (\hat{\kappa}_2 \cdot \hat{\eta})^2). \]  

(C59)

Thus
\[ \cos \phi = \frac{\hat{\kappa}_1 \cdot \hat{\kappa}_2 - \hat{\kappa}_1 \cdot \hat{\eta}\hat{\kappa}_2 \cdot \hat{\eta}}{\sqrt{\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2} \sqrt{\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2}}. \]  

(C60)

and
\[
\frac{r^{2n+1}}{\eta^{4n+2}} \left(1 - \frac{x^2}{\eta^2}\right)^{m-n} \kappa_1^{2m+1} \kappa_2^{2n+1} = \\
= (1 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^n (1 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^n (\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^{1/2} (\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^{1/2},
\]

(C61)

\[
(\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^{m-n} \kappa_1^{2n} \kappa_2^{2n} = \\
= (\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^{\frac{1}{2}} (\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^{1/2} (\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^n (\kappa_1^2 - \hat{\kappa}_1 \cdot \hat{\eta})^m.
\]

(C62)

So finally factoring out \( \cos \phi \) we obtain
\[
(\tilde{\kappa}_1 \cdot \tilde{\eta})^{2m+1} (\tilde{\kappa}_2 \cdot \tilde{\eta})^{2n+1} \eta^{2m+2n+1} = \\
= 2 \left[ \frac{[2m + 2n + 1]!!}{(2m + 2n + 1)!} \right]^2 \left[ (n + m + 1)! \right]^2 \\
\times \frac{\eta^4 (1 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^n (1 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^n (\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^{1/2} (\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2)^{1/2}}{\sqrt{\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2} \sqrt{\kappa_2^2 - (\hat{\kappa}_2 \cdot \hat{\eta})^2}}
\]

\[
\times \sum_{l=0}^{n} \sum_{k=0}^{n-l} \sum_{j=0}^{k} \binom{2n+1}{l} \binom{2m+1}{l+m-n} \binom{2(n-l)+1}{2k} \binom{2k}{j} (-1)^k \eta^{l+j+k} (\cos \phi)^{n-l+j-k}.
\]

(C63)

Check once again the limit \( \tilde{\kappa}_1 = -\tilde{\kappa}_2 \) where Eq.(63) becomes
\[
\text{Eq.(C63)} = 2 \left[ \frac{[2m + 2n + 1]!!}{(2m + 2n + 1)!} \right]^2 \left[ (n + m + 1)! \right]^2 \frac{(\kappa_1^2 - (\hat{\kappa}_1 \cdot \hat{\eta})^2)^{m+n+1}}{\eta} \\
\sum, \quad (C64)
\]

where the triple sum \( \sum \) collapses to a single sum:
\[
\sum (\cos \phi = 1) \Rightarrow j = k = 0,
\]

so that
\[
\sum = \sum_{l=0}^{n} \binom{2n+1}{l} \binom{2m+1}{l+m-n} = \frac{1}{2} \frac{(2m + 2n + 1)!!}{(n + m + 1)!}^2.
\]

(C65)
APPENDIX D: DERIVATION OF THE INVARIANT MASS $M$ USING THE NEW DIRAC BRACKETS

Kerner has shown [26] that it is possible to develop a single-time Hamiltonian formulation of Wheeler-Feynman dynamics. His idea, basically is to replace the infinitude of “field” coordinates by an infinity of mechanical ones and then with the high order of the equations of motion replaced by higher powers of the momentum in the interaction. In Ref. [48], Crater and Yang give a modification of his approach to obtain a Hamiltonian expression for both scalar and vector interactions through order $1/c^4$. The approach taken in this Appendix is similar to that given in [48] with two important distinctions: 1) terms of all order in $1/c^2$ are included and 2) the effects of the new Dirac brackets are included. The net result is an expression for $M$ that agrees exactly with the results obtained in Eq.(6.14). This result would not be obtained without the use of these brackets in working our Hamilton’s equations.

We begin with the Lagrangian expression for the invariant mass [see Eq.(4.10) with $\bar{\lambda}(\tau) = 0$; $h_1$ is defined in Eq.(6.4)]

$$E_{rel} = h(\dot{\eta}_1, \dot{\eta}_2, \eta) = \frac{m_1}{\sqrt{1 - \dot{\eta}_1^2}} + \frac{m_2}{\sqrt{1 - \dot{\eta}_2^2}} + \frac{Q_1 Q_2}{4\pi |\eta|} + Q_1 Q_2 h_1(\dot{\eta}_1, \dot{\eta}_2, \eta) :=$$

$$= h_0(\dot{\eta}_1, \dot{\eta}_2, \eta) + Q_1 Q_2 h_1(\dot{\eta}_1, \dot{\eta}_2, \eta).$$

(D1)

In order to find the Hamiltonian $H(\vec{\kappa}_1, \vec{\kappa}_2; \vec{\eta})$ from $h_1(\dot{\eta}_1, \dot{\eta}_2, \eta)$ we must demand that Hamilton’s equation be satisfied. We use the Dirac bracket since we have used the constraint as a strong condition on the dynamical variables. Thus we begin with

$$\dot{\eta}_k = \{\eta_k, H\}^* = \{\eta_k, H\} -$$

$$-\int d^3\sigma \{\eta_k, -\sum_i Q_i \vec{A}_{\perp Si}(\tau, \vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i)\} \cdot \{-\sum_j Q_j \vec{\Pi}_{\perp Sj}(\tau, \vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j), H\} -$$

$$-\{\eta_k, -\sum_j Q_j \vec{\Pi}_{\perp Sj}(\tau, \vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j)\} \cdot \{-\sum_i Q_i \vec{A}_{\perp Si}(\tau, \vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i), H\},$$

(D2)

in which
\[ H = H_0 + Q_1 Q_2 H_1(\vec{\kappa}_1, \vec{\kappa}_2; \vec{\eta}), \]  
\[
\text{(D3)}
\]

and

\[ H_0 = \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}. \]
\[
\text{(D4)}
\]

Substituting this Hamiltonian into the above bracket and using Grassmann truncation yields

\[
\dot{\vec{\eta}}_1 = \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_1} - \left[ \int d^3 \sigma \{ \vec{\eta}, -Q_1 \vec{A}_{\perp S1}(\tau, \vec{\sigma} - \vec{\eta}_1, \vec{\kappa}_1) \} \cdot \{ -Q_2 \vec{P}_{\perp S2}(\tau, \vec{\sigma} - \vec{\eta}_2, \vec{\kappa}_2), \sqrt{m_2^2 + \vec{\kappa}_2^2} \} - \{ \vec{\eta}_1, -Q_1 \vec{P}_{\perp S1}(\tau, \vec{\sigma} - \vec{\eta}_1, \vec{\kappa}_1) \} \cdot \{ -Q_2 \vec{A}_{\perp S2}(\tau, \vec{\sigma} - \vec{\eta}_2, \vec{\kappa}_2), \sqrt{m_2^2 + \vec{\kappa}_2^2} \} = \right.
\]
\[
\frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_1} \left( \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} \cdot \frac{\partial}{\partial \vec{\kappa}_2} \right) \vec{\kappa}_{12},
\]
\[
\text{(D5)}
\]

\[
\dot{\vec{\eta}}_2 = \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_2} - \left[ \int d^3 \sigma \{ \vec{\eta}_2, -Q_2 \vec{A}_{\perp S2}(\tau, \vec{\sigma} - \vec{\eta}_2, \vec{\kappa}_2) \} \cdot \{ -Q_1 \vec{P}_{\perp S1}(\tau, \vec{\sigma} - \vec{\eta}_1, \vec{\kappa}_1), \sqrt{m_1^2 + \vec{\kappa}_1^2} \} - \{ \vec{\eta}_2, -Q_2 \vec{P}_{\perp S2}(\tau, \vec{\sigma} - \vec{\eta}_2, \vec{\kappa}_2) \} \cdot \{ -Q_1 \vec{A}_{\perp S1}(\tau, \vec{\sigma} - \vec{\eta}_1, \vec{\kappa}_1), \sqrt{m_1^2 + \vec{\kappa}_1^2} \} = \right.
\]
\[
\frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_2} + \frac{\partial}{\partial \vec{\kappa}_2} \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} \cdot \frac{\partial}{\partial \vec{\kappa}_1} \right) \vec{\kappa}_{12}.
\]
\[
\text{(D6)}
\]

But

\[
\int d^3 \sigma (\vec{E}_{\perp S} \times \vec{B}_S)(\tau, \vec{\sigma}) = \frac{\partial}{\partial \vec{\eta}_2} \vec{\kappa}_{12} = -\frac{\partial}{\partial \vec{\eta}_1} \vec{\kappa}_{12} := -\frac{\partial}{\partial \vec{\eta}} \vec{\kappa}_{12}.
\]
\[
\text{(D7)}
\]

Hence

\[
\dot{\vec{\eta}}_1 = \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_1} - \frac{\partial}{\partial \vec{\kappa}_1} \left( \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} \cdot \int d^3 \sigma (\vec{E}_{\perp S} \times \vec{B}_S)(\tau, \vec{\sigma}) \right),
\]
\[
\dot{\vec{\eta}}_2 = \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} + Q_1 Q_2 \frac{\partial H_1}{\partial \vec{\kappa}_2} - \frac{\partial}{\partial \vec{\kappa}_2} \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} \cdot \int d^3 \sigma (\vec{E}_{\perp S} \times \vec{B}_S)(\tau, \vec{\sigma}) \right).
\]
\[
\text{(D8)}
\]

Substituting this into \( h_0 \) we find that
\[ h_0 = H_0 + \frac{Q_1 Q_2}{4\pi \eta} + \]
\[ + \frac{(m_1^2 + \kappa_1^2)}{m_1^2} \kappa_1 \cdot \frac{\partial}{\partial \kappa_1} (Q_1 Q_2 H_1) - \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \int d^3\sigma (\tilde{E}_{\perp S} \times \tilde{B}_S)(\tau, \tilde{\sigma}) + \]
\[ + \frac{(m_2^2 + \kappa_2^2)}{m_2^2} \kappa_2 \cdot \frac{\partial}{\partial \kappa_2} (Q_1 Q_2 H_1) - \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \int d^3\sigma (\tilde{E}_{\perp S} \times \tilde{B}_S)(\tau, \tilde{\sigma}) = \]
\[ = H - Q_1 Q_2 h_1 = H_0 + Q_1 Q_2 H_1 - Q_1 Q_2 h_1. \tag{D9} \]

Thus letting
\[ H_1 = \frac{1}{4\pi \eta} + \tilde{H}_1, \tag{D10} \]
and using [see Eqs.(6.4) and (6.5) for the expressions of \( h_1 \) and \( \tilde{h}_1 \)]
\[ h_1(\dot{\eta}_1, \dot{\kappa}_2, \eta) = h_1(\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \eta) + O(Q_1 Q_2), \]
\[ \tilde{h}_1(\dot{\eta}_1, \dot{\kappa}_2, \eta) = \tilde{h}_1(\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \eta) + O(Q_1 Q_2), \tag{D11} \]
where
\[ \int d^3\sigma (\tilde{E}_{\perp S} \times \tilde{B}_S)(\tau, \tilde{\sigma}) := Q_1 Q_2 \tilde{h}_1(\dot{\eta}_1, \dot{\kappa}_2, \eta), \tag{D12} \]
then we obtain the following differential equation for \( \tilde{H}_1 \)
\[ \tilde{H}_1 - \frac{\partial}{\partial \kappa_1} \cdot \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{(m_1^2 + \kappa_1^2)}{m_1^2} \tilde{H}_1 - \frac{\partial}{\partial \kappa_2} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \frac{(m_2^2 + \kappa_2^2)}{m_2^2} \tilde{H}_1 = \]
\[ = h_1(\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \eta) - \]
\[ - \frac{(m_1^2 + \kappa_1^2)}{m_1^2} \kappa_1 \cdot \frac{\partial}{\partial \kappa_1} (\frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}) \cdot \tilde{h}_1(\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \eta) - \]
\[ - \frac{(m_2^2 + \kappa_2^2)}{m_2^2} \kappa_2 \cdot \frac{\partial}{\partial \kappa_2} (\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}) \cdot \tilde{h}_1(\frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \eta) + O(Q_1 Q_2). \tag{D13} \]
The \( O(Q_1 Q_2) \) term gives a vanishing contribution as both sides are multiplied by \( Q_1 Q_2 \).

Using this and \( \nabla_{\eta}^2 |\eta|^l = l(l - 1)|\eta|^{l-2} \) we obtain, in addition to the expression given for \( h_1 \) in Eq.(6.12),
\[
\int d^3 \sigma (\vec{E}_{LS} \times \vec{B}_S)(\tau, \sigma) = Q_1 Q_2 \tilde{h}_1 \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \tilde{\eta} \right) =
\]
\[
= \frac{Q_1 Q_2}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \nabla_\eta \left[ \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \times \left( \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2m+1} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2n} + \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2n} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2m+1} \right) \right] \right)
\]
\[
= \frac{1}{(2n + 2m + 2)!} \eta^{2n + 2m + 1} \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2m+2} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2n+1} + \frac{1}{(2n + 2m + 4)!} \eta^{2n + 2m + 3} \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2m+2} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2n+1} \right) \right].
\]

The differential equation for \( \tilde{H}_1 \) is of the form
\[
\tilde{H}_1 - \mathcal{O}\tilde{H}_1 = \tilde{h}_1(\mathcal{O}) \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \tilde{\eta} \right) - \mathcal{O} \cdot \tilde{h}_1(\mathcal{O}) \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}}, \tilde{\eta} \right),
\]
where we define the linear operator \( \mathcal{O} \) by
\[
\mathcal{O} = \left( \frac{m_1^2 + \kappa_1^2}{m_1^2} \right) \frac{\partial}{\partial \kappa_1} + \left( \frac{m_2^2 + \kappa_2^2}{m_2^2} \right) \frac{\partial}{\partial \kappa_2},
\]
and the linear operator \( \mathcal{O} \) by
\[
\mathcal{O} = \left( \frac{m_1^2 + \kappa_1^2}{m_1^2} \right) \frac{\partial}{\partial \kappa_1} + \left( \frac{m_2^2 + \kappa_2^2}{m_2^2} \right) \frac{\partial}{\partial \kappa_2}.
\]

In order to solve for \( \tilde{H}_1 \) we need first to work out the right hand side of its differential equation. First note
\[
\mathcal{O} \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2m+1} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2n+1} =
\]
\[
= \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \nabla_\eta \right)^{2m+1} \left( \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \nabla_\eta \right)^{2n+1}\right).
\]

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So Eq. (6.5) imply

\begin{equation}
\vec{\boldsymbol{\Omega}} \cdot Q_1Q_2\vec{h}_1(\frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}}, \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}}, \vec{\eta}) = \frac{Q_1Q_2}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( (2n+1) \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^2 + \\
+ (2n + 2m + 4) \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right) \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right) + (2m+1) \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^2 \right) \times \\
\times \left[ \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2n} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2n} \eta^{2n+2m+1} - \\
\left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+1} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2n+1} \right] \\

\frac{1}{(2n + 2m + 4)!} \frac{1}{(2n + 2m + 4)!} \frac{\eta^{2n+2m+3}}{\eta^{2n+2m+3}} \right)
\end{equation}

(D20)

In analogy to our decomposition of the field energy integral we decompose this into single
and double sum pieces giving

\begin{equation}
\vec{\boldsymbol{\Omega}} \cdot Q_1Q_2\vec{h}_1 = \frac{Q_1Q_2}{4\pi} \sum_{m=1}^{\infty} \frac{1}{(2m)!} \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m} + (2n+1) \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+1} - \\
= \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+1} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+1} \eta^{2n+2m+3} \\
+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( (2n + 2m + 4) \left[ \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right]^2 \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} - \\
\left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} \right] \\
+ \left( (2n + 2m + 4) \left[ \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right]^2 \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} - \\
\left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} \right] \\
= \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} \\
+ \left( (2n + 2m + 4) \left[ \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right]^2 \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} - \\
\left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^{2m+2} \eta^{2n+2m+3} \right] \\
\end{equation}

(D21)

Combining like terms we obtain

\begin{equation}
\vec{h}_1 - \vec{\boldsymbol{\Omega}} \cdot \vec{h}_1 = \frac{Q_1Q_2}{4\pi} \left\{ \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right)^2 \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right)^2 - \left( \frac{\vec{k}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_{\eta} \right) \left( \frac{\vec{k}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_{\eta} \right) \right\} + \\
\end{equation}

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Based on the above expression we assume a particular solution of the form

\[
Q_1 Q_2 H_1(\vec{\kappa}_1, \vec{\kappa}_2, \vec{\eta}) = k \frac{Q_1 Q_2}{4\pi} \times \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \frac{1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_\eta (-\frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_\eta \eta) \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{a_{mn}} \left( \frac{\kappa_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \frac{1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_\eta (-\frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_\eta \eta) \right) \times \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+1}(\vec{\kappa}_2 \cdot \vec{\nabla}_\eta)^{2n+1}}{(2n + 2m)!} \frac{(\vec{\kappa}_2 \cdot \vec{\nabla}_\eta)^{2m+2}(\vec{\kappa}_2 \cdot \vec{\nabla}_\eta)^{2n+2}}{(2n + 2m + 4)!} \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+2}(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2n+2}}{(2n + 2m + 6)!} \right] + \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \frac{1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_\eta (-\frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_\eta \eta) \right) \times \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+1}(\vec{\kappa}_2 \cdot \vec{\nabla}_\eta)^{2n+1}}{(2n + 2m + 4)!} \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+2}(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2n+2}}{(2n + 2m + 6)!} \right] + \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \frac{1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_\eta (-\frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_\eta \eta) \right) \times \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+1}(\vec{\kappa}_2 \cdot \vec{\nabla}_\eta)^{2n+1}}{(2n + 2m + 4)!} \frac{(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2m+2}(\vec{\kappa}_1 \cdot \vec{\nabla}_\eta)^{2n+2}}{(2n + 2m + 6)!} \right] .
\]

Using

\[
\mathcal{O} \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}} \right) = 2 \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \frac{\kappa_2}{\sqrt{m_2^2 + \kappa_2^2}},
\]

and

\[
\mathcal{O} \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \kappa_1^2}} \cdot \vec{\nabla}_\eta \right)^{2m+1} \left( \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \kappa_2^2}} \cdot \vec{\nabla}_\eta \right)^{2n+1} =
\]

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Comparing the two sides of the equation leads to

\[ (1 - \mathcal{O})Q_1Q_2\tilde{H}_1(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\eta}) = -k \frac{Q_1Q_2}{8\pi} \times \]

\[
\left( \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \frac{1}{\eta} \left( \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \tilde{\nabla}_\eta \right) \left( \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \cdot \tilde{\nabla}_\eta \right) \right)
\]

\[
= \frac{Q_1Q_2}{4\pi} \sum_{m = 0}^{\infty} \sum_{n = 0}^{\infty} \left[ -a_{mn}(2n + 2m + 3) \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \times \frac{(2n + 2m + 2)!}{(2n + 2m + 4)!} \right. 
\]

\[
+ b_{mn}(2n + 2m + 3) \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \times \frac{(2n + 2m + 4)!}{(2n + 2m + 6)!} \left. \right]
\]

\[
+ c_{mn}(2n + 2m + 5) \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \times \frac{(2n + 2m + 4)!}{(2n + 2m + 6)!} \left[ . \right.
\]

\[
+ d_{mn}(2n + 2m + 5) \frac{\tilde{\kappa}_1}{\sqrt{m_1^2 + \tilde{\kappa}_1^2}} \cdot \frac{\tilde{\kappa}_2}{\sqrt{m_2^2 + \tilde{\kappa}_2^2}} \times \frac{(2n + 2m + 4)!}{(2n + 2m + 6)!} \left] . \right.
\]

Comparing the two sides of the equation leads to

\[ k = -1; \ a_{mn} = b_{mn} = c_{mn} = d_{mn} = 1, \]
\[\begin{align*}
&+ \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} \cdot \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} \cdot \vec{\nabla}_\eta \right)^{2m + 2} \left( \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} \cdot \vec{\nabla}_\eta \right)^{2n + 2m + 3} \,(2n + 2m + 4)! \\
&- \left( \frac{\vec{\kappa}_1}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} \cdot \vec{\nabla}_\eta \right)^{2m + 3} \left( \frac{\vec{\kappa}_2}{\sqrt{m_2^2 + \vec{\kappa}_2^2}} \cdot \vec{\nabla}_\eta \right)^{2n + 3} \eta^{2n + 2m + 5} \,(2n + 2m + 6)! \right],
\end{align*}\]

which agrees exactly with the Darwin portion of \( M \) obtained earlier Eq.(6.14).
APPENDIX E: SCHILD-LIKE SOLUTION FOR THE TWO-BODY PROBLEM IN THE CASE OF EQUAL MASSES

Here we present the semiclassical Hamilton equations for the equal mass case restricted to circular orbits (we suppress the tilde notation). First note that the Hamiltonian Eq.(6.37)

\[
H = 2\sqrt{m^2 + \vec{\kappa}^2} + \frac{Q_1 Q_2}{4\pi \eta} + \frac{Q_1 Q_2}{8\pi \eta} \times \\
\left[ m^2(3\vec{\kappa}^2 + (\vec{\kappa} \cdot \hat{\eta})^2) - 2\vec{\kappa}^2[\vec{\kappa}^2 - 3(\vec{\kappa} \cdot \hat{\eta})^2]\sqrt{\frac{m^2 + \vec{\kappa}^2}{m^2 + (\vec{\kappa} \cdot \hat{\eta})^2}} - 2[\vec{\kappa}^2 + (\vec{\kappa} \cdot \hat{\eta})^2]\right] \left[ m^2 + (\vec{\kappa} \cdot \hat{\eta})^2 \right]. 
\]  
(E1)

is of the form

\[
H = 2\sqrt{m^2 + \vec{\kappa}^2} + \frac{Q_1 Q_2}{4\pi \eta} + \frac{Q_1 Q_2}{8\pi \eta} f(\vec{\kappa}^2, (\vec{\kappa} \cdot \hat{\eta})^2). 
\]  
(E2)

Thus, Hamilton’s equations are

\[
\dot{\vec{\kappa}} = \frac{\partial H}{\partial \eta} = -\frac{Q_1 Q_2}{4\pi \eta^3} \eta - \frac{Q_1 Q_2}{8\pi \eta^3} \eta f(\vec{\kappa}^2, (\vec{\kappa} \cdot \hat{\eta})^2) + \frac{Q_1 Q_2}{4\pi \eta} f_2(\vec{\kappa}^2, (\vec{\kappa} \cdot \hat{\eta})^2) \vec{\kappa}(\vec{\kappa} \cdot \hat{\eta}),
\]

\[
\dot{\eta} = \frac{\partial H}{\partial \vec{\kappa}} = \frac{2\vec{\kappa}}{\sqrt{\vec{\kappa}^2 + m^2}} + \frac{Q_1 Q_2}{4\pi \eta} \left[ f_{,1}(\vec{\kappa}^2, (\vec{\kappa} \cdot \hat{\eta})^2) \vec{\kappa} + f_{,2}(\vec{\kappa}^2, (\vec{\kappa} \cdot \hat{\eta})^2) \eta(\vec{\kappa} \cdot \hat{\eta}) \right].
\]  
(E3)

From this we can see that circular orbits defined by \( \dot{\eta} \cdot \vec{\eta} = 0 \) are implied by \( \vec{\kappa} \cdot \eta = 0 \). This furthermore implies that \( -\dot{\vec{\kappa}} \cdot \vec{\kappa} = 0 \). Thus not only is \( \eta^2 = \text{const.} \) but also \( \vec{\kappa}^2 = \text{const.} \). Imposing these conditions on the above Hamilton equations we can simplify our equations above to

\[
\dot{\vec{\kappa}} = \frac{\partial H}{\partial \eta} = -\frac{Q_1 Q_2}{4\pi \eta^3} \eta - \frac{Q_1 Q_2}{8\pi \eta^3} \eta f(\vec{\kappa}^2, 0) = B\vec{\eta},
\]

\[
\dot{\eta} = \frac{\partial H}{\partial \vec{\kappa}} = \frac{2\vec{\kappa}}{\sqrt{\vec{\kappa}^2 + m^2}} + \frac{Q_1 Q_2}{4\pi \eta} f_{,1}(\vec{\kappa}^2, 0) \vec{\kappa} = A\vec{\kappa}.
\]  
(E4)

with \( B \) and \( A \) constants. Combine the two equations and we find

\[
\frac{\dot{\vec{\kappa}}}{A} = \frac{\dot{\eta}}{B} = -\vec{\eta} \Rightarrow \vec{\kappa} = \vec{\eta} = -AB\eta := -\Omega^2 \eta,
\]  
(E5)

so that

\[
\ddot{\eta} = -AB\dot{\eta} = -\Omega^2 \eta,
\]  
(E6)
with

\[ \Omega^2 = -\frac{Q_1 Q_2}{4\pi \eta^3 \sqrt{\kappa^2 + m^2}} [1 + f(\kappa^2, 0)] = -\frac{Q_1 Q_2}{4\pi \eta^3 \sqrt{\kappa^2 + m^2}} [1 - \kappa^2 \frac{2\kappa^2 \sqrt{\kappa^2 + m^2} - m^2}{2m^3(m^2 + \kappa^2)}]. \] (E7)

The frequency is defined by the initial data \( \eta \) and \( |\kappa| \) and is real for \( Q_1 Q_2 < 0 \) for \( 0 \leq \kappa^2 \leq \kappa_{\text{max}}^2 \) in which \( \kappa_{\text{max}}^2 \) is the value at which \( \Omega^2 = 0 \). Let us remark that at the semiclassical level \( Q_i \) are Grassmann variables: therefore \( \Omega \sim \sqrt{Q_1 Q_2} \) is to be interpreted as an even algebraic object satisfying \( \Omega^4 = 0, Q_1 \Omega^2 = Q_2 \Omega^2 = 0 \).

We also find

\[ H = 2\sqrt{\kappa^2 + m^2} [1 - \frac{1}{4}\Omega^2 \eta^2]. \] (E8)

Our solution to Eq.(E6) is

\[ \vec{\eta}(\tau) = \vec{\alpha} \cos \Omega \tau + \vec{\beta} \sin \Omega \tau = \vec{\alpha} (1 - \frac{1}{2!}\Omega^2 \tau^2) + \vec{\beta} (\Omega \tau - \frac{1}{3!}\Omega^3 \tau^3), \] (E9)

and thus

\[ \dot{\vec{\eta}}(\tau) = -\vec{\alpha} \Omega^2 \tau + \vec{\beta} (\Omega - \frac{1}{2}\Omega^3 \tau^2). \] (E10)

But \( \dot{\vec{\eta}} \cdot \vec{\eta} = 0 \). This implies that

\[ (\vec{\beta}^2 - \vec{\alpha}^2) \Omega^2 \tau + \vec{\alpha} \cdot \vec{\beta} \Omega (1 - 2\Omega^2 \tau^2) = 0, \]

so that

\[ \vec{\beta}^2 = \vec{\alpha}^2, \quad \vec{\alpha} \cdot \vec{\beta} = 0, \] (E11)

and therefore

\[ \vec{\alpha}^2 = \vec{\eta}^2. \] (E12)

Otherwise the vectors \( \vec{\alpha} \) and \( \vec{\beta} \) are arbitrary.
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