Hyperkähler Quotients, Mirror Symmetry, and F-theory

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Abstract

Using the graphical method developed in hep-th/9908082, we obtain the full curve corresponding to the hyperkähler quotient from the extended $E_7$ Dynkin diagram. As in the $E_6$ case discussed in the same paper above, the resulting curve is the same as the one obtained by Minahan and Nemeschansky. Our results seem to indicate that it is possible to define a generalized Coulomb branch such that four dimensional mirror symmetry would act by interchanging the generalized Coulomb branch with the Higgs branch of the dual theory. To understand these phenomena, we discuss mirror symmetry and F-theory compactifications probed by D3 branes.

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1 Introduction

String theory provides a very powerful setting for the study of gauge theories. Gauge theories can be constructed through the geometrical engineering [1] by compactifying string theory on Calabi-Yau manifolds with appropriate Hodge numbers and singularities. They can also be realized as the world volume theories on extended objects such as D-branes [2]. The authors of [3] studied the gauge theories obtained by placing D-branes on orbifold singularities and introduced ‘quiver diagrams’, which summarize the gauge group structures and matter contents of the gauge theories. They considered D-5 branes and noticed that the moduli space of D-brane ground states is a ALE space described by a hyperkähler quotient.

The hyperkähler quotient was introduced in [4]; it was mathematically refined in [5]. One way to construct it is to gauge isometries of a non-linear sigma model in such a way as to preserve $\mathcal{N}=2$ supersymmetry. In the spirit of [6], a graphical method was invented in [7] and used to obtain the curve that corresponds to a hyperkähler quotient of a linear space. In particular, it was applied to the hyperkähler quotients constructed from the extended Dynkin diagrams of $A_k, D_k$ series and $E_6$ case.

Remarkably, the Higgs branch of a quiver gauge theory based on the extended $E_6$ Dynkin diagram turned out to be identical, when it was expressed in terms of $E_6$ Casimir invariants, to the curve\(^1\) with $E_6$ global symmetry obtained by Minahan and Nemeschansky some time ago [8] and later by the authors of [10]. In this article, we work out the full\(^2\) curve corresponding to the $E_7$ extended Dynkin diagram. The resulting curve is again equal to the generalized Coulomb branch with $E_7$ global symmetry computed in [9] and [10]\(^3\).

We understand the origin of these phenomena through mirror symmetry and F-theory [11] compactifications [12, 13]. D3 branes are used to probe the singularities of the backgrounds under consideration [14]. The relevant F-theory compactifications for our purpose are the ones which give rise to $E_7$ gauge group. The $E_7$ global symmetry is realized on the world volume theory of the D3 branes.

It is the physics near such singularities that is responsible for the field theory limit of string compactifications [15]. The mirror geometry of ADE singularities was discussed [16] in the context of type II strings. On the dual backgrounds, the gauge groups of the dual superconformal field theories are given by a product of $U(n_i)$ groups. The $n_i$’s are given by multiples of the Dynkin numbers of the nodes in the corresponding Dynkin diagrams.

\(^1\)We will call this curve the generalized Coulomb branch.

\(^2\)The orbifold limits of $E_7$ and $E_8$ (and some other higher order quiver diagrams) were considered in [7].

\(^3\)In $E_7$ case, it is easier to compare with [10] since the authors used $E_7$ Casimir invariants, while the authors of [9] expressed their curve in terms of the $SO(12)\times SU(2)$ Casimir invariants, as we discussed in section 2.
Mirror symmetry is well understood in three dimensions \([17, 18]\) where both the Higgs branch and the Coulomb branch are hyperkähler manifolds. They get interchanged under the action of mirror symmetry. In the four dimensional models we consider in this paper, the Coulomb branch of the original theory is a Riemann surface, which is real two dimensional, whereas the Higgs branch of the dual theory has real four dimensions. What we find in this paper seems to indicate that mirror symmetry in these four dimensional models acts in such a way that it is the generalized Coulomb branch (rather than the Coulomb branch) of the original gauge theory that gets interchanged with the Higgs branch of the mirror dual theory.\(^4\)

More intuitive understanding of the origin of the identity between the curve we compute and the generalized Coulomb branch seems possible by applying various string dualities to the system under consideration. We illustrate this point in section 3 with a heuristic example using the D7-D3 brane system.

The organization is as follows. After briefly reviewing the hyperkähler quotients, we present the calculation of \(E_7\) case in section 2. The final form of the curve is given in Appendix A. It is expressed in terms of \(E_7\) Casimirs, \(P_i\), whose definition is given in Appendix A. As in the case of \(E_6\), the curve obtained is the generalized Coulomb branch with \(E_7\) global symmetry. The generalized Coulomb branch can also be expressed in terms of \(E_7\) Casimir invariants which we also denote as \(P_i\). However, the \(P_i\)'s of our curve are functions of Fayet-Iliopoulos (FI) parameters, \(b_j\), while the \(P_i\)'s of the generalized Coulomb branch are functions of mass parameters\(^5\), \(m_k\). In anticipation of mirror symmetry, we find the relations between \(b\)'s and \(m\)'s which render \(P_i(b) = P_i(m)\).\(^6\) In section 3, we discuss mirror symmetry and F-theory compactifications probed by D3 branes. Section 4 includes summary and open problems.

2 The Hyperkähler Quotient For \(E_7\) Case

We begin by briefly reviewing the hyperkähler quotients and refer the reader to \([4, 5, 7]\) (and the references therein) for more details. Gauging isometries of a non-linear sigma model while preserving \(\mathcal{N}=2\) susy gives rise to the hyperkähler quotient. More specifically, consider a sigma model with isometries. To elevate the isometries to local symmetries, introduce an \(\mathcal{N}=2\) vector multiplet, which consists of an \(\mathcal{N}=1\) vector multiplet and an \(\mathcal{N}=1\)

\(^4\)There is a natural relation between the generalized Coulomb branch in four dimensions and the Coulomb branch in three dimensions, as discussed in section 3.

\(^5\)They are associated with relevant deformations of the superconformal field theory under consideration.

\(^6\)These relations reflect the fact that under mirror symmetry FI and mass parameters get interchanged.
Figure 1: The bug calculus. $b_i$ is the Fayet-Iliopoulos parameter associated to the $i$'th node, and a vertical bar through the $i$'th node represents a $U(N_i)$ Kronecker-$\delta$.

chiral multiplet, denoted respectively as $V$, $S$ in [4]. In $\mathcal{N}=1$ superspace, one integrates out $V$ and $S$ by their field equations. Inserting the solution for $V$ field equations into the gauged Lagrangian and keeping the $S$ field equations as constraints gives the Kähler potential of the quotient space. The constraints from $S$ field equations can be represented graphically and are given in figure 1\(^7\). The gauge groups and the representations appropriate for the the construction of ALE spaces are summarized by the extended Dynkin diagrams [19], as in Figure 2.

Now, we compute the hyperkähler quotient corresponding to the $E_7$ extended Dynkin Diagram given in Figure 3(a). In addition to the Dynkin numbers in the same figure, we label the nodes by assigning 1 to the far left node and 2 to the next one, etc. The upper node in the middle is referred to as the eighth node. We closely follow [7] with a convenient

\(^7\)Figure 1 and Figure 3 are taken from [7].
Figure 2: The $A_k$, $D_k$, and $E_8$ Dynkin diagrams.

Figure 3: The $E_7$ invariants and some useful matrices.
set of variables defined in Figure 3(b).

Consider the highest order invariant, $U$, and its orientation reversed diagram $\bar{U}$. The product of these two diagrams can be written as\(^8\)

$$U\bar{U} = W\text{Tr}(MNKN)$$

(1)

One can use the so called Schouten identity to rewrite $\bar{U}$ in terms of the variables defined in Figure 3(b): The relevant Schouten identity is

$$\text{Tr}(\{M,N\}K) = \text{Tr}(MN)\text{Tr}(K) + \text{Tr}(MK)\text{Tr}(N) - \text{Tr}(M)\text{Tr}(N)\text{Tr}(K)$$

(2)

Noting the following relations,

$$\text{Tr}(\{M,N\}K) = U + \bar{U}$$

$$\text{Tr}(M) = b_1$$

$$\text{Tr}(MN) = V$$

$$\text{Tr}(MK) = W$$

(3)

one obtains,

$$\bar{U} = -U + \text{Tr}(K)V + \text{Tr}(N)W + b_1\text{Tr}(NK) - b_1\text{Tr}(N)\text{Tr}(K)$$

(4)

where $b_1$ is the FI term associated with the first node of the Dynkin diagram. To rewrite the right hand side of (1), consider

$$0 = N^{k_2}_{k_1}K^{k_3}_{k_2}N^{k_4}_{k_3}M^{k_1}_{k_4}$$

$$= \text{Tr}(MNKN) + \text{Tr}(NNKM) - \text{Tr}(NNM)\text{Tr}(K) - \text{Tr}(NKM)\text{Tr}(N)$$

$$+ \text{Tr}(NM)\text{Tr}(K)\text{Tr}(N) - \text{Tr}(NM)\text{Tr}(KN)$$

(5)

Applying the Schouten identity (2) again to $\text{Tr}(NNKN)$ and $\text{Tr}(NNM)$ leads to

$$\text{Tr}(MNKN) = \text{Tr}(MN)\text{Tr}(NK) - \frac{1}{2}\text{Tr}(MK)\text{Tr}(NN) + \frac{1}{2}\text{Tr}(MK)\text{Tr}(N)^2$$

In the orbifold limit, $U = -\bar{U}$, but this is not true in the presence of the Fayet-Iliopoulos terms.
\[ + \frac{1}{2} \text{Tr}(NN)\text{Tr}(M)\text{Tr}(K) - \frac{1}{2} \text{Tr}(M)\text{Tr}(K)\text{Tr}(N)^2 \]
\[ = \text{Tr}(NK)V + \left[ -\frac{1}{2} \text{Tr}(NN) + \text{Tr}(N)^2 \right] W \]
\[ + \frac{1}{2} \text{Tr}(K)\text{Tr}(NN) - \frac{1}{2} \text{Tr}(K)\text{Tr}(N)^2 \]  

(6)

where the second equality follows from (3). Substituting (4) and (6) into (1) gives

\[ U\left(-U + \text{Tr}(K)V + \text{Tr}(N)W + b_1\text{Tr}(NK) - b_1\text{Tr}(N)\text{Tr}(K)\right) \]
\[ = W \left(\text{Tr}(NK)V + \left[ -\frac{1}{2} \text{Tr}(NN) + \text{Tr}(N)^2 \right] W + \frac{1}{2} \text{Tr}(K)\text{Tr}(NN) - \frac{1}{2} \text{Tr}(K)\text{Tr}(N)^2 \right) \]  

(7)

Therefore the whole task of finding the curve is reduced to the computation of \(\text{Tr}(N), \text{Tr}(K), \text{Tr}(NN)\) and \(\text{Tr}(NK)\). It is a simple exercise to compute \(T(N)\): It is expressed purely in terms of \(b_i\)'s. The other three quantities are more complicated to obtain: The final forms are,

\[ \text{Tr}(K) = k_v(b_i)V + k \]
\[ \text{Tr}(NN) = 2W + n_v(b_i)V + n(b_i) \]
\[ \text{Tr}(NK) = m_w(b_i)W - V^2 + m_v(b_i)V + m(b_i) \]  

(8)

where the coefficients are functions of the FI parameters, \(b_i\), as indicated. Since these coefficients are lengthy, we will not present them explicitly. However, once we make the change of variables discussed below, the coefficients can be expressed in terms of \(E_7\) Casimir invariants. This makes the curve simple enough to present. Upon substitution of (8) into (7), we obtain,

\[ U^2 - W^3 - V^3W \]
\[ + U\left(-b_1m_w - l\right)W + (b_1 - k_v)V^2 + (-b_1m_v + b_1lk_v - k)V + b_1(lk - m) \]
\[ + W^2 \left( m_w - \frac{1}{2}n_v + b_1k_v \right)V + \frac{1}{2}l^2 + b_1k - \frac{n}{2} \]
\[ + W \left( \frac{b_1}{2}n_vk_v + m_v \right)V^2 + \left[ m + \frac{b_1}{2}nk_v + \frac{b_1}{2}n_vk - \frac{b_1}{2}l^2k_v \right)V - \frac{b_1}{2}l^2k + \frac{b_1}{2}nk = 0 \]  

(9)

where \(l \equiv \text{Tr}(N)\). To put this curve into the standard form, we perform the following change of variables,
$$U = X - \frac{1}{2} \left[ (b_1 m_w - l) W + (b_1 - k_v) V^2 + (-b_1 m_v + b_1 k_v - k) V + b_1 (l - k - m) \right]$$

$$V = Z + \frac{1}{6} b_1^2 m_w + \frac{1}{18} b_1 k_v n_v + \frac{1}{18} b_1 k_v m_w + \frac{1}{3} m_v - \frac{1}{9} n_v m_w + \frac{1}{36} n_v^2 - \frac{1}{6} k_v l + \frac{1}{6} b_1 l + \frac{1}{9} b_1^2 k_v^2 + \frac{1}{9} m_w^2$$

$$W = Y + \left[ -\frac{1}{6} n_v + \frac{1}{3} m_w + \frac{1}{3} b_1 k_v \right] Z + \frac{1}{3} b_1 k - \frac{1}{12} b_1^2 m_w^2 - \frac{1}{6} b_1 l m_w + \frac{1}{12} l^2 - \frac{1}{6} n$$

$$+ \frac{1}{6} [2b_1 k_v - n_v + 2m_w] \left[ \frac{1}{6} b_1^2 m_w + \frac{1}{18} b_1 k_v n_v + \frac{1}{18} b_1 k_v m_w + \frac{1}{3} m_w 
- \frac{1}{9} n_v m_w + \frac{1}{36} n_v^2 + \frac{1}{6} k_v l + \frac{1}{6} b_1 l + \frac{1}{9} b_1^2 k_v^2 + \frac{1}{9} m_w^2 \right]$$

(10)

In terms of the new variables, \(X, Y\) and \(Z\), the curve becomes

$$X^2 = Y^3 + f(Z)Y + g(Z)$$

(11)

where

\[
f(Z) = Z^3 + \alpha_1(b_i)Z + \alpha_0(b_i)
\]

\[
g(Z) = \beta_4(b_i)Z^4 + \beta_3(b_i)Z^3 + \beta_2(b_i)Z^2 + \beta_1(b_i)Z + \beta_0(b_i)
\]

(12)

The coefficients \(\alpha\) and \(\beta\) are expressible in terms of \(E_7\) Casimir invariants and are given in the Appendix A.

We discuss the comparison of (11) with the curve of [9] in Appendix B in more detail. Here we only present the relations\(^9\) between \(b_i\)'s and \(m_i\)'s:

\[
b_1 = \phi
\]

\[
b_3 = m_5 - m_6,
\]

\[
b_4 = m_4 - m_5,
\]

\[
b_5 = m_3 - m_4,
\]

\[
b_6 = m_2 - m_3,
\]

(13)

\(^9\)It is easier to compare to [10] since they also use \(E_7\) Casimirs. It is straightforward to check that our curve is equal to that of [10] if we identify our \(P_i\) with their \(P_i\).
where $\phi$ is the simple root of SU(2). Upon substitution in (11), we find exactly the curve of [9]. The mass parameters have a group theoretical interpretation as an orthonormal basis for the root space.

3 Mirror Symmetry and F-theory

In [7], it was observed that the hypermultiplet moduli space of a model constructed from the $E_6$ extended Dynkin diagram is equal to the generalized Coulomb branch of SU(2) gauge theory with $E_6$ global symmetry. In this letter, we have extended this observation to the $E_7$ case. Given this remarkable correspondence it is natural to conjecture that there is a mirror symmetry acting on four dimensional gauge theories analogous to the mirror symmetry acting on three dimensional gauge theories [17]. However, the Coulomb branch in four dimensions is not, in general, a Hyperkähler manifold so it cannot be directly exchanged with the Higgs branch of the gauge theory which is a Hyperkähler manifold. Instead we conjecture that what is exchanged with the Higgs branch is the full four dimensional elliptically fibered space that one obtains by fibering the Seiberg-Witten torus over the usual Coulomb branch as the base. In other words, the Higgs branch gets interchanged with the four dimensional space given by the equation

$$X^2 = Y^3 + f(Z)X + g(Z)$$

where $Z$ is now interpreted as the usual coordinate on the Coulomb branch of the gauge theory. We now try to collect some evidence for this conjecture.

Closely following [20], we can connect our generalized four dimensional mirror symmetry with the mirror symmetry that acts on three dimensional gauge theories [17] by performing a dimensional reduction of the four dimensional Seiberg-Witten theory to three dimensions$^{10}$

In four dimensions the effective action that we are interested in can be written

$$\int d^4x \left( \frac{1}{4e^2} F_{\mu \nu} F^{\mu \nu} + \frac{i \theta}{64\pi^2} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} + g_{u \bar{u}} \partial_{\mu} u \bar{\partial}^\mu \bar{u} \right)$$

where $u = \text{Tr} \phi^2$ is the usual Seiberg-Witten moduli space coordinate. Going to three dimensions by compactifying on a circle of radius $R$, the gauge vector splits into a scalar and

$^{10}$Since it is believed that the world volume theory with exceptional global symmetry does not admit a Lagrangian description, this discussion may not directly apply to our case. However, we believe that our conjecture is more general than the cases of exceptional global symmetry. Therefore we consider a case where such a description does exist.
a tree dimensional vector which, in turn, can be dualized to another scalar. The correct normalization for the first scalar is

$$A_4 = \frac{b}{\pi R}$$

which gives an action

$$\int d^3x \left( \frac{1}{\pi R e^2} \partial_i b \partial^i b + \frac{\pi R}{2e^2} F_{ik} F^{ik} + \frac{i\theta}{8\pi^2} \epsilon^{ikl} F_{ik} \partial_l b + 2\pi R g_{uu} \partial_i u \partial^i \bar{u} \right)$$

Then we dualize the 3D photon by the usual “gauging of isometries” trick. The “isometry” we want to gauge is $A_k \rightarrow A_k + \epsilon_k$ where $\epsilon_k$ is a constant, but after gauging it can be arbitrary. To gauge we introduce a gauge field $B_{ik}$ which transforms as $B_{ik} \rightarrow B_{ik} - \partial_i [\epsilon_k]$. The whole model becomes invariant if we replace $F_{ik}$ with $F_{ik} + B_{ik}$. We also have to impose the constraint that the field strength of $B_{ik}$ is equal to zero so that we don’t change the model. This gives us the action

$$\int d^3x \left( \frac{1}{\pi R e^2} \partial_i b \partial^i b + \frac{\pi R}{2e^2} (F_{ik} + B_{ik})^2 + \frac{i\theta}{8\pi^2} \epsilon^{ikl} (F_{ik} + B_{ik}) \partial_l b + 2\pi R g_{uu} \partial_i u \partial^i \bar{u} - \frac{i}{8\pi} \sigma \epsilon^{ikl} \partial_l B_{kl} \right)$$

If we integrate over $\sigma$, we get back to the original model ($B$ is equal to zero modulo a gauge transformation). If we instead integrate over $B$, we obtain

$$B_{ik} = -F_{ik} + \frac{i e^2}{8\pi^2} \epsilon^{ikl} \left( \partial_l \sigma - \frac{\theta}{\pi} \partial_l b \right)$$

and inserting this in the action gives

$$\int d^3x \left( \frac{1}{\pi R e^2} |db|^2 + \frac{e^2}{64\pi^3 R} \left| d\sigma - \frac{\theta}{\pi} db \right|^2 + 2\pi R g_{uu} |du|^2 \right)$$

From this we can read off the metric on the moduli space. It is

$$ds^2 = \frac{e^2}{64\pi^3 R} \left[ d\sigma^2 + \left( \frac{64\pi^2}{e^4} + \frac{\theta^2}{\pi^2} \right) db^2 - 2\frac{\theta}{\pi} db d\sigma \right] + g_{uu} du d\bar{u}$$

We see that the $b$ and $\sigma$ coordinates are coordinates on a torus with $\tau$ parameter $\tau = \frac{\theta}{\pi} + i \frac{2\pi^2}{\pi}$ which is the same as the Seiberg-Witten torus. We thus see that going to three dimensions we naturally have to put the Seiberg-Witten moduli space (spanned by $u$) together with the Seiberg-Witten torus to get the full space spanned by the complex curve. It should also be noted that, in the cases where we can compare, the map between the FI parameters and the mass terms is exactly the same as in the three dimensional case [21].
We now discuss F-theory compactifications probed by D3 branes. F-theory [11] can be viewed as a novel compactification of type IIB string theory with the dilaton, \( \phi \), and the Ramond-Ramond (RR) scalar, \( a \), varying over a manifold \( B \) on which IIB string theory is compactified. By considering the subspace of the moduli space where \( \tau (\equiv a + i \exp(-\phi/2)) \) remains constant, Sen [12] showed that F-theory in this regime is described by the IIB orientifold in the weak coupling limit.

In particular, an interesting observation was made that the description of the physics near one of the four orientifold planes is identical to the Seiberg-Witten results [22] of \( \mathcal{N}=2 \) SU(2) gauge theory with SO(8) global symmetry. The SU(2) gauge theory, in turn, was interpreted [14] as the world volume theory of D3 brane probes parallel to the 7 branes.

In [13, 23, 24], other branches of the moduli space were studied where the \( \tau \) remains constant. In particular, it was noted that in these branches exceptional gauge groups arise on the \((p,q)\) 7 branes [25]. The branch we need to consider is the one which gives \( E_7 \) group. The \((p,q)\) 7-brane configuration of \( E_7 \) singularity is often referred to as \( A_6 B_2 C_2 \) where \( A_6 \), \( B_2 \), and \( C_2 \) are associated with \((p,q)\) charges of the 7-branes. Since the singularity occurs at \( \tau = i \), the theory is strongly coupled.

It will be very intuitive if one can relate a Seiberg-Witten theory and the corresponding quiver theory by string dualities. More specifically, one may start with a F-theory compactification that realizes a Seiberg-Witten theory and apply various string dualities to turn it into the corresponding quiver theory. We illustrate this point with a heuristic example using a D7-D3 brane system.

We know that the quiver gauge theory associated with a particular ADE Dynkin diagram is realized on D3 branes probing the corresponding ALE singularity. We also know that the gauge theory with the global symmetry corresponding to the ADE Dynkin diagram can be realized on D3 branes probing a system of D7-branes. One way that these two pictures might be related is through the following sequence of T-dualities and S-dualities. We begin with the D3-D7 brane system. We can assume that the branes are oriented as follows (where a \( \times \) means of infinite extent and a \(-\) means pointlike in the particular dimension)

\[
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 \\
D7 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
D3 & \times & \times & \times & \times & \times & \times & \times & \times & \times
\end{align*}
\]

(22)

We now perform two T-duality transformations on the 6, 7 coordinates. This will take us from type IIB theory to type IIB theory and change the system into a system of intersecting
D5-branes with the following orientation

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D5 & & & & & & & & & \\
\times & \times & \times & \times & \times & \times & - & - & - & - \\
D5' & & & & & & \times & \times & - & - \\
\end{array}
\]  \quad (23)

Since we are in type IIB we can perform an S-duality transformation which transforms the D5-branes into NS5-branes with the same orientation

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
NS5 & & & & & & & & & \\
\times & \times & \times & \times & \times & \times & - & - & - & - \\
NS5' & & & & & & \times & \times & - & - \\
\end{array}
\]  \quad (24)

Then we perform another T-duality transformation on the fourth coordinate which will take us to type IIA theory while turning the first set of NS-branes into type IIA NS-branes and the second set of NS-branes into an A\(k\) orbifold singularity [26, 27, 28]. Subsequently we perform another T-duality on the third coordinate which, since it is parallel to both the orbifold singularity and the NS-branes, will only have the effect of taking us back to type IIB theory and changing the NS-branes back to type IIB NS-branes leaving us with the type IIB brane configuration

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
NS5 & & & & & & & & & \\
\times & \times & \times & \times & \times & \times & - & - & - & - \\
ORB & & & & & & \times & \times & - & - \\
\end{array}
\]  \quad (25)

Finally we perform another S-duality transformation which leaves the orbifold singularity invariant while changing the NS-branes into D5-branes

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D5 & & & & & & & & & \\
\times & \times & \times & \times & \times & \times & - & - & - & - \\
ORB & & & & & & \times & \times & - & - \\
\end{array}
\]  \quad (26)

The D5-branes should be thought of as wrapping the two-cycles of the singularity, i.e. they can be thought of as a set of fractional D3-branes. Hence they can be recombined into a D3-brane which is free to leave the singularity, exactly the brane probe picture that we wanted to reach.

This rather heuristic discussion can hopefully be improved to also be valid for the exceptional cases treated in this paper and in [7]. In those cases, the 7-branes that one starts with carry (p,q) charges and it is not always clear what their T- and S- duals are. It would be interesting to elucidate this point.
We could also imagine constructing the Seiberg-Witten theory through geometrical engineering. In that case we could study the how string theory mirror symmetry acts along the lines of [16]. It is not a priori clear that the mirror theory obtained this way is the mirror theory proposed in this paper but since the three dimensional mirror symmetry can be explained in this fashion we expect a connection also in our case. If this picture is true we could take the viewpoint that what we have been doing in this paper is to "solve" the superconformal field theory with $E_7$ global symmetry using the method of geometrical engineering and mirror symmetry as outlined in [16].

4 Summary and Open Problems

We have extended the observation made in [7] to $E_7$ case: The curve corresponding to the hyperkähler quotient based on the $E_7$ extended Dynkin diagram is equal, when it is expressed in terms of $E_7$ Casimirs, to the generalized Coulomb branch with $E_7$ global symmetry. The relations between FI parameters and mass parameters were obtained. The identity of the two curves led us to conjecture that mirror symmetry in the four dimensional field theories we considered should act in such a way to interchange the generalized Coulomb branch of the original theory with the Higgs branch of the dual quiver gauge theory. For evidence, we discussed the connection of the generalized Coulomb branches in four dimensions to the Coulomb branches of the three dimensional theories obtained by compactifying one dimension. We also discussed F-theory compactifications, and IIA/B mirror symmetry.

What we have shown in this article is that the complex structures of the Higgs branch and the generalized Coulomb branch are the same. To confirm the mirror hypothesis, we also need to show that the metrics are the same. It will be worth studying whether our conjecture is true in more general context. It will be also interesting to consider other quivers and study if the resulting curves can be interpreted as the generalized Coulomb branches of higher genera of some gauge theories. We hope to come back to these issues and others in [29].

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Appendix A

The coefficients in (12) can be expressed in terms of $E_7$ Casimir invariants, $P_i$, which appear as the coefficient of $x^{56-i}$ in the expansion of $\det(x - v \cdot H)$. One can express $v \cdot H$ as $v \cdot H = (v \cdot \lambda_1 \ldots v \cdot \lambda_{56})$. $\lambda$'s are the weights of the fundamental representation. Defining $\chi_n = \frac{1}{2} Tr[(v \cdot H)^n]$, the coefficients in (12) are

$$\alpha_1 = \frac{1}{240} \chi_8 - \frac{11}{6480} \chi_6 \chi_2 + \frac{25}{2239488} \chi_2^4$$

$$= -\frac{2405}{2239488} P_2 + \frac{5}{432} P_2 P_6 - \frac{1}{60} P_8$$

$$\alpha_0 = -\frac{3240 \chi_{12}}{60949905408} + \frac{13}{81648} \chi_{10} \chi_2 - \frac{97}{3265920} \chi_8 \chi_2^2 + \frac{19}{233280} \chi_6^2$$

$$+ \frac{13}{16796160} \chi_6 \chi_2^3 + \frac{103}{4353564720} \chi_2^6$$

$$= \frac{63713}{19} \chi_6 - \frac{7838208}{7838208} P_2^6 - \frac{431}{408240} P_2 P_10$$

$$+ \frac{58320}{58320} P_2 P_8 - \frac{1}{5184} P_6^2 + \frac{1}{540} P_{12}$$

$$\beta_4 = -\frac{1}{36} \chi_2 = \frac{1}{36} P_2$$

$$\beta_3 = \frac{1}{216} \chi_6 - \frac{7}{93312} \chi_2^3 = -\frac{1}{72} P_6 + \frac{169}{93312} P_2^3$$

$$\beta_2 = -\frac{1}{2520} \chi_{10} + \frac{1}{715} \chi_8 \chi_2 - \frac{123280}{233280} \chi_6 \chi_2^2 + \frac{17}{67184640} \chi_2^5$$

$$= \frac{1}{504} P_{10} - \frac{94058496}{479} P_2^5 + \frac{1}{27216} P_2 P_6^2 - \frac{1}{1080} P_2 P_8$$

$$\beta_1 = \frac{1}{2679641} \chi_{14} - \frac{2}{1860480} \chi_{12} \chi_2 + \frac{1}{426202560} \chi_{10} \chi_2^2$$

$$- \frac{1}{1503360} \chi_8 \chi_6 - \frac{6893}{13638481920} \chi_8 \chi_2^3 - \frac{9233}{70140764160} \chi_6 \chi_2^4$$

$$+ \frac{1}{121772160} \chi_6 \chi_2^5 + \frac{78346801}{636317012459520} \chi_2^7$$

$$= \frac{1}{127263402491904} P_2^7 - \frac{3828}{1517} P_{14} - \frac{96277}{4688228160} P_2 P_{10} - \frac{1370167}{120018640896} P_2^4 P_6$$

$$+ \frac{56233}{5357975040} P_2^3 P_8 + \frac{29766528}{29766528} P_2 P_6^2 + \frac{331}{3100680} P_2 P_{12} - \frac{91}{1378080} P_8 P_6$$

$$\beta_0 = -\frac{1}{265464} \chi_{18} + \frac{18577}{21340120032} \chi_{14} \chi_2^2 + \frac{172020672}{21340120032} \chi_{12} \chi_6$$

$$- \frac{1}{118529123834880} \chi_{12} \chi_2^3 + \frac{551}{278737200} \chi_{10} \chi_8 - \frac{241907}{541865116800} \chi_{10} \chi_6 \chi_2^2$$

\(^{111}\)The $\chi$’s defined here with the factor $\frac{1}{2}$ in front are more convenient because the last twenty eight weights are given by minus the first twenty eight weights as discussed below.
There are the following relations between χ’s,

\[
\begin{align*}
\chi_4 &= \frac{\chi_2^2}{12} \\
\chi_{16} &= \frac{13}{27} \chi_6 \chi_{10} + \frac{13}{80} \chi_8^2 + \frac{590}{957} \chi_2 \chi_{14} - \frac{8567}{31320} \chi_2 \chi_8 \chi_6 \\
&\quad - \frac{15925}{103356} \chi_2^2 \chi_{12} + \frac{61607}{1691280} \chi_2^2 \chi_6^2 + \frac{5291}{338256} \chi_2^3 \chi_{10} \\
&\quad + \frac{10824192}{97417728} \chi_2^4 \chi_8 - \frac{36127}{11222522656} \chi_2^5 \chi_6 + \frac{111449}{11222522656} \chi_2^8
\end{align*}
\]

(28)

We also have the relations between λ’s and FI parameters. We only give the expressions for the first twenty eight weights out of fifty six since the other twenty eight λ’s are minus the weights given below. This reflects the fact that the 56 is a real representation.

\[
\begin{align*}
v \cdot \lambda_1 &= -\frac{3}{4} b_1 - \frac{1}{2} b_3 - \frac{1}{4} b_5 + \frac{1}{4} b_6 + \frac{1}{2} b_8 + \frac{3}{4} b_7 \\
v \cdot \lambda_2 &= -\frac{3}{4} b_1 - \frac{1}{2} b_2 - \frac{1}{4} b_3 + \frac{1}{4} b_5 + \frac{1}{2} b_6 - \frac{1}{4} b_7 \\
v \cdot \lambda_3 &= -\frac{3}{4} b_1 - \frac{1}{2} b_2 - \frac{1}{4} b_3 + \frac{1}{4} b_5 - \frac{1}{2} b_6 - \frac{1}{4} b_7 \\
v \cdot \lambda_4 &= -\frac{3}{4} b_1 - \frac{1}{2} b_2 - \frac{1}{4} b_3 - \frac{3}{4} b_5 - \frac{1}{2} b_6 - \frac{1}{4} b_7 \\
v \cdot \lambda_5 &= -\frac{1}{2} b_1 + \frac{1}{2} b_3 + \frac{1}{2} b_8 \\
v \cdot \lambda_6 &= -\frac{1}{2} b_1 - \frac{1}{2} b_3 + \frac{1}{2} b_8
\end{align*}
\]
$$v \cdot \lambda_7 = -\frac{1}{2} b_1 + \frac{1}{2} b_3 - \frac{1}{2} b_8$$
$$v \cdot \lambda_8 = -\frac{1}{2} b_1 - b_2 - \frac{1}{2} b_3 + \frac{1}{2} b_5$$
$$v \cdot \lambda_9 = -\frac{1}{2} b_1 - \frac{1}{2} b_3 - \frac{1}{2} b_8$$
$$v \cdot \lambda_{10} = -\frac{1}{2} b_1 - b_2 - \frac{1}{2} b_3 - \frac{1}{2} b_8$$
$$v \cdot \lambda_{11} = -\frac{1}{4} b_1 + \frac{1}{2} b_2 + \frac{1}{2} b_3 + \frac{3}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{12} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 + \frac{1}{4} b_3 + \frac{3}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{13} = -\frac{1}{4} b_1 + \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{14} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 + \frac{3}{4} b_3 + \frac{3}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{15} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{16} = -\frac{1}{4} b_1 + \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{17} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 - \frac{3}{4} b_3 - \frac{1}{4} b_5 + \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{18} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{19} = -\frac{1}{4} b_1 + \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 - \frac{3}{4} b_7$$
$$v \cdot \lambda_{20} = \frac{1}{2} b_5 + b_6 + \frac{1}{2} b_7 + \frac{1}{2} b_8$$
$$v \cdot \lambda_{21} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 - \frac{3}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{22} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 + \frac{1}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 + \frac{1}{4} b_7$$
$$v \cdot \lambda_{23} = \frac{1}{2} b_5 + \frac{1}{2} b_7 + \frac{1}{2} b_8$$
$$v \cdot \lambda_{24} = \frac{1}{2} b_5 + b_6 + \frac{1}{2} b_7 - \frac{1}{2} b_8$$
$$v \cdot \lambda_{25} = -\frac{1}{4} b_1 - \frac{1}{2} b_2 - \frac{3}{4} b_3 - \frac{1}{4} b_5 - \frac{1}{2} b_6 - \frac{3}{4} b_7$$
$$v \cdot \lambda_{26} = -\frac{1}{2} b_5 + \frac{1}{2} b_7 + \frac{1}{2} b_8$$
$$v \cdot \lambda_{27} = \frac{1}{2} b_5 - \frac{1}{2} b_7 + \frac{1}{2} b_8$$
$$v \cdot \lambda_{28} = \frac{1}{2} b_5 + \frac{1}{2} b_7 - \frac{1}{2} b_8$$

(29)
Appendix B

To directly compare our curve with the one in [9], we should express our result in terms of Casimir invariants of the SO(12) × SU(2) subgroup of E7. More specifically, let us consider the subgroup we get by removing the simple root corresponding to $b_2$. Then the simple root corresponding to $b_1$ becomes the simple root of SU(2) and the rest becomes associated with the roots of SO(12). The mass parameters in [9] can be thought of as an orthonormal basis for the root space. The standard way of choosing such a basis for SO algebras would in our case correspond to

\[
\begin{align*}
b_3 &= m_5 - m_6, \\
b_4 &= m_4 - m_5, \\
b_5 &= m_3 - m_4, \\
b_6 &= m_2 - m_3, \\
b_7 &= m_1 - m_2, \\
b_8 &= m_5 + m_6,
\end{align*}
\]

and since $b_1$ is already orthogonal to everything else it is simply equal to the SU(2) simple root

\[
b_1 = \phi
\]

Similar relations were found in [21] for three dimensional theories. Inserting these expressions into our formulas we find that the curves are equal up to the following rescalings of the basic variables in (11)

\[
\begin{align*}
X &\rightarrow \frac{iX}{8} \\
Y &\rightarrow -\frac{Y}{4} \\
Z &\rightarrow -\frac{Z}{2}
\end{align*}
\]

which turns (11) into

\[
X^2 = Y^3 - \left[+2Z^3 + 8\alpha_1 Z - 16\alpha_0\right] Y \\
- \left[4\beta_1 Z^4 - 8\beta_3 Z^3 + 16\beta_2 Z^2 - 32\beta_1 Z + 64\beta_0\right],
\]

which, after a shift in $Z$ (using the notation of [9])

\[
Z \rightarrow Z + \frac{1}{6} \left(\frac{T_2^2}{12} + T_4\right),
\]

becomes exactly the curve given in [9].
References

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