We introduce the definition of generic bound entanglement for the case of continuous variables. We provide some examples of bound entangled states for that case, and discuss their physical sense in the context of quantum optics. We rise the question of whether the entanglement of these states is generic. As a byproduct we obtain a new many parameter family of bound entangled states with positive partial transpose. We also point out that the "entanglement witnesses" and positive maps revealing the corresponding bound entanglement can be easily deduced.

Pacs Numbers: 03.65.Bz

Entanglement is a fascinating property of quantum states evoking fundamental [1,2], as well as practical questions. In the context of information theory, it has been proved to be useful in quantum cryptography [3], quantum dense coding [4], quantum teleportation [5] and quantum computation. The idea of distillation of noisy entanglement shared by Alice and Bob between distant laboratories has been introduced [6] to make noisy entanglement useful in spite of the noise coming from interaction with environment. The distillation problem, i.e. the question which states are distillable, has a simple solution for low dimensional quantum systems: two spin-$\frac{1}{2}$ particles, or spin-1 plus spin-$\frac{1}{2}$ systems [7]. In those cases any noisy entanglement can be distilled to maximally entangled form, for larger spins the existence bound entangled (BE) i.e. entanglement which is not distillable has been revealed [8]. BE represents the result of nontrivial irreversible process in which entanglement is confined to the physical system. It was shown that there is connection of BE with other very interesting quantum phenomena, called nonlocality without entanglement [9] (see [10] for discussion).

It is not trivial to provide examples of states which are BE. It has been shown [8] that any state which is entangled and at the same time satisfies positive partial transpose (PPT) condition [11] is bound entangled. The existence of PPT entangled states was discussed in [13] and the first explicite examples were provided in [14]. The first systematic procedure of constructing such states, employing nonextendible product basis was provided in Refs. [15]. On the mathematical ground however, the first example of matrices which can be treated as prototypes of PPT entangled states were provided by Choi already in Ref. [16], as a result of analysis of cones of positive matrices. Here, we shall use the generalised structure of the Choi matrix to provide the first examples of PPT entangled states for continuous variables.

Let us recall that the PPT separability condition [11], applied to a density matrix $\rho$ requires that the partial transposed matrix $\rho_T$ constructed from original one is still the legitimate state. The matrix $\rho_T$ associated with an arbitrary product orthonormal $|i,j\rangle$ basis is defined in this basis as:

$$\rho_{T,n,n'} = \langle m,\mu|\rho_T|n,\nu\rangle = \rho_{mn,\mu\nu}.$$ (1)

It has been pointed out that for any separable state $\rho$, the partial transpose $\rho_T$ has to be also a state [11], i.e. has to be positively definite.

This statement is valid also for the cases when the state is defined on infinite dimensional Hilbert space [12].

Although the existence of BE states for finite dimensions has been proved, it has not been known so far whether nontrivial examples for the latter exist in the infinite dimensional case. In fact, main investigations of entanglement in the continuous variables area were performed for pure states resulting in nonlocality effects [18], new versions of teleportation [19], quantum computation [20], quantum error correction [21] and quantum dense coding [22]. For mixed states, however, there were so far only analysis of PPT condition in situations, where it provides necessary and sufficient condition of separability, like it happens in the low spin systems [13]. In particular, it has been shown for continuous variables that this is the case for Gaussian states [23,24].

In this paper we discuss bound entanglement for continuous variables. We define the requirement any generic bound entangled state must satisfy in that case. We provide the first examples of nontrivial PPT entangled states, ergo BE states for continuous variables. We rise the question how generic they are, and discuss also the problem of physical realization of such states.

Of course, one can simply construct a trivial example. Consider, say $3 \otimes 3$ BE state $\sigma$, and the infinitely dimensional Hilbert space $H$. Let us define infinitely many "copies" of $\sigma$ labeled by $\sigma_n$, each of which has the matrix elements of the original $\sigma$, but in basis $S_n = \{|i,j\rangle\}_{i,j=3n}^{3n+3}$. Let $\{p_i\}_{i=1}^{\infty}$ be an infinite sequence of nonzero probabilities, $\sum_{i=1}^{\infty} p_i = 1$. Then the following state

$$\tilde{\sigma} = \sum_{n=1}^{\infty} p_n \sigma_n,$$ (2)

is bound entangled, but it has a trivial form from the continuous variables point of view. Actually, it can be re-
produced with arbitrary accuracy performing local transformations on states which are of the $3 \otimes 3$ type. Moreover, they can be produced in a reversible way. This follows from the fact that the states $\sigma_n$ and $\sigma_{n'}$ are locally orthogonal [17]. It means that Alice and Bob can use only local quantum actions and classical communication (LQCC) to distinguish between them. This can be done in a reversible way as both experimentalists can forget the results of measurements. In effect, there is no entanglement between states belonging to sets of Alice’s vectors $|i\rangle_{3n+3}$ and Bob ones $|i\rangle_{3n'+3}$ for $n \neq n'$.

Thus, in the case of the states (2) we deal effectively with $3 \otimes 3$ type entanglement only. What does that mean from the formal, and more rigorous point of view? One should ask first what does that mean that a state represents a generic $N \otimes N$ type entanglement. The answer to this question can be obtained immediately using the recently introduced definition of Schmidt rank [25] for mixed states. Let us recall the definition:

**Definition.** Bipartite density matrix $\rho$ has Schmidt rank $K$ if (i) for any decomposition of $\rho$, $\{p_i, \{|i\rangle \}$ with $\rho = \sum p_i |i\rangle \langle i|$ at least one of vectors $|i\rangle$ has Schmidt rank at least $K$, and (ii) there exists a decomposition of $\rho$ with all vectors $\{|i\rangle\}$ of Schmidt rank at most $K$.

Thus it is natural from the physical point of view to say that the state represents generic rank $K$ entanglement if it has Schmidt rank $K$. We introduce therefore:

**Definition.** A state $\rho$ represents generic continuous variables or infinite Schmidt rank entanglement if it is the limit of states $\rho_n$ of Schmidt rank $K_n$, with $\lim_n K_n = \infty$.

In the following we shall focus on the question of existence of generic continuous variables entanglement, which would be at the same time bound entanglement, i.e. entanglement which cannot be distilled. We will construct PPT states for continuous variables and argue that they represent generic infinite Schmidt rank entanglement.

For this aim consider first the state

$$|\Psi\rangle = \sum_{n=1}^{\infty} a_n |n, n\rangle, \quad ||\Psi||^2 = \sum_{n=1}^{\infty} a_n^2 = q < \infty,$$

and the family of states

$$|\Psi_{mn}\rangle = c_m a_n |n, m\rangle + (c_m)^{-1} a_m |m, n\rangle,$$

$$n = 1, 2, ..., m > n, \quad 0 < a_{n+1} < a_n < 1.$$

Let us assume that the sum $\sum_{n=1}^{\infty} \sum_{m>n} ||\Psi_{mn}\rangle^2$ is finite. This can be achieved for example by setting $a_n = a^n$, $c_n = c^n$ for some $0 < a < c < 1$. The sum is given then by $a^4 c^4 (1 - a^2)(1 - a^2c^2) - a^6 (c^2 - a^2)^{-1}(c^2 - a^4)^{-1}$. Under the above assumptions the matrix

$$g = \frac{1}{A} (|\Psi\rangle \langle \Psi| + \sum_{n=1}^{\infty} \sum_{m>n} |\Psi_{mn}\rangle \langle \Psi_{mn}|),$$

with the normalizing factor

$$A \equiv ||\Psi||^2 + \sum_{n=1}^{\infty} \sum_{m>n} ||\Psi_{mn}\rangle^2,$$

represents a legitimate quantum state in the space $l^2(C) \otimes l^2(C)$, where $l^2(C)$ is the space of all complex sequences $\{|z_n\rangle\}$, $\sum_{n=1}^{\infty} |z_n|^2 < \infty$. It can be seen by inspection that the above state has PPT property. It follows immediately that pure state entanglement cannot be distilled from (5).

For this aim simple arguments from the Ref. [8] can be recalled, and applied to the separable superoperators in infinitely dimensional space.

Subsequently we shall show that the above states are entangled, and, as PPT states represent bound entanglement, the entanglement of the states is bound. To this aim we shall prove that the state (5) has the following property:

**Property.** Any local measurement of state $\rho$

$$\rho \rightarrow \rho' = \frac{P \otimes Q \rho P \otimes Q}{\text{Tr}(P \otimes Q \rho P \otimes Q)}$$

by means of $P$, $Q$ projecting onto the space span$\{|n_1\rangle, ..., |n_K\rangle\}$ on Alice and Bob’s sides respectively results in generic $K \otimes K$ bound entangled state $\rho'$. From the above property it follows immediately that $\rho$ is a bound entangled state.

**Proof.** Let us first prove that for any $K$ the state $\rho'$ is a $K \otimes K$ BE state. To this aim we observe that after local filter operation corresponding to the operator $V = \text{diag}[a_n^{-1}, ..., a_n^{-1}]$ on the Alice side, and a suitable unitary transformation $U_1 \otimes U_2$ (that transforms $|n_m\rangle \rightarrow |m\rangle$ on both Alice and Bob’s sides), the state becomes proportional to the matrix of the form:

$$\Sigma = |\Phi\rangle \langle \Phi| + \sum_{n=1}^{K} \sum_{m>n} |\Phi_{mn}\rangle \langle \Phi_{mn}|,$$

with $|\Phi\rangle = \sum_{n=1}^{K} |n, n\rangle$, and $|\Phi_{mn}\rangle = \alpha_m |n, m\rangle + \alpha_m^{-1} |m, n\rangle$. Subsequently, we shall use the general definition of $\Sigma$ as well, but after the above mentioned local actions the state is proportional to the matrix with parameters $\alpha_m = c_n$. It means in particular that $0 < \alpha_{i+1} < \alpha_i < 1$. We shall prove now that $\Sigma$ does not have any product vector in its range, ergo that, if normalised, $\Sigma$ is an entangled state. That will mean, however, also that the state $\rho'$ is bound entangled, since the local filtering and the local unitary operations are reversible with nonzero probability.

Suppose that there was a product state in the range of the matrix (7). Then, there would exist some $g, g_{ij}$, $i = 1, ..., K, j > i$ such that:

$$g|\Phi\rangle + \sum_{i=1}^{K} \sum_{j>i} g_{ij} |\Phi_{ij}\rangle = |\psi, \phi\rangle.$$
for some nonzero product vector \( \langle \psi, \phi \rangle \). Suppose first that we would have \( g \neq 0 \) in (8). Then, we could set \( g = 1 \), and the following constraints would immediately follow from (8):

\[
\begin{align*}
\psi &= [x_1, ..., x_K], \\
\phi &= [(x_1)^{-1}, ..., (x_K)^{-1}]
\end{align*}
\]

for some numbers \( \{x_i\} \) which are all nonzero. Substituting (9) into (8) leads to the equations:

\[
\begin{align*}
g_{ij} \alpha_j &= \frac{x_i}{x_j}, \\
g_{ij} (\alpha_j)^{-1} &= \left( \frac{x_i}{x_j} \right)^{-1},
\end{align*}
\]

for \( i = 1, ..., K, \ j > i \).

Thus we have \( \alpha_j^2 = (\frac{x_j}{x_i})^2 \) for every \( i = 1, ..., K - 1, \ j > i \).

We can, however, put \( x_1 = 1 \), and then we get that all \( x_i^2 \)'s are equal, and that

\[
\alpha_j^2 = 1, \text{ for } j = 2, ..., K - 1.
\]

This is in contradiction with the condition \( 0 < \alpha_{i+1} < \alpha_i < 1 \) fulfilled by \( \Sigma \).

Consider now the case when \( g = 0 \) holds in equation (8). That would mean, keeping the same notation for \( \psi \), i.e. \( |\psi\rangle = [x_1, ..., x_K] \) that we could have \( |\phi\rangle = [y_1, ..., y_K] \) with \( y_i \neq 0 \) iff \( x_i = 0 \). But, if we examine the equation (8) under those conditions we get immediately that all \( g_{ij} \) parameters must vanish, so that the whole LHS of the equation becomes then equal to zero. It means that there is no product vector in the range of the matrix \( \Sigma \). Following previous discussion it is not difficult to see that the same holds for states \( \varrho' \), which are thus (by virtue of the range criterion of Ref. [14]) entangled. Collecting all the above observations, we see that the Property of the original matrix \( \varrho \) holds. \( \square \)

It is unfortunately not easy to see that the latter represents the generic rank \( K \) entanglement. In fact it contains in the mixture the pure state of Schmidt rank \( K \) which cannot be distinguished from the rest content of the mixture in a reversible manner, since its reduced density matrix has full rank \( K \) (let us recall that all parameters \( a_i \) are nonzero).

Finite dimensional bound entangled states .- It is remarkable that as a byproduct we have obtained here a new family of \( K \otimes K \) bound entangled states for an arbitrary \( K \). These are the states \( \sigma = \frac{1}{\Sigma} (\Sigma \varrho\varphi) \), with \( \Sigma \) violating one (or more) of the \( K - 2 \) conditions (11). In this notation we recall the Choi matrix as a special case of \( \Sigma \) with \( K = 3 \), and all \( \alpha \)'s equal to 2 (see [16]).

The corresponding “entanglement witnesses” and positive maps .- It should be pointed out that any BE state from the last paragraph (i.e. \( \frac{1}{\Sigma} (\Sigma \varrho\varphi) \) violating condition (11)) has no product vector in its range. Thus the projector \( P \) onto its range has no product vector in its range as well. This is the same as in the projector orthogonal to UPB set of vectors [15], and thus mutatis mutandis the approach from the paper [26] can be immediately applied to reproduce both entanglement witnesses, as well as the corresponding positive maps.

Possibility of physical realisation.- Let us now discuss a possibility of physical realization of the states of the type of \( \varrho \), as states of two photon modes of electromagnetic field of equal frequency, but orthogonal polarizations.

Let us set \( a_n = e^{-\beta n}, c_n = e^{-\gamma n}, \gamma < \beta \), and let us denote the corresponding photon creation and annihilation operators of the two modes considered as \( A^\dagger, A, B^\dagger, B \), respectively. The state \( \varrho \) can be represented as a mixture

\[
\varrho \sim |\Psi\rangle\langle \Psi | + \sum_{k=1}^{\infty} \varrho_k,
\]

where

\[
\varrho_k = V \delta(B^\dagger B - A^\dagger A - k)V^\dagger
\]

where \( V = e^{-\beta A^\dagger A - \gamma B^\dagger B} + U e^{- (\beta - \gamma) B^\dagger B} \), \( U \) is the unitary operator that transforms \( A \) photons into \( B \) photons, while the operator function \( \delta(.) \) is the operator valued Kronecker delta, \( \delta(x) = 0 \), except for \( x = 0 \), when \( \delta(x) = 1 \). We will now discuss the possibilities of generating states corresponding to the subsequent terms in the mixture (12).

- The state \( |\Psi\rangle \sim \exp(-\gamma B^\dagger B)|0,0\rangle \) can be can be created as a two mode squeezed state, for instance in the process of degenerate parametric amplification [18,22]. In fact, this kind of states have been used for the teleportation schemes with continuous variables [22].

- Each of the terms \( \varrho_k \) can be obtained by applying the positive operator valued map to the states \( \delta_k = \delta(B^\dagger B - A^\dagger A - k) \), that transforms (with some probability)

\[
\delta(B^\dagger B - A^\dagger A - k) \to V \delta(R^\dagger B - A^\dagger A - k)V^\dagger
\]

- The operator \( V \) can be realised by splitting coherently the photon beam, applying corresponding (different) thermal noise to the two splitted beams, applying then the unitary operator \( U \) to one of the beams, and finally combining them together. The unitary operator \( U \) is easily realised using linear optics for equal frequency photons with different polarizations.

- The main problem seems to consist thus in generating the states \( \delta_k \). These states can be formally written as

\[
\sum_n \sum_{k=1}^{\infty} |n, n+k\rangle\langle n, n+k|.
\]
These states cannot be normalized, but that does not pose a problem since we can always regularize them by including part of the thermal noise operators into the definition of $\delta_k$. The main question is how to realize, at least in an approximate way, the Kronecker delta. We propose to send the photon field through an appropriately designed array of nonlinear phase shifters such that

- (i) each phase shifter acting on $|n,m\rangle$ state produces a shift $\Delta_i = x_i(n-m-k)$, where the first two terms in $\Delta_i$ represent Kerr effect of $A$ and $B$ photons respectively. Kerr effect should be here of the opposite sign for the two modes in question, but of the same magnitude. The last term in $\Delta_i$ represents normal linear phase shift.

- (ii) the probability amplitude of passing the $i$-th shifter should be $\alpha(x_i)$. Note that in such case, the action of the phase shifter on the state $|n,m\rangle$ would be

$$|n,m\rangle \rightarrow \sum_i \alpha(x_i)e^{ix_i(n-m-k)}|n,m\rangle,$$

so that designing $\alpha(x_i)$ to be a broad, for instance Gaussian function would indeed project $|n,m\rangle \rightarrow \delta_k(n-m-k)|n+k,m\rangle$, where $\delta_k$ is the finite bandwidth approximation of the Kronecker delta.

Finally, let us ask the question concerning the character of the entanglement rank of the states (5). Certainly, they do not consist of locally orthogonal representation of finite Schmidt rank entanglement. This can bee easily seen from the fact that local orthogonality is a stronger property than orthogonality in case of pure states vectors. If the above statement about local orthogonality was true, then it would follow that eigenvectors of $\rho$ would be locally orthogonal, and of finite Schmidt rank, which is obviously not true, since one of the eigenvectors of $\rho$ is of infinite Schmidt rank. Nevertheless, we have not been able so far to show that the Schmidt rank of the proposed states is infinite. It is, however, quite likely that either these states, or some modification of them possess that property.

PH acknowledges support from Deutscher Akademischer Austauschdienst and partial support from Polish Committee for Scientific Research, grant No. 2 PO3B 103 16. This work has been also supported by the Deutsche Forschungsgemeinschaft (Schwerpunkt “Quanteninformationsverarbeitung”).

* E-mail address: pawel@mifgate.pg.gda.pl
** E-mail address: lewen@itp.uni-hannover.de

[12] This statement (which is not obvious for infinite dimensional case) holds for any positive map (R. Werner, private communication).