\[
\left(\frac{1}{r^2} + \frac{\gamma}{r} \right) f(r) \sim \frac{1}{r^2} \left( \frac{\gamma}{r} \right)^2, \quad r \ll \frac{\gamma}{\gamma_0}
\]

The above equation shows that the field amplitude is proportional to the inverse square of the distance, with an additional term that depends on \(\gamma/r\). This term becomes negligible for small values of \(r\) compared to \(\gamma\), which is consistent with the behavior observed in polarization and interference experiments.

The above expression is derived from the quantum mechanical description of the system. It is a limiting case of the more general equation that describes the behavior of the field in the presence of a nonlinear medium.

This equation is useful in understanding the behavior of light in nonlinear optical materials, where the nonlinear response of the medium can significantly affect the propagation of light.

\[\text{Freezing light via hot atoms} \]
Eq. (1) yields the same result, since the susceptibility [5,6] depends only on the combination ω + kv,
\[ \chi_v(\omega, k) = \chi(\omega + kv) = \frac{\mu_0^2 N \alpha_{ab} \Gamma_{ac} + \Omega^2 n_{ce}/\Gamma_{ac}}{\hbar (\omega - i k_d + \omega_c)}, \]
Here \( n_{ab} = \rho_{ad} - \rho_{bd}, \ n_{ce} = \rho_{ec} - \rho_{ae}, \ \rho_{cd} \) is the population of the \( \ell \)-th level, \( \gamma \) and \( \gamma_{cd} \) are the relaxation rates of excited state and \( c - b \) respectively (\( \gamma \gg \gamma_{cd} \)); \( \omega_c \) and \( \omega_d \) are the frequencies of the optical and low frequency transitions (\( \omega_d \gg \omega_c \)); \( k_d \) and \( \omega_c \) are the frequency and wavenumber of the driving and probe fields respectively. \( N \) is the atomic density. \( \Omega = \mu_0 E_0 / 2\hbar \) is the Rabi frequency of drive field \( (1/2) E_0 \exp(\omega_d - i k_d z) + c.c. \). \( \mu_{ac} \) and \( \mu_{cd} \) are the dipole moments of \( a - c \) and \( a - b \) transitions respectively, \( \Gamma_{ac} = \gamma + i(\Delta \omega_d + k_d v), \ \Gamma_{cb} = \gamma + i(\Delta \omega + k_d v), \ \Gamma_{cd} = \gamma_{cd} + i(\Delta \omega - \Delta \omega_d + \Delta k v), \ \Delta \omega = \omega - \omega_c, \ k_d = \omega_d - \omega_c, \ c, \ k = \omega_d + \omega_c/\gamma \). We use a standard model, with incoherent pump and loss rates \( (v_c = \gamma_{ac}) / 2 \), assuming time of flight broadening of \( b - c \) transition [6] (Fig. 1), so that in the absence of fields \( p_{bc} = \rho_{bb} = 1/2 \). According to Eqs. (1), (2), we again obtain \( v_c = \tilde{v}_c = v - v \). The physical reason for this drift is that the field is basically “seized” by the atoms in the form of atomic polarization.

An important question is how to input the light pulse into the gas. There are different possibilities. One example uses a grid mirror that has the grid stripes of small area, so that atoms can freely fly through the mirror, and small spacing between the grid stripes as compared to the wavelength of light to provide efficient reflection, as in Fig. 1b. If atoms are at rest, the light would propagate in the forward direction. However, if the velocity of atoms is equal to (or larger than) \( \tilde{v}_c \), one should see a frozen (or backward) pulse.

Depending on the mechanism of pulse input into the medium, one should look for the solution of the problem with initial (time), boundary (space), or mixed (time-space) conditions. In the case of the initial value problem, we solve the dispersion equation for \( \Omega = \omega(k) \). Fig. 3a. Galilean transformation ensures the same EIT half width, \( \Delta k_{EIT} = 
\omega \gamma \tilde{v}_c \), for as the atoms at rest since \( 1/m(\bar{\omega}[k]) = 1/m(\omega[k]) \). In the case of the boundary value problem, we find \( k = k(\omega) \). The result shows narrowing of the EIT dispional to the kinematic factor \( \alpha = (\tilde{v}_c - \tilde{v}) / \tilde{v}_c \). Indeed, in the accompanying frame the dispersion relation near resonance can be decomposed in a form of a quadratic polynomial, \( \Delta k = \Delta k_{\Omega} + \xi(\Delta \omega - \Delta \omega_d)^2 + (\Delta \omega - \Delta \omega_d) / \tilde{v}_c \). Its Galilean transformation to the laboratory frame yields
\[ \Delta k = \Delta k_{\Omega} + \frac{1}{\alpha} \left[ \frac{\omega}{\tilde{v}_c} - i k_{cd} - i \xi \left( \frac{\omega}{\alpha} \right)^2 \right], \]
where \( \omega = \Delta \omega - \Delta \omega_d \). Coefficients in Eq. (3) can be easily deduced using Eq. (2). For example, for the case of one-photon resonance \( \Delta \omega_d = 0 \) at \( \Omega^2 \gg \gamma_{cd} \), we have \( \tilde{v}_c = \hbar \Omega / 2 \mu_0^2 k_N D \). \( \Delta k_0 = 0 \), residual absorption coefficient at the center of EIT dip is \( \kappa_0 = \gamma_{cd} / \tilde{v}_c \), and a coefficient determining the parabolic profile of absorption in the EIT dip is \( \kappa = \gamma \Omega / \tilde{v}_c \). This approximation is valid if residual absorption is small, \( k_0 \xi \ll (1 - v^2 / \tilde{v}_c^2)^2 / \tilde{v}_c^2 \). Absorption increases twice as much as EIT minimum value at detuning \( \delta \Omega_{EIT} = |\tilde{v}_c - v| \Omega / \gamma_{cd} / \tilde{v}_c \) that is much less than the EIT half width \( \Delta \omega_{EIT} = \Delta k_{EIT} / \tilde{v}_c - v \).

Eq. (3) shows that the absorption coefficient \( 1/m k \) is increased and sharpened by a factor \( (\tilde{v}_c - v) / \tilde{v}_c \) as compared to that in the co-moving frame. Since the spectrum of the pulse cannot be transformed on the stationary boundary, only those spectral components that are within the sharpened EIT dip penetrate deep into the medium. For drift velocity \( v > \tilde{v}_c \), the backward EIT polariton can be excited from inside a cell (Fig. 1b).

In the case of an atomic beam with a moving boundary (or moving sample), i.e., for the mixed boundary-initial value problem, the spectrum (inverse duration) of the pulse shrinks at the moving boundary exactly in the same way as the EIT width in Eq. (3), \( \Delta \omega = \Delta \Omega |\tilde{v}_c - v| / \tilde{v}_c \). This is not a coincidence, but is necessary for consistency of viewing of the same process from different frames. The pulse within the EIT dip decays in time with the same rate independently of whether it propagates through atoms at rest or through a beam, since this decay is predetermined by atomic relaxation \( \gamma_{cd} \).

**Atoms with a thermal velocity distribution.** Let us consider a stationary cell of hot atoms. If the intensity of the drive is strong enough to provide EIT for the resonant group of atoms (see Fig. 4) but at the same time weak enough to avoid an interaction with off-resonant atoms, moving with “wrong” velocities, it is mainly this drifting beam that would support the ultra-fast EIT polariton with zero or even small negative group velocity.

To prove this we calculate the dispersion law \( \omega(k) \) for the EIT polariton in a hot gas in a cell at rest. The susceptibility is given by an average of the beam susceptibility over a velocity distribution \( F(v) \) of atoms in a gas with thermal velocity \( v_T \), \( \chi(\omega, k) = \int_{-\infty}^{+\infty} dv \chi(v, \omega, k) \). Instead of the Maxwellian thermal distribution we use Lorentzian, \( F(v) = \nu_T / (\pi v_T^2 + v^2) \), since the far-off-resonant tails are not important. This allows us to obtain simple analytical results because an integration over velocities is reduced to a sum of a few residues in the simple poles, \( v = v_T \). Only those poles count that lay in the lower half plane \( v^2 \)-plane in the formal limit of infinitely large growth rate \( \lim \omega \to \infty \). For a positive wavenumber detuning, \( \Delta k > 0 \), there are two such poles. One originates from Lorentzian, \( v_T = iv \), and the other from the velocity dependent populations, \( v_T = -(\gamma G + \Delta \omega_d) / \Delta k \). Here \( \gamma G = \gamma (1 + \Omega^2 / \gamma_{cd})^{1/2} \) determines the velocity width of an effective drifting beam of atoms that are driven by an external field into a coherent “dark”
\[
\Delta \omega = \Delta \omega_d - v_d \Delta k + \gamma_k - \frac{\Omega^2}{\gamma (1 + G)} \left[ \frac{\gamma G \Delta k}{2 \pi \mu_{eb} k_d N_f + i} \right]^{-1}
\]
shown in Fig. 3b. The EIT half width is \(\Delta k_{\text{EIT}} = \gamma_k / v_d^2\).
For small detuning \(|\Delta \ell| \ll \Delta k_{\text{EIT}}\), Eq. (6) yields linear dispersion and parabolic decay profile, \(\Delta \omega = \Delta \omega_d + \Delta k(v_d - v_d) + i \gamma_k + \Delta k^2 \gamma_k^2 / \gamma_k\). Decay increases twice as much as EIT minimum value, \(\ln \Delta \omega = 2 \gamma_k\), at very small detuning \(\delta k_{\text{EIT}} = \sqrt{\gamma_k} / v_d\). The group velocity describes pulse kinematics if \(\omega_d \Delta k / \hbar\) has negligible imaginary part, i.e., near the center of the EIT dip where \(|\Delta \ell| < v_d - v_d \gamma_k / v_d^2\). The last inequality does not mean that the pulse cannot be stopped. It just means that when the pulse is frozen, \(v_d = v_d^2 - v_d = 0\), its evolution is governed by the dispersion of absorption.

Fig. 3 clearly shows that the ultra-slow EIT polaron in a hot gas is similar to that in a mono-velocity beam, since detuning of driving field picks a beam with velocity \(v_d = \Delta \omega_d / k_d\). However, effective density of atoms supporting EIT polaron \(N^*\) and EIT width \(\Delta k_{\text{EIT}} = \gamma_k / v_d^2\) in a hot gas are different because of factors \(\gamma_k\) and \(F(v_d)\).

As a result, the group velocity at the EIT resonance, according to Eq. (6), in terms of a critical density is

\[
v_d = \frac{\beta N_{cr}}{N F(v_d) - v_d}; \quad N_{cr} = \frac{\pi \Omega}{2 \pi^2 \beta \mu_{eb} \gamma_k} \left( \frac{\gamma_k}{v_d} \right)^2 - 1
\]
where \(\beta = \max[v_d F(v_d)]\). For Lorentzian \(F(v_d)\), we have \(\beta = 1 / 2 \pi\), and \(v_d = (v_d - v_d^{(1)}) (v_d - v_d^{(2)}) N_{cr} / 2 N v_f\) is a quadratic polynomial over \(v_d\) i.e., the group velocity is zero for drive detunings \(v_d^{(1,2)} = v_d [N / N_{cr} \pm \sqrt{(N / N_{cr})^2 - 1}]\) and negative between them for a density higher than the critical value, \(N > N_{cr}\), as is shown in Fig. 2. To achieve minimal group velocity, \(\min v_d = -(v_d / 2 N_{cr}) [1 - (N / N_{cr})^2] \), one has to tune at \(v_d = v_d / N_{cr} \cdot \). The condition to freeze or reverse the light \((v_d \leq 0)\) means that the group velocity supported by the drifting beam with the density \(N^* = \pi F(v_d) G / k_d\) should be equal to or less than the velocity of atoms in the beam, i.e., \(v_d = v_d / N_{cr} / 4\). If we compare a mono-velocity beam with a hot gas at \(v_d = v_d^*\) and the same \(N^*\) as the total density \(N / N_{cr}\) in a beam to provide the same group velocity, \(v_d = v_d^*\), we find that the EIT width and the residual decay in a hot gas are \(G \propto \Omega / \sqrt{\gamma_k} \) times less than in a beam. To minimize \(N_{cr}\) the critical density, the group velocity should be as low as possible to decrease \(v_d^*\) due to power broadening effect and to avoid EIT contribution from the atoms with “wrong” (positive) velocities. That is, the drive intensity should be just above a threshold of the EIT effect at resonance, \(\Omega^2 > \gamma_k \gamma_k\). Under realistic parameters relevant to the experiments with \(^{87}\)Rb vapor \([4]\) and chosen in Figs. 2-5, the critical density is \(N_{cr} \sim 10^{11}\) cm\(^{-3}\).

Absorption or time variation of the drive field results in a spatial or time dependence of the group velocity in the

\[\Delta \omega = \Delta \omega_d - v_d \Delta k + \gamma_k - \frac{\Omega^2}{\gamma (1 + G)} \left[ \frac{\gamma G \Delta k}{2 \pi \mu_{eb} k_d N_f + i} \right]^{-1}\]

\[
v_d = \frac{\beta N_{cr}}{N F(v_d) - v_d}; \quad N_{cr} = \frac{\pi \Omega}{2 \pi^2 \beta \mu_{eb} \gamma_k} \left( \frac{\gamma_k}{v_d} \right)^2 - 1
\]

where \(v_d = \Delta \omega_d / k_d\).
cell. This allows us to control input and parameters of the pulse in the cell. According to geometrical optics, the parameters of the EIT polariton adiabatically follow the local properties of the driven atoms. Fig. 5 demonstrates how the ultra-slow pulse decelerates up to the point \( v_d = 0 \) where it becomes frozen.

The important conclusion is that the drifting beam provides large enough drift spatial dispersion \( \partial n / \partial k \) (see Eq.(1)) to ensure \( v_d \leq 0 \). Although the density of drifting atoms is small, \( N' \ll N \), their resonant contribution dominates. This allows us to make the group velocity zero or even negative [7]. To observe freezing or backward light one can look, e.g., for a scattering, luminescence, delay, or enhanced nonlinear mixing caused by ultra-slow pulse.

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![Fig. 1.](image1) (a) Three-level atomic \( \Lambda \)-system. (b) Geometry of ultra-slow EIT pulse propagation in the gas of atoms.

\[
\text{FIG. 2. Ultra-slow and negative group velocity of EIT polariton vs detuning of drive laser: } \Omega = 0.25 \gamma, k_a, \nu_T = 100 \gamma, \gamma_{ch} = 0.001 \gamma. \text{ (a) } N = 0.6 N_r; \text{ (b) } N = N_r; \text{ (c) } N = 1.5 N_r.
\]

\[
\text{FIG. 3. Dispersion, } Re(\Delta \omega) = Re(\omega - \omega_0), \text{ and decay, } Im(\Delta \omega), \text{ spectra of the ultra-slow EIT polariton according to numerical solution of the dispersion equation for: (a) atomic beam } \left( N = 1.1 N_r, \pi F(v_d) \gamma G / k_d \right) \text{ with susceptibility (2); (b) stationary cell of hot gas } \left( N = 1.1 N_r \right) \text{ with exact susceptibility (4); } \Omega = 0.25 \gamma, v = v_d = v_T, k_a, \nu_T = 100 \gamma, \gamma_{ch} = 0.001 \gamma.
\]

\[
\text{FIG. 4. The velocity distribution of atoms in a cell (solid line). Effective drifting beam (dotted) selected by drive laser.}
\]

\[
\text{FIG. 5. Kinematics of the deceleration of the ultra-slow pulse to the point of freezing } (v_d = 0) \text{ along a cell with decreasing group velocity } v_d(z). \text{ Positions of pulse are shown at subsequent moments of time } t = m \tau (\tau = 3 L / 2 v_d(0), m = 0, 1, 2). v_d(z) \text{ is calculated numerically according to decreasing drive intensity found from the wave equation for the same parameters as in Fig. 3(b). } L = 10 \text{ cm.}
\]


