Fock representations from $U(1)$ holonomy algebras

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ABSTRACT
We revisit the quantization of $U(1)$ holonomy algebras using the abelian $C^*$ algebra based techniques which form the mathematical underpinnings of current efforts to construct loop quantum gravity. In particular, we clarify the role of “smeared loops” and of Poincare invariance in the construction of Fock representations of these algebras. This enables us to critically re-examine early pioneering efforts to construct Fock space representations of linearised gravity and free Maxwell theory from holonomy algebras through an application of the (then current) techniques of loop quantum gravity.
1. Introduction

In the early nineties [1, 2, 3] linearised gravity in terms of connection variables and free Maxwell theory on flat spacetime, were treated as useful toy models on which to test techniques being developed for loop quantum gravity[4]. Significant progress has been made in the field of loop quantum gravity since then[5]. Hence, it is useful to revisit these systems using current techniques to clarify certain questions which arise in the context of those pioneering but necessarily non-rigorous efforts.

Two important (and related) questions are:

(I) How did similar techniques for the quantization of general relativity and for its linearization about flat space, result in a non Fock representation for the (kinematic sector) of the former and a Fock representation for the latter? In particular, what is the role of Poincare invariance in obtaining the Fock representation? (This last point was a puzzle to the authors themselves [1]).

(II) What is the role of “smeared” loops in [1] in obtaining a Fock representation?

In this work, we use the abelian $C^*$ algebra techniques [6, 8] which constitute the mathematically rigorous framework of the loop quantum gravity program today, to investigate (I) and (II) above. It is also our aim to clarify the role of the different mathematical structures in the quantization procedure which determine whether a Fock or non Fock representation results. Although we restrict attention to $U(1)$ theory on a flat spacetime, we believe that our results should be of some relevance to the case of linearised gravity.

This work is motivated by the following question in loop quantum gravity: how do Fock space gravitons on flat spacetime arise from the non-Fock structure of the Hilbert space which serves as the kinematical arena for loop quantum gravity? Admittedely, the answer to this question must await the construction of the full physical state space (i.e. the kernel of all the constraints) of quantum gravity. Nevertheless, this work may illuminate some facets of the issues involved.

The starting point for our analysis is the abelian Poisson bracket algebra of $U(1)$ holonomies around loops on a spatial slice. This algebra is completed to the abelian
$C^*$ algebra, $\mathcal{A}$ of [6, 8]. Hilbert space representations of $\mathcal{A}$ are in determined by continuous positive linear functions (PLFs) on $\mathcal{A}$. We review the construction of $\mathcal{A}$ and of the PLF introduced in [6, 8] (which we shall call the Haar PLF) in section 2. The resulting representation is a non Fock representation in which the Electric flux is quantized [7].

In section 3 we construct an abelian $C^*$ algebra $\mathcal{A}_r$, based on the Poisson bracket algebra of holonomies around the “Gaussian smeared” loops of [1]. Next, we derive the key result of this work, namely that there exists a natural $C^*$ algebraic isomorphism, $I_r : \mathcal{A} \rightarrow \mathcal{A}_r$ with the property that $I_r(\mathcal{A}) = \mathcal{A}_r$.

The standard flat spacetime Fock vacuum expectation value restricts to a positive linear function on $\mathcal{A}_r$. We are unable to show the continuity or lack thereof, of this Fock PLF on $\mathcal{A}_r$. Nevertheless, since the GNS construction needs only a $*$ algebra (as opposed to a $C^*$ algebra), we can use the Fock PLF to construct a representation of the $*$ algebra $\mathcal{A}_r$. In section 4 we show that this representation is indeed the standard Fock representation even though $\mathcal{A}_r$ is a proper subalgebra of the standard Weyl algebra for $U(1)$ theory.

Using the map, $I_r$, we can define a Haar PLF on $\mathcal{A}_r$. We construct the resulting representation in section 5a. Finally, we use $I_r$ to define a Fock PLF on $\mathcal{A}_r$. The resulting representation is, in a precise sense, an approximation to the standard Fock representation. We study it in section 5b.

Section 6 is devoted to a discussion of our results in the context of the questions (I) and (II). Some useful lemmas are proved in Appendices A1 and A2.

In this work the spacetime of interest is flat $\mathbb{R}^4$ and we use global cartesian coordinates $(t, x^i)$, $i = 1, 2, 3$. The spatial slice of interest is the initial $t = 0$ slice and all calculations are done in the spatial cartesian coordinate chart $(x^i)$. We use units in which both the velocity of light and Planck’s constant, $\hbar$, are equal to 1. We freely raise and lower indices with the flat spatial metric. The Poisson bracket between the $U(1)$ connection $A_a(x), a = 1, 2, 3$ and its conjugate electric field $E^b(y)$ is $\{A_a(x), E^b(y)\} = e\delta^b_a \delta(x, y)$ where $e$ is a constant with units of electric charge.

\[1^r\] is a small length which characterises the width of the Gaussian smearing function in [1].
2. Review of the construction and representation theory of $\mathcal{HA}$.

We quickly review the relevant contents of [6, 8]. We refer the reader to [6, 8], especially appendix A2 of [8] for details.

The mathematical structures of interest are as follows.

$\mathcal{A}$ is the space of smooth $U(1)$ connections on the trivial $U(1)$ bundle on $R^3$. We restrict attention to connections $A_a(x)$ whose cartesian components are functions of rapid decrease at infinity.

$\mathcal{L}_{x_0}$ is the space of unparametrized, oriented, piecewise analytic loops on $R^3$ with basepoint $x_0$. Composition of a loop $\alpha$ with a loop $\beta$ is denoted by $\alpha \circ \beta$. Given a loop $\alpha \in \mathcal{L}_{x_0}$, the holonomy of $A_a(x)$ around $\alpha$ is

$$H_\alpha(A) := \exp(i \int_\alpha A_a dx^a).$$

$\tilde{\alpha}$ is the holonomy equivalence class (hoop class) of $\alpha$ i.e. $\alpha, \beta$ define the same hoop iff $H_\alpha(A) = H_\beta(A)$ for every $A_a(x) \in \mathcal{A}$.

$\mathcal{H}G$ is the group generated by all hoops $\tilde{\alpha}$, where group multiplication is hoop composition i.e. $\tilde{\alpha} \circ \tilde{\beta} := \tilde{\alpha \circ \beta}$.

$\mathcal{HA}$ is the abelian Poisson bracket algebra of $U(1)$ holonomies.

$\mathcal{FL}_{x_0}$ is the free algebra generated by elements of $\mathcal{L}_{x_0}$, with product law $\alpha \beta := \alpha \circ \beta$. With this product, all elements of $\mathcal{FL}_{x_0}$ are expressible as complex linear combinations of elements of $\mathcal{L}_{x_0}$.

$K$ is a 2 sided ideal of $\mathcal{FL}_{x_0}$, such that

$$\sum_{i=1}^N a_i \alpha_i \in K \text{ iff } \sum_{i=1}^N a_i H_{\alpha_i}(A) = 0 \text{ for every } A_a(x) \in \mathcal{A},$$

where $a_i$ are complex numbers.

$\mathcal{FL}_{x_0}$ is quotiented by $K$ to give the algebra $\mathcal{FL}_{x_0}/K$. The $K$ equivalence class of $\alpha$ is denoted by $[\alpha]$. As abstract algebras, $\mathcal{HA}$ and $\mathcal{FL}_{x_0}/K$ are isomorphic.

$$\left( \sum_{i=1}^N a_i [\alpha_i] \right)^* := \sum_{i=1}^N a_i [\alpha_i^{-1}]$$

$^2$Thus a minor change of notation from A2 of [8] is that we denote $\mathcal{A}_0$ of that reference by $\mathcal{A}$.

$^3$This is in contrast to the $C^1$ loops of A2 of [8].
defines a relation on $\mathcal{H}\mathcal{A}$. $||\sum_{i=1}^{N} a_{i}[\alpha_{i}]|| := \sup_{A \in \mathcal{A}} |\sum_{i=1}^{N} a_{i}H_{\alpha_{i}}(A)|$ (3)

defines a norm on $\mathcal{H}\mathcal{A}$. $\mathcal{H}\mathcal{A}$ is the abelian $C^{*}$ algebra obtained by defining $*$ on $\mathcal{H}\mathcal{A}$ and completing the resulting $*$ algebra with respect to $|| \cdot ||$.

$\Delta$ is the spectrum of $\mathcal{H}\mathcal{A}$. $\Delta$ is also denoted by $\mathcal{A}/\mathcal{G}$ where $\mathcal{G}$ denotes the $U(1)$ gauge group and is a suitable completion of the space of connections modulo gauge, $\mathcal{A}/\mathcal{G}$. From Gel’fand theory, $\Delta$ is the space of continuous, linear, multiplicative $*$ homeomorphisms, $h$, from $\mathcal{H}\mathcal{A}$ to the ($C^{*}$ algebra of) complex numbers $\mathbb{C}$. From [8] the elements of $\Delta$ are also in 1-1 correspondence with homeomorphisms from $\mathcal{H}\mathcal{G}$ to $U(1)$.

Given $X \in \mathcal{H}\mathcal{A}$, $h(X)$ is a complex function on $\Delta$. $\Delta$ is endowed with the weakest topology in which $h(X)$ for all $X \in \mathcal{H}\mathcal{A}$ are continuous functions on $\Delta$. In this topology, $\Delta$ is a compact, Hausdorff space and the functions $h(\{\alpha\})$, $\alpha \in \mathcal{L}_{x_{0}}$ are dense in the $C^{*}$ algebra, $C(\Delta)$, of continuous functions on $\Delta$. Further, $C(\Delta)$ is isomorphic to $\mathcal{H}\mathcal{A}$. Every continuous cyclic representation of $\mathcal{H}\mathcal{A}$ is in 1-1 correspondence with a continuous positive linear functional (PLF) on $\mathcal{H}\mathcal{A}$. Since $\mathcal{H}\mathcal{A} \cong C(\Delta)$, every continuous PLF so defined on $C(\Delta)$ is in correspondence, by the Riesz lemma, with some regular measure $d\mu$ on $\Delta$ and $\hat{H}_{\alpha}$ is represented on $\psi \in L^{2}(\Delta, d\mu)$ as unitary operator through $(\hat{H}_{\alpha}\psi)(h) = h(\{\alpha\})\psi(h)$.

In particular, the continuous ‘Haar’ PLF [8]

$$\Gamma(\alpha) = 1 \text{ if } \hat{\alpha} = \hat{\delta}$$
$$= 0 \text{ otherwise}$$ (4)

(where $\delta$ is the trivial loop), corresponds to the Haar measure on $\Delta$.

$\Delta = \mathcal{A}/\mathcal{G}$ can also be constructed as the projective limit space [9] of certain finite dimensional spaces. Each of these spaces is isomorphic to $n$ copies of $U(1)$ and is labelled by $n$ strongly independent hoops. Recall from [8] that $\hat{\alpha}_{i}, i = 1..n$ are strongly independent hoops iff $\alpha_{i} \in \mathcal{L}_{x_{0}}$ are strongly independent loops; $\alpha_{i}, i = 1..n$
are strongly independent loops iff each $\alpha_i$ has at least one segment which intersects $\alpha_{j\neq i}$ at most at a finite number of points. The Haar measure on $\Delta$ is the projective limit measure of the Haar measures on each of the finite dimensional spaces. Then the considerations of [10] show that the electric flux $\int_S E^a ds_a$ through a surface $S$ can be realised as an essentially self adjoint operator on the dense domain of cylindrical functions as

$$\int_S \hat{E}^a ds_a \psi_{\{[\alpha_i]\}} = e \sum_i N(S, \alpha_i) h([\alpha_i]) \frac{\partial \psi_{\{[\alpha_i]\}}}{\partial h([\alpha_i])}$$

where $N(S, \alpha_i)$ is the number of intersections between $\alpha_i$ and $S$.

3. $\overline{HA}_r$ and the isomorphism $I_r$

In section 3a we recall the definition of ‘smeared’ loops and their holonomies from [1] and construct the ‘smeared’ loop related structures $\tilde{\alpha}_r$, $K_r$, $HA_r$, $\overline{HA}_r$ and $\Delta_r$. In section 3b, using Appendix A2, we show that an isomorphism exists between the structures $\tilde{\alpha}$, $K$, $HA$, $\overline{HA}, \Delta$ and their ‘smeared’ versions.

3a. The construction of $\overline{HA}_r$

In the notation of [1],

$$H_\alpha(A) = \exp i \int_{R^3} X^\alpha_\gamma(\vec{x}) A_a(\vec{x}) d^3x,$$

$$X^\alpha_\gamma(\vec{x}) := \oint_{\gamma} ds^3(\vec{\gamma}(s), \vec{x}) \gamma^a,$$

where $s$ is a parametrization of the loop $\gamma$, $s \in [0, 2\pi]$. $X^\alpha_\gamma(\vec{x})$ is called the form factor of $\gamma$. Its Fourier transform is

$$X^\alpha_\gamma(\vec{k}) := \frac{1}{2\pi^2} \int_{R^3} d^3x X^\alpha_\gamma(\vec{x}) e^{-ik\cdot\vec{x}}$$

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4Note that the proof of continuity of the Haar PLF in [8] is incomplete in that it applies only if the loops $\alpha_j$ of A.7 of [8] are holonomically independent. Nevertheless, if as in this work, we restrict attention to piecewise analytic loops, continuity of the Haar PLF can immediately be inferred from its definition through the Haar measure.

5Cylindrical functions on $\Delta$ are of the form $\psi_{\{[\alpha_i]\}} := \psi(h([\alpha_1]), \ldots, h([\alpha_n]))$, where $\alpha_i, i = 1..n$, are a finite number of strongly independent loops and $\psi$ is a complex function on $U(1)^n$.agli 5
The Gaussian smeared form factor \cite{1} is defined as

\[ X^a_{\gamma(r)}(\bar{x}) := \int_{R^3} d^3 y f_r(\bar{y} - \bar{x}) X^a_{\gamma}(\bar{y}) = \int d \gamma \gamma(s) e^{-i \bar{k} \cdot \gamma(s)} =: \int d \gamma \gamma(s) e^{-i \bar{k} \cdot \gamma(s)} \] (8)

where

\[ f_r(\bar{x}) = \frac{1}{2\pi^{3/2}} e^{-\frac{x^2}{2}} e^{-i \bar{k} \cdot \bar{x}} \] (10)

approximates the Dirac delta function for small \( r \). The Fourier transform of the smeared form factor is

\[ X^a_{2\gamma(r)}(\bar{k}) = \frac{1}{2\pi^{3/2}} X^a_{\gamma}(\bar{k}) \] (11)

and the smeared holonomy is defined as

\[ H_{\gamma(r)}(A) = \exp i \int_{R^3} X^a_{\gamma(r)}(\bar{x}) A_a(\bar{x}) d^3 x = \exp i \int_{R^3} X^a_{\gamma(r)}(\bar{x}) A_a(\bar{x}) d^3 k. \] (12)

where \( A_a(\bar{k}) \) is the Fourier transform of \( A_a(\bar{x}) \).

We define \( \tilde{\alpha}_r, K_r, H\mathcal{A}_r, \Gamma\mathcal{A}_r, \Delta_r \) as follows.

\( \tilde{\alpha}_r \) is the \( r \)-hoop class of \( \alpha \) i.e. \( \alpha, \beta \) define the same \( r \)-hoop iff \( H_{\alpha(r)}(A) = H_{\beta(r)}(A) \) for every \( A_a(x) \in \mathcal{A} \). \( \mathcal{H}\mathcal{G}_r \) is the group generated by all \( r \)-hoops \( \tilde{\alpha}_r \) where group multiplication is \( r \)-hoop composition i.e.

\[ \tilde{\alpha}_r \circ \tilde{\beta}_r := (\tilde{\alpha} \circ \tilde{\beta})_r. \] (13)

Note that the above definition is consistent because, from (12) and the definition of \( r \)-hoop equivalence, it follows that

\[ H_{\alpha(r)}(A) H_{\beta(r)}(A) = H_{(\alpha, \beta)(r)}(A) \] (14)

Note that from (13), it follows that the identity element of \( \mathcal{H}\mathcal{G}_r \) is \( \tilde{\alpha}_r \) and that \( (\tilde{\alpha}_r)^{-1} = \tilde{\alpha}^{-1}_r \).

\( \mathcal{H}\mathcal{A}_r \) is the abelian Poisson bracket algebra of the \( r \)-holonomies, \( H_{\alpha(r)}(A), A_a \in \mathcal{A}, \alpha \in L_{x_0} \).
Recall that with the product law defined in section 2, all elements of $\mathcal{F}L_{x_0}$ are expressible as complex linear combinations of elements of $L_{x_0}$. We define the 2 sided ideal of $K_r \in \mathcal{F}L_{x_0}$, through

$$\sum_{i=1}^{N} a_i \alpha_i \in K_r \text{ iff } \sum_{i=1}^{N} a_i H_{\alpha_i(r)}(A) = 0 \text{ for every } A_a(x) \in \mathcal{A},$$

where $a_i$ are complex numbers. The $K_r$ equivalence class of $\alpha$ is denoted by $[\alpha]_r$. It can be seen that, as abstract algebras, $\mathcal{H}A_r$ and $\mathcal{F}L_{x_0}/K_r$ are isomorphic.

It can be checked that the relation $*$, defined on $\mathcal{H}A_r$ by

$$\left(\sum_{i=1}^{N} a_i [\alpha_i]_r\right)^* := \sum_{i=1}^{N} a_i^* [\alpha_i^{-1}]_r,$$

is a $*$ relation. Note that from (12), the complex conjugate of $H_{\alpha(r)}(A)$ is $H_{\alpha^{-1}(r)}(A)$ and hence the abstract $*$ relation just encodes the operation of complex conjugation on the algebra $\mathcal{H}A_r$.

Next we define the norm $|| \cdot ||_r$ as

$$||\sum_{i=1}^{N} a_i [\alpha_i]_r||_r := \sup_{A \in \mathcal{A}} |\sum_{i=1}^{N} a_i H_{\alpha_i(r)}(A)|.$$

It is easily verified that $|| \cdot ||_r$ is indeed a norm on the $*$ algebra $\mathcal{H}A_r$ with $*$ relation defined by (16). Completion of $\mathcal{H}A_r$ with respect to $|| \cdot ||_r$ gives the abelian $C^*$ algebra $\overline{\mathcal{H}A}_r$.

Next, we characterize the spectrum $\Delta_r$ of $\overline{\mathcal{H}A}_r$ as the space of all homomorphisms from $\mathcal{H}G_r$ to $U(1)$.

Let $h \in \Delta_r$. Thus $h$ is a linear, multiplicative, continuous $*$ homorphism from $\overline{\mathcal{H}A}_r$ to $\mathbb{C}$.

$$\Rightarrow h([\alpha]_r)h([\alpha^{-1}]_r) = h([\alpha]_r).$$

Choosing $\alpha = \varpi$, $\Rightarrow h([\varpi]_r)^2 = h([\varpi]_r) \Rightarrow h([\varpi]_r) = 1.$

$$\Rightarrow h([\alpha^{-1}]_r) = \frac{1}{h([\alpha]_r)} = h^*([\alpha]_r).$$
implies that $|h(\alpha)| = 1$ and this, coupled with the fact that $H^r$ is commutative, shows that every $h \in \Delta_r$ defines a homomorphism from $H^r$ to $U(1)$.

Conversely, let $h$ be a homomorphism from $H^r$ to $U(1)$. Its action can be extended by linearity to elements of $H^A$, so that $h(\sum_{i=1}^{N} a_i(\alpha)_r) := \sum_{i=1}^{N} a_i h(\alpha)_r$. It is also easy to see that $\alpha(\alpha^{-1}) = h^*(\alpha)$. These properties and the fact that $h$ is a homomorphism from $H^r$ to $U(1) \subset C$, imply that $h$ is a linear, multiplicative, * homomorphism from $H^A$ to $C$.

Finally we show that $h$ extends to a continuous homomorphism on $H^A$. From [6] it follows that for $\alpha_i \in L_{x_0}$, $i = 1..n$, there exist strongly independent $\beta_j$, $j = 1..m$ such that each $\alpha_i$ is the composition of some of the $\{\beta_j\}$. From this fact and Lemma 2 of Appendix A1, it can be shown that, for a given $\sum_{i=1}^{N} a_i(\alpha)_r \in H^A$ and any $\delta > 0$, there exists $\mathbb{A}^a(\alpha, \delta, r) \in A$ such that

$$| \sum_{i=1}^{N} a_i(h(\alpha)_r - H_{\alpha_i(r)}(A^{(a, \delta, r)}))| < \delta. \quad (21)$$

From (21), it is straightforward to show that

$$| \sum_{i=1}^{N} a_i h(\alpha)_r | \leq \sup_{A \in A} | \sum_{i=1}^{N} a_i H_{\alpha_i(r)}(A)|. = || \sum_{i=1}^{N} a_i(\alpha)_r ||_r. \quad (22)$$

Since $H^A$ is dense in $H^A$, (22) implies that $h$ can be extended to a continuous (linear, multiplicative) homomorphism from $H^A$ to $C$.

Thus $\Delta_r$ can be identified with the set of all homomorphisms from $H^r$ to $U(1)$.

### 3b. The isomorphism $I_r$

We show that

(i) $K = K_r$ : Let

$$\sum_{i=1}^{N} a_i H_{\alpha_i}(A) = 0 \text{ for every } A_a(x) \in A. \quad (23)$$

$h$ can be defined on $[\alpha]_r$ because $K_r$ equivalence subsumes $r$-hoop equivalence.
From Lemma 3 of Appendix A1, given \(A_a \in \mA\), there exists \(A_{\alpha(r)} \in \mA\) such that

\[
\sum_{i=1}^{N} a_i H_{\alpha_i(r)}(A) = \sum_{i=1}^{N} a_i H_{\alpha_i}(A_{\alpha(r)}).
\]  

(24)

(23) and (24) imply that \(K \subset K_r\).

Let

\[
\sum_{i=1}^{N} a_i H_{\alpha_i}(A) = 0 \text{ for every } A_a(x) \in \mA.
\]  

(25)

\(\Rightarrow\) Given \(A_a, B_a \in \mA\),

\[
|\sum_{i=1}^{N} a_i H_{\alpha_i}(A)| = |\sum_{i=1}^{N} a_i H_{\alpha_i}(A) - \sum_{i=1}^{N} a_i H_{\alpha_i}(B)|.
\]  

(26)

Choose, \(B_a = A_a^{\alpha}\) where \(A_a^{\alpha}\) is defined in Lemma 1, A1.

Then

\[
|\sum_{i=1}^{N} a_i H_{\alpha_i}(A)| \leq \sum_{i=1}^{N} |a_i| \varepsilon \text{ for every } \varepsilon > 0. \Rightarrow \sum_{i=1}^{N} a_i H_{\alpha_i}(A) = 0 \text{ and hence, } K_r \subset K.
\]

Thus \(K = K_r\), \([\alpha] = [\alpha]_r\) and \(\bar{\alpha} = \bar{\alpha}_r\)

(ii) \(|\sum_{i=1}^{N} a_i [\alpha_i]| = || \sum_{i=1}^{N} a_i [\alpha_i]||_r\):

Let \(|| \sum_{i=1}^{N} a_i [\alpha_i]||_r = c_r\). Then \(c_r \geq |\sum_{i=1}^{N} a_i H_{\alpha_i(r)}(A)|\) for every \(A_a \in \mA\). Further, for every \(\tau > 0\) there exists \(^{(r)}A_a \in \mA\) such that \(c_r - |\sum_{i=1}^{N} a_i H_{\alpha_i(r)}(^{(r)}A)| \leq \tau\). Then, from Lemma 3, A1, there exists \(^{(r)}A_a^{(r)} \in \mA\) such that

\[
0 \leq c_r - \sum_{i=1}^{N} a_i H_{\alpha_i}^{(r)} A_{\alpha_i(r)} \leq \tau.
\]  

(27)

\[
\Rightarrow \sup_{A \in \mA} \left|\sum_{i=1}^{N} a_i H_{\alpha_i}(A)\right| \geq c_r \Rightarrow \left|\sum_{i=1}^{N} a_i [\alpha_i]\right|_r \geq \left|\sum_{i=1}^{N} a_i [\alpha_i]_r\right|_r
\]  

(28)

Let \(|| \sum_{i=1}^{N} a_i [\alpha_i]|| = c\). Then for every \(\tau > 0\) there exists \(^{(\tilde{r})}A_a \in \mA\) such that

\[
c - |\sum_{i=1}^{N} a_i H_{\alpha_i}^{(\tilde{r})} A| \leq \frac{\tau}{2}.
\]  

(29)
From Lemma 1 A1, there exists $(\tilde{T})A_a^i \in A$ such that

$$|H_{\alpha_i(r)}((\tilde{T})A^r) - H_{\alpha_i}((\tilde{T})A)| \leq \epsilon.$$  

$$\Rightarrow |\sum_{i=1}^N a_i(H_{\alpha_i(r)}((\tilde{T})A^r) - H_{\alpha_i}((\tilde{T})A))| \leq \sum_{i=1}^N |a_i|\epsilon. \quad (30)$$

Choose $0 < \epsilon < \frac{\tau}{2\sum_{i=1}^N |a_i|}$. From (29) and (30) it follows that, for every $\tau > 0$,

$$c - |\sum_{i=1}^N a_i H_{\alpha_i(r)}((\tilde{T})A^r)| < \tau. \quad (31)$$

$$\Rightarrow c \leq \sup_{A \in A} |\sum_{i=1}^N a_i H_{\alpha_i(r)}(A)| \Rightarrow \|\sum_{i=1}^N a_i[\alpha_i]\| \leq \|\sum_{i=1}^N a_i[\alpha_i]_r\|. \quad (32)$$

Thus, for any finite $N$, $\|\sum_{i=1}^N a_i[\alpha_i]\| = \|\sum_{i=1}^N a_i[\alpha_i]_r\|$

From (i) and (ii) it follows that the structures $K_r, [\alpha_r], \tilde{\alpha}_r, \mathcal{HG}_r, *, \mathcal{HA}_r, \overline{\mathcal{HA}}_r, \Delta_r$ are isomorphic to $K, [\alpha], \tilde{\alpha}, \mathcal{HG}, *, \mathcal{HA}, \overline{\mathcal{HA}}, \Delta$.

Thus a $C^*$ isomorphism $I_r : \overline{\mathcal{HA}} \to \overline{\mathcal{HA}}_r$ exists such that

$$I_r(\sum_{i=1}^N a_i[\alpha_i]) = \sum_{i=1}^N a_i[\alpha_i]_r. \quad (33)$$

$I_r$ defines a natural 1-1 map from $\Delta$ to $\Delta_r$ (which we shall also call $I_r$). Given the $*$ isomorphism $h \in \Delta$, from $\overline{\mathcal{HA}}$ to $C$, its image is $h_r \in \Delta_r$ where

$$h_r(\sum_{i=1}^N a_i[\alpha_i]_r) := h(\sum_{i=1}^N a_i[\alpha_i]_r). \quad (34)$$

Note that if $h_r$ is defined by some smooth, non-flat $A_a \in A$ then it is not true that $h$ is associated with (the gauge equivalence class of) the same connection.

4. **Fock representation from $\overline{\mathcal{HA}}_r$**

The standard Fock space vacuum expectation value restricted to $\mathcal{HA}_r$ defines the Fock PLF on $\mathcal{HA}_r$ as

$$\Gamma_F(\sum_{i=1}^N a_i[\alpha_i]_r) := \sum_{i=1}^N a_i \exp -\left(\int \frac{d^3k}{k} |X_{\alpha(r)}^a(k')|^2 \right). \quad (35)$$
Since $\mathcal{H}_\mathcal{A}_r$ is a proper subalgebra of the standard Weyl algebra for $U(1)$ theory, it is not clear that its quantization (through the GNS construction based on the Fock PLF) reproduces the full Fock space. We prove that the full Fock space is indeed obtained.

Let the GNS Hilbert space (based on $\Gamma F$) be $\mathcal{H}$. Let $D$ be the linear subspace of $\mathcal{H}$ spanned by elements of the form $^H_\mathcal{R}(r)\Omega$ where $\Omega$ is the GNS vacuum. It can be seen that $D$ is dense in $\mathcal{H}$. $D$ is naturally embedded in the Fock space, $F$, through the map $U : D \to F$ defined by

$$U(\Omega) = |0> \quad \text{and} \quad U(\sum_{i=1}^N a_i \hat{H}_{\alpha_{(r)}} \Omega) = \sum_{i=1}^N a_i \exp i \int_{\mathbb{R}^3} X_{\alpha_{(r)}}(x) \hat{A}_a(x) d^3x |0>.$$ (36)

Here $\hat{A}_a$ is the standard Fock space operator valued distribution at $t = 0$

$$\hat{A}_a(x) = \frac{1}{2\pi^2} \int \frac{d^3k}{\sqrt{k}} (e^{i\hat{x} \cdot \hat{k}} \hat{a}_a(k) + e^{-i\hat{x} \cdot \hat{k}} \hat{a}_a(k))$$ (37)

where

$$\hat{a}_a(k) k^a = 0, \quad [\hat{a}_a(k), \hat{a}_b^\dagger (\vec{l})] = \delta_{ab} \delta(k, \vec{l}).$$ (38)

By construction, $U$ is a unitary map and can be uniquely extended to $\mathcal{H}$ so that it embeds $\mathcal{H}$ in $\mathcal{F}$. We show that Cauchy limits of states in $U(D)$ span a dense set in $\mathcal{F}$ - this suffices to show that the entire Fock space is indeed obtained, i.e. that $U(\mathcal{H}) = \mathcal{F}$.

Define the ‘occupation number’ states

$$|\phi, p> := \int d^3k_1..d^3k_p \phi^{a_1..a_p} (\vec{k}_1..\vec{k}_p) \hat{a}_{a_1}^\dagger (\vec{k}_1) .. \hat{a}_{a_p}^\dagger (\vec{k}_p) |0>.$$ (39)

$\phi^{a_1..a_p} (\vec{k}_1..\vec{k}_p)$ (with $p$ a positive integer) is such that

(a) $\int d^3k_i |\phi^{a_1..a_p} (\vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_p)|^2 < \infty$ and $\phi^{a_1..a_p} (\vec{k}_1..\vec{k}_p)$ falls of faster than any inverse power of $k_i$ as $k_i \to \infty$, $\vec{k}_j \neq \vec{k}_i$ fixed.

(b) $\phi^{a_1..a_p} (\vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_p) (k_i)_{a_i} = 0$ i.e. it is transverse.

(c) it is symmetric under interchange of $(a_i, \vec{k}_i)$ with $(a_j, \vec{k}_j)$ for all $i, j = 1..p$.

$|\phi, p> >$ for all $p$ together with $|0>$, span a dense set, $D_0 \subset \mathcal{F}$. 

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Given 2 vectors \( \vec{x}, \vec{v} \), define the operator

\[
\hat{O}_{\vec{x},\vec{v}} := \frac{i}{2\pi} \int \frac{d^3k}{\sqrt{2k}} e^{ik\cdot\vec{x}} e^{-k^2/2} (\vec{v} \times \vec{k})^a (\hat{a}_a(\vec{k}) + \hat{a}^+_a(-\vec{k})).
\] (40)

As argued in Appendix A2, states of the form \( |\psi_{\{\vec{x},\vec{v}\}}\rangle := \prod_{i=1}^p \hat{O}_{(\vec{x}_i,\vec{v}_i)}|0\rangle, p = 1, 2.. \) together with \( |0\rangle \) span \( \mathcal{D}_0 \).

Our proof that \( \psi_{\{\vec{x},\vec{v}\}} \in U(\mathcal{H}) \) is as follows.

(i) Note that \( \psi_{\{\vec{x},\vec{v}\}} \in \mathcal{D}_0 \).

(ii) Let \( \gamma^{(m,\vec{x},\vec{n})} \) be a circular loop of radius \( \epsilon_m := \frac{1}{2m} (m \text{ is a positive integer}), \) centred at \( \vec{x} \) and let its plane have unit normal \( \vec{n} \). \(^7\) The image of \( \hat{H}_{\gamma^{(m,\vec{x},\vec{n})}} \) on \( U(\mathcal{D}) \) is \( \exp(i \int X^a_{\gamma^{(m,\vec{x},\vec{n})}}(\vec{y}) \hat{A}_a(\vec{y})d^3y) \). Define

\[
\hat{O}_{\vec{x},\vec{n},m} := \frac{i \int X^a_{\gamma^{(m,\vec{x},\vec{n})}}(\vec{y}) \hat{A}_a(\vec{y})d^3y}{i\pi \epsilon_m^2} - 1.
\] (41)

The formal limit of \( \hat{O}_{\vec{x},\vec{n},m} \) as \( m \to \infty \) is \( \hat{O}_{\vec{x},\vec{n}} \). We show below that \( \hat{O}_{\vec{x},\vec{n},m}|\psi\rangle, |\psi\rangle \in \mathcal{D}_0, \) \(^8\) form a Cauchy sequence with limit \( \hat{O}_{\vec{x},\vec{n}}|\psi\rangle \). Then, choosing \( |\psi\rangle = |0\rangle \), we see that \( \hat{O}_{\vec{x},\vec{n}}|0\rangle \) is in the completion of \( U(\mathcal{D}) \).

(iii) From (i) above, \( \hat{O}_{\vec{x},\vec{n}}|0\rangle \in \mathcal{D}_0 \). We can repeat the argument in (ii) above to conclude that \( \hat{O}_{\vec{x},\vec{n}} \hat{O}_{\vec{x},\vec{n},m} |0\rangle \) is obtained as the Cauchy limit of the states \( \hat{O}_{\vec{x},\vec{n},m} \hat{O}_{\vec{x},\vec{n},l} |0\rangle \). Iterating this argument we see that \( |\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle, |\vec{n}_i| = 1 \) is in the completion of \( U(\mathcal{D}) \). Finally, set \( \vec{v}_i := v_i \vec{n}_i \), where \( v_i \) are real numbers. Then it follows that \( |\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle = (\prod_{i=1}^p v_i) |\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle > \) and hence that \( |\psi_{\{\vec{x}_i,\vec{n}_i\}}\rangle \in U(\mathcal{H}) \).

Thus, it remains to show (see (ii) above) that:

Given \( \psi \in \mathcal{D}_0, \hat{O}_{\vec{x},\vec{n}} \) as defined in (40) and \( \hat{O}_{\vec{x},\vec{n},m} \) as defined in (41),

\[
\lim_{m \to \infty} ||\hat{O}_{\vec{x},\vec{n}} - \hat{O}_{\vec{x},\vec{n},m}|\psi\rangle|| = 0
\] (42)

\(^7\)Although \( \gamma^{(m,\vec{x},\vec{n})} \) is not in \( \mathcal{L}_{x_0} \), the loop formed by joining \( \gamma^{(m,\vec{x},\vec{n})} \) to the base point \( x_0 \) and retracing, is. We shall continue to denote this loop, which represents the same hoop, by \( \gamma^{(m,\vec{x},\vec{n})} \).

\(^8\)Note that since \( \hat{O}_{\vec{x},\vec{n},m} \) are bounded operators defined on the entire Fock space, \( \hat{O}_{\vec{x},\vec{n},m} \) are well defined on \( \mathcal{D}_0 \).
Proof: Let
\[
\hat{D} := \int X_{\gamma(r)}^a \, \hat{A} \, d^3 y - \pi \epsilon_m^2 \, \hat{O}_{\vec{x}, \vec{n}}.
\] (43)

Thus
\[
\hat{O}_{\vec{x}, \vec{n}, m} = \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} + i \hat{D}} - 1}{i \pi \epsilon_m^2}.
\] (44)

Since both \(e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}}}\) and \(e^{i \hat{D}}\) are commuting elements of the standard Weyl algebra,
\[
\hat{O}_{\vec{x}, \vec{n}, m} = \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} e^{i \hat{D}} - 1}}{i \pi \epsilon_m^2}.
\] (45)

\[\Rightarrow \| \hat{O}_{\vec{x}, \vec{n}} - \hat{O}_{\vec{x}, \vec{n}, m} \psi \| \]
\[
= \| \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} (e^{i \hat{D}} - 1)}}{i \pi \epsilon_m^2} \psi > + \| \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} - 1}}{i \pi \epsilon_m^2} - \hat{O}_{\vec{x}, \vec{n}, m} \| \psi > \|
\]
\[
\leq \| \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} - 1}}{i \pi \epsilon_m^2} - \hat{O}_{\vec{x}, \vec{n}, m} \| \psi > \| + \| \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} (e^{i \hat{D}} - 1)}}{i \pi \epsilon_m^2} \| \psi > \|
\] (46)

From Lemma 2 of Appendix A2, \(\hat{O}_{\vec{x}, \vec{n}}\) is a densely defined symmetric operator on \(D_0\) and admits self adjoint extensions. Hence, from [11], the first term in (46) vanishes in the \(\epsilon_m \to 0\) limit. Further, since \(e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}}}\) is a unitary operator, we have
\[
\| \frac{e^{i \pi \epsilon_m^2 \hat{O}_{\vec{x}, \vec{n}} (e^{i \hat{D}} - 1)}}{i \pi \epsilon_m^2} \| \psi > \| = \| \frac{(e^{i \hat{D}} - 1)}{i \pi \epsilon_m^2} \| \psi > \|.
\] (47)

But
\[
\| \frac{(e^{i \hat{D}} - 1)}{i \pi \epsilon_m^2} \| \psi > \|^2 = - \left( < \psi | \frac{(e^{i \hat{D}} - 1)}{i \pi \epsilon_m^2} \| \psi > + < \psi | \frac{e^{-i \hat{D}} - 1}{i \pi \epsilon_m^2} \| \psi > \right).
\] (48)

From Lemma 3, A2 and (48), \(\| \frac{(e^{i \hat{D}} - 1)}{i \pi \epsilon_m^2} \| \psi > \| \to 0\) as \(\epsilon_m \to 0\) and then (46) implies (42).
Thus we have shown above that the GNS representation of $\mathcal{H}_\mathcal{A}$ on the GNS Hilbert space $\mathcal{H}$, is unitarily equivalent to the standard Fock representation on $\mathcal{F} = L^2(S',d\mu_G)$ ($S'$ denotes the appropriate space of tempered distributions and $\mu_G$ is the standard Gaussian measure with covariance $\frac{1}{2}(-\nabla^2)^{-\frac{1}{2}}$ [14]) via the unitary map $U$.

The action of the smeared electric field operator, $\hat{E}(\tilde{f}) := \int d^3x f_a(\bar{x})\hat{E}^a(\bar{x})$, on $\psi \in \mathcal{C}_F \subset L^2(S',d\mu_G)$ is written in the standard way [14] as

$$\hat{E}(\tilde{f})\psi = e \int d^3x (-i f_a(\bar{x})\frac{\delta}{\delta A_a(\bar{x})} + i((-\nabla^2)^{\frac{1}{2}} f^a)(\bar{x})A_a(\bar{x}))\psi.$$  \hspace{1cm} (49)

Here $f_a(\bar{x})$ is real, divergence free, smooth and of rapid decrease, and $\mathcal{C}_F \subset L^2(S',d\mu_G)$ is the standard dense domain of cylindrical functions appropriate to Fock space. The smeared electric field operator on $\mathcal{H}$ is defined as the unitary image of $\hat{E}(\tilde{f})$ by $U^{-1}$ i.e. for $\psi \in U^{-1}(\mathcal{C}_F) \subset \mathcal{H}$

$$\hat{E}(\tilde{f})\psi = U^{-1}e \int d^3x (-i f_a(\bar{x})\frac{\delta}{\delta A_a(\bar{x})} + i((-\nabla^2)^{\frac{1}{2}} f^a)(\bar{x})A_a(\bar{x}))U\psi.$$  \hspace{1cm} (50)

With this action, $\hat{E}(\tilde{f})$ is densely defined on the dense domain $U^{-1}(\mathcal{C}_F) \subset \mathcal{H}$, and just like its unitary image on $\mathcal{C}_F$, admits a unique self adjoint extension.

5. Induced representations through $I_r$.

It can be verified that $I_r$ is a topological homomorphism from $\Delta$ to $\Delta_r$ (where $\Delta$ and $\Delta_r$ are equipped with their Gel’fand topologies). Hence, $I_r$ defines a measurable isomorphism $I_r : \mathcal{B} \rightarrow \mathcal{B}_r$ where $\mathcal{B}$ and $\mathcal{B}_r$ are the Borel sigma algebras associated with $\Delta$ and $\Delta_r$ respectively. Any regular Borel measure $\mu$ on $\Delta$ induces a regular Borel measure $\mu_r$ on $\Delta_r$, with $\mu_r := \mu I^{-1}_r$. It follows that $I_r$ defines a unitary map $U_r$ from $L^2(\Delta, d\mu)$ to $L^2(\Delta_r, d\mu_r)$.

$U_r$ can be explicitly defined through its action on the dense set $\mathcal{C} \in L^2(\Delta, d\mu)$, of cylindrical functions (cylindrical functions in the context of $\mathcal{H}_\mathcal{A}$ have been defined in section 2). Denote the dense set of cylindrical functions in $L^2(\Delta_r, d\mu_r)$ by $\mathcal{C}_r$.  

\footnote{Cylindrical functions are of the form $\psi_{([\alpha],r)}(h) := \psi(h([\alpha_1],..h([\alpha_n],r)))$, for $\alpha_i \in \mathcal{L}_{x_0}, i = 1..n$, $h \in \Delta_r$, and they span $\mathcal{C}_r$.}
maps $\mathcal{C}$ to $C_r$ through

$$U_r(\psi_{\{a_i\}}) = \psi_{\{a_i\}_r}.$$

(51)

It also follows that

$$U_r \hat{H}_\alpha U_r^{-1} = \hat{H}_{\alpha(r)}.$$

(52)

Thus $I_r$ induces a representation of $\overline{\mathcal{H}A_r}$ from a representation of $\overline{\mathcal{H}A}$. In section 5a, we induce a Haar like representation of $\overline{\mathcal{H}A_r}$ from the Haar representation of $\overline{\mathcal{H}A}$.

Since the image of $I_r$ restricted to $\mathcal{H}A$ is $\mathcal{H}A_r$, $I_r$ (or $I_r^{-1}$) can also be used to induce representations of $\mathcal{H}A$ from those of $\mathcal{H}A_r$ and vice versa. In section 5b, we induce a Fock like representation of $\mathcal{H}A$ from the Fock representation of $\mathcal{H}A_r$. The elements of $\mathcal{H}A_r$ define a dense subspace of $\mathcal{H}$ through the GNS construction and a map $U_r$ is defined through (51) and (52). $U_r^{-1}$ induces a Fock like representation of $\mathcal{H}A$.

5a. Haar representation of $\overline{\mathcal{H}A_r}$

We denote both the Haar measure on $\Delta$ as well as its image on $\Delta_r$ by $d\mu_H$. The induced PLF (corresponding to $d\mu_H$) on $\overline{\mathcal{H}A_r}$ is defined by

$$\Gamma(\alpha) = \begin{cases} 
1 & \text{if } \bar{\alpha}_r = \bar{\alpha}_r \\
0 & \text{otherwise.}
\end{cases}$$

(53)

From (51) and (52) it follows that $\hat{H}_{\alpha(r)}$ are represented by unitary operators on $\psi \in L^2(\Delta_r, d\mu_H)$ by

$$(\hat{H}_{\alpha(r)} \psi)(h) = h(\{a\}_r)\psi(h), \ h \in \Delta_r.$$

(54)

We construct electric field operators on $L^2(\Delta_r, d\mu_H)$ as unitary images of appropriate electric field operators on $L^2(\Delta, d\mu_H)$ as follows. Define the classical Gaussian smeared electric field as

$$E^a_r(\vec{x}) := \int d^3 y f_r(y - \vec{x}) E^a(y)$$

(55)
where \( f_r \) has been defined in section 3. Given \( \psi_{([\alpha],r)} \in \mathcal{C} \subset L^2(\Delta, d\mu_H) \) it can be checked that

\[
(\hat{E}_r^a(\vec{x})\psi_{([\alpha],r)})(h) = e^\sum_{i=1}^n X^a_{\alpha_i(r)}(\vec{x})h([\alpha_i])\frac{\partial \psi_{([\alpha],r)}(h)}{\partial h([\alpha_i])}.
\] (56)

The methods of [10] can be used to show that \( \hat{E}_r^a(\vec{x}) \) is essentially self adjoint on \( \mathcal{C} \). Note that (56) implies that

\[
[\hat{E}_r^a(\vec{x}), \hat{H}_\alpha] = eX^a_{\alpha_i(r)}(\vec{x})\hat{H}_\alpha.
\] (57)

The unitary image of (57) is

\[
[U_r\hat{E}_r^a(\vec{x})U_r^{-1}, \hat{H}_{\alpha(r)}] = eX^a_{\alpha_i(r)}(\vec{x})\hat{H}_{\alpha(r)}.
\] (58)

Denote the classical counterpart of \( U_r\hat{E}_r^a(\vec{x})U_r^{-1} \) by \( F^a(\vec{E}) \). Then (58) provides a quantum representation of the classical Poisson bracket,

\[
\{F^a(\vec{E}), H_{\alpha(r)}(A)\} = -ieX^a_{\alpha_i(r)}(\vec{x})H_{\alpha(r)}(A).
\] (59)

Note that \( \{E^a(\vec{x}), H_{\alpha(r)}(A)\} = -ieX^a_{\alpha_i(r)}(\vec{x})H_{\alpha(r)}(A) \). Hence, we can consistently identify \( F^a(\vec{E}) \) with \( E^a(\vec{x}) \). Thus, \( U_r\hat{E}_r^a(\vec{x})U_r^{-1} = \hat{E}_r^a(\vec{x}) \) and from (56),

\[
(\hat{E}_r^a(\vec{x})\psi_{([\alpha],r)})(h) = e^\sum_{i=1}^n X^a_{\alpha_i(r)}(\vec{x})h([\alpha_i])\frac{\partial \psi_{([\alpha],r)}(h)}{\partial h([\alpha_i])}.
\] (60)

Since \( U_r \) is unitary, \( \hat{E}_r^a(\vec{x}) \) is essentially self adjoint on \( \mathcal{C}_r \subset L^2(\Delta_r, d\mu_H) \).

To summarize: The induced Haar representation of \( \mathcal{H}_r \) provides a quantum representation of the classical Poisson bracket algebra of smeared holonomies \( H_{\alpha(r)}(A) \) and (divergence free) electric field \( E^a(\vec{x}) \). \( \hat{H}_{\alpha_r} \) are represented by unitary operators through (54) and the unsmeared electric field operator, \( \hat{E}_r^a(\vec{x}) \), is represented through (60) as an essentially self adjoint operator on the dense domain of cylindrical functions, \( \mathcal{C}_r \subset L^2(\Delta_r, d\mu_H) \). Note that \( \hat{E}_r^a(\vec{x}) \) is a genuine operator as opposed to an operator valued distribution!
5b. Fock representation of $\mathcal{HA}$

We denote the ‘Fock’ PLF on $\mathcal{HA}$, as well as its image on $\mathcal{HA}$ by $\Gamma_F$. Note that the induced PLF on $\mathcal{HA}$ is defined by

$$\Gamma_F(\sum_{i=1}^{N} a_i |a_i\rangle) := \sum_{i=1}^{N} a_i \exp \left( - \int \frac{d^3k}{k} |X_{\alpha(r)}^a (\vec{k})|^2 \right). \quad (61)$$

Since $\hat{H}_{\alpha(r)}$ are represented as unitary operators on $\mathcal{H}$, it follows that

$$\hat{H}_{\alpha} := U_r^{-1} \hat{H}_{\alpha(r)} U_r \quad (62)$$

are represented as unitary operators on $U_r^{-1}(\mathcal{H})$.

It remains to construct, following the strategy of section 5a, electric field operators on $U_r^{-1}(\mathcal{H})$ as unitary images of appropriate electric field operators on $\mathcal{H}$. On $U^{-1}(\mathcal{C}_F) \subset \mathcal{H}$,

$$[\hat{E}(\vec{f}), \hat{H}_{\alpha(r)}] = e \int d^3x f_a(\vec{x}) X_{\alpha(r)}^a (\vec{x}) \hat{H}_{\alpha(r)}. \quad (63)$$

$$\Rightarrow [U_r^{-1} \hat{E}(\vec{f}) U_r, \hat{H}_{\alpha}] = e \int d^3x f_a(\vec{x}) X_{\alpha(r)}^a (\vec{x}) \hat{H}_{\alpha}. \quad (64)$$

Define the classical function

$$E_r(\vec{f}) := \int d^3x f_a(\vec{x}) E_r^a(\vec{x}), \quad (65)$$

where $E_r^a(\vec{x})$ is defined by (55). Since

$$\{ E_r(\vec{f}), H_{\alpha}(A) \} = -ie \int d^3x f_a(\vec{x}) X_{\alpha(r)}^a (\vec{x}) H_{\alpha}(A), \quad (66)$$

we identify

$$\hat{E}_r^a(\vec{f}) := U_r^{-1} \hat{E}(\vec{f}) U_r. \quad (67)$$

To summarize: The induced Fock representation of $\mathcal{HA}$ provides a quantum representation of the classical Poisson bracket algebra of holonomies $H_{\alpha}(A)$ and “Gaussian-smeared, smeared” electric fields $E_r(\vec{f})$. The ‘unsmeared’ holonomy operators $\hat{H}_{\alpha}$ are represented by unitary operators through (62) and $\hat{E}(\vec{f})$ is represented as a self adjoint operator through (67). Note that the “Gaussian smeared object” $\hat{E}_r(\vec{x})$ is represented as an operator valued distribution (as opposed to a genuine operator) on $U_r^{-1}(\mathcal{H})$. 

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6. Discussion

Preliminary remarks: In this paper, representations of the Poisson algebra of $U(1)$ theory were constructed in 2 steps. First, Hilbert space representations of the abelian Poisson algebra of configuration functions (i.e. functions of $A_a$) were constructed by specifying a PLF. Second, real functions of the conjugate electric field were represented by self adjoint operators on this Hilbert space. The Haar representation of $\mathcal{HA}$ and its image on $\mathcal{HA}_r$ support a representation of the electric field wherein, formally,

$$\hat{E}^a(\vec{x}) = -i\frac{\delta}{\delta A^a(\vec{x})}.$$  \hfill (68)

This action is not connected with Poincare invariance and it is not surprising that the resulting representations of section 2 and 5a, are non Fock representations. On the Fock representation of $\mathcal{H}_A$ \footnote{We remind the reader that we displayed a fairly rigorous argument that the entire Fock space is obtained in such a representation through the constructions of section 4 and Appendix A2. We reiterate our belief that the formal equation (89) can be rendered mathematically well defined in a more careful treatement.} and its image on $\mathcal{H}_A$, equation (68) is incompatible with the requirement of self adjointness of the electric field operators. Their action necessarily contains a term dependent on the Gaussian measure (see (50)) to ensure self adjointness. The choice of Gaussian measure is intimately associated with the properties of the de Alembertian, $\frac{\partial^2}{\partial t^2} - \nabla^2$, and hence with Poincare invariance.

A rephrasing of the above remarks which brings them closer to the strategy of [1, 3] is as follows. Given a representation in which (smeared or unsmeared) holonomies are represented by multiplication by unitary operators and the electric field acts, as in (68), purely by functional differentiation, the requirement of self adjointness of the electric field operator determines the Hilbert space measure to be the Haar measure. The self adjointness of electric field operators results in the Gaussian measure only if their action has a contribution dependent on the Gaussian measure. Thus, to obtain the standard Fock representation or the induced one of section 5b, the Gaussian measure and hence, Poincare invariance, plays an essential and explicit role.

Note that this work concerns the ‘connection’ representation of a theory of a real $U(1)$ connection. In contrast [1] constructs the loop representation of a description
of linearised gravity based on a \textit{self dual} connection\textsuperscript{11}. Despite these differences, there is also a certain amount of shared mathematical structure in our work and [1]. Therefore, the delineation of the structures involved in the construction of $U(1)$ theory as spelt out in this paper, allows us to identify the role of the key structures in [1].

\textit{Discussion of (I) and (II):} We use the notation of [1] and [3] when discussing those papers. We first discuss (I). In [1] the action of the linearised metric variable in the loop representation is deduced, ultimately, from its action of the form $-i\frac{\delta}{\delta A_0}$ in the connection representation. The loop representation then becomes an ‘electric field’ type representation in which the magnetic field operator acts purely by functional differentiation with respect to the loop form factor. Yet a Fock representation (of the positive and negative helicity gravitons) results in apparent contradiction to our claims that such a representation cannot result without using Poincare invariance explicitly.

The resolution of this apparent contradiction for the positive helicity graviton sector seems to lie, in what appears at first sight, to be a mere mathematical nicety. In [1] the Gaussian measure contribution to the $\hat{B}^+$ operator is absorbed (and hidden) in the rescaling of the wave function. Such a rescaling is permissible for finite dimensional systems but results in a mathematically ill defined ‘measure’ for the field theory in question. In spite of the fact that for most applications this formal treatment suffices, it is crucial to realise, in the context of (I), that it hides the role of Poincare invariance in constructing the Fock representation. To obtain a well defined (Gaussian) measure, the wave functions ((101) of [1]) need to be rescaled and a Gaussian measure term needs to be added to the action of the $\hat{B}^+$ - this, of course, feeds explicit Poincare invariance back into the construction.

Note that this argument does not apply to the negative helicity sector. There, the choice of self dual connection results in the negative helicity magnetic field operator, $\hat{B}^-$ being the same as the negative helicity annihilation operator. $\hat{B}^-$ is naturally represented as a functional derivative ((70) of [1]) and, in this aspect, matches the

\textsuperscript{11}Note, however that the description reduces to one in terms of a triplet of abelian connections.
standard Fock representation of the annihilation operator as a pure functional derivative term. The resulting representation is the Fock representation for negative helicity gravitons and indeed, for this sector, it seems that explicit Poincare invariance is not invoked.

Thus the Fock representation of linearised gravity seems to result partly due to explicit Poincare invariance (which is suppressed in [1] by a mathematically ill defined operation) and partly due to the use of self dual connections.

Considerations similar to those for the positive helicity gravitons also apply to the treatment of free Maxwell theory in [3]. There, it is shown that the extended loop representation coincides, formally, with the electric field representation. Again, an ill defined measure is used and a proper mathematical treatment restores the explicit role of Poincare invariance.

We turn now, to a discussion of (II). In the loop representation of [1] the two sets of important operators are the magnetic field, $\hat{B}^\pm$, and the linearised metric, $\hat{\mathcal{h}}^\pm$. They are represented by functional differentiation and multiplication, on the representation space of functionals of loop form factors. This representation space supports the holonomies as operators. Indeed, the action of the $\hat{B}^\pm$ operators is deduced from the fact that the classical magnetic flux is the lowest non trivial term in the expansion of the holonomy of a small loop ((57) of [1]). However, all these constructions are rendered formal because of the distributional nature of the loop form factor and the resulting divergence of the ground state functional. Therefore a regularization procedure is adopted wherein the loop form factors are replaced by their Gaussian smeared, $r$- versions (see (9)) and $\hat{B}^\pm$, $\hat{\mathcal{h}}^\pm$ are represented as functional differentiation and multiplication operators on the space of functionals of $r$-loop form factors. An important question is: Are the holonomy operators or some regularized version thereof, represented on this space?

One may choose to ignore this question and simply postulate the action of $\hat{B}^\pm$, $\hat{\mathcal{h}}^\pm$ in terms of $r$-loop form factors. Then the primary configuration variables of the theory are $\mathcal{B}^\pm$ and the construction does not seem to have much to do with loops and holonomies. Since holonomies and loops are the primary objects in the loop approach
to full-blown quantum gravity, an interpretation of the regularization which allows for the representation of holonomy operators is of interest. It seems to us that such an interpretation must regard the $r$-form factor representation as an *approximation* to the standard Fock representation, which becomes better as $r \to 0$.

A precise formulation of such an interpretation in the context of $U(1)$ theory is provided by the induced Fock representation of section 5b. There, the holonomy operators are represented on the Hilbert space and the magnetic field operators can be constructed by a “shrinking of loop” limit, as the image of $U^{-1}\hat{O}_{\bar{x},\bar{n}}U$ via $U_r^{-1}$. That representation, although *not* the standard Fock representation, is a good approximation to it for small $r$. The nature of the approximation is as follows. For sufficiently small $r$, the holonomies $H_\gamma(A)$, the electric field $E^a(x)$ and their Gaussian smeared counterparts, $H_{\gamma(r)}(A)$, $E_r^a(x)$ approximate each other well. An approximate Fock representation can be constructed in which the operators corresponding to $H_\gamma(A)$, $E_r^a(x)$ act in the same way as the operators corresponding to $H_{\gamma(r)}(A)$, $E^a(x)$ in the standard Fock representation. This approximate Fock representation is the induced Fock representation of section 5b.

To summarize: The standard Fock representation for $U(1)$ theory is obtained only when the algebra of smeared holonomies is used *and* explicit Poincare invariance is invoked. However, the role of Poincare invariance (or equivalently, the choice of PLF) seems to be more important than that of smeared loops. If the requirement of smeared loops is dropped, it is still possible to construct an approximate Fock representation; but dropping Poincare invariance results in the non Fock representations of section 2 and 5a.

**Comments:**

(i) The ‘area derivative’ plays an important role in some approaches to loop quantum gravity [4, 3]. Our construction of $\hat{O}_{\bar{x},\bar{n}}|\psi>$ (or its image in the induced Fock representation of 5b) as a Cauchy limit is a rigorous realization of the area derivative in the context of Fock like representations. Note that the required limits do not exist in the Haar representation and hence the area derivative is ill defined there.

(ii) As noted above, self duality of the connection plays a key role in obtaining the
(negative helicity) graviton Fock representation without explicit recourse to Poincare invariance (see [15] for a detailed examination of the relation between self duality and helicity). However, recent efforts in loop quantum gravity use real (as opposed to self dual) connections. It would be useful to reformulate linearised gravity in terms of real connections and construct its quantization.

(iii) Note that we have mainly been concerned with the kinematics of $U(1)$ theory. The Fock representation, of course, supports the Maxwell Hamiltonian as an operator. Note that the normal ordering prescription adopted in [1] is, of course, connected with Poincare invariance. It is an open question as to how to express (presumably an approximation of) the Hamiltonian as an operator in the Haar representation.

(iv) We have not been able to show continuity of the Fock PLF on $\mathcal{H}A_r$ or lack thereof. If the Fock PLF is continuous, a ‘Fock’ measure, $d\mu_F$, can be constructed on $\Delta_r$ and $\mathcal{H}$ can be identified with $L^2(\Delta_r, d\mu_F)$. The considerations of section 5b can then be extended to the $C^*$ algebras $\overline{\mathcal{H}A}$ and $\mathcal{A}$.

If, however, the Fock PLF on $\mathcal{H}A_r$ turns out not to be continuous, then a corresponding Fock measure on $\Delta_r$ does not exist and it is incorrect to identify $\Delta_r$ with the ‘quantum configuration space’. If this is indeed the case, then the emphasis on continuous cyclic representations of $\overline{\mathcal{H}A}$ in loop quantum gravity [6] would seem unduly restrictive.

(v) The representation of kinematic loop quantum gravity is the $SU(2)$ counterpart of the Haar representation for $U(1)$ theory. An important question is how the Fock space-graviton description of linearised gravity arises out of loop quantum gravity. It is possible that some insight into this issue may be obtained by considering the following (simpler) question in the context of $U(1)$ theory. Is there any way in which an approximate Fock structure can be obtained from the Haar representation of $U(1)$ theory? Since the PLFs play a key role in determining the type of representation, this work suggests that to get an approximate Fock structure, it may be a good strategy to try to approximate (in some, yet unknown way) the Fock PLF by the Haar PLF.

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Lemma 1: Given
(i) $\gamma_i \in L_{x_0}, i = 1..n$, $n$ finite, (ii) $A_a(\vec{x}) \in \mathcal{A}$ and (iii) $\epsilon > 0$, there exists a connection $A'_a(\vec{x}) \in \mathcal{A}$ such that
\[
|H_{\gamma_i(r)}(A') - H_{\gamma_i}(A)| < \epsilon
\] (69)
for $i = 1..n$.

Proof: For a single loop $\gamma$, from (8)
\[
|X^a_{\gamma}(\vec{k})| < C_\gamma \quad C_\gamma := \frac{3}{(2\pi)^{\frac{3}{2}}} L_\gamma
\] (70)
where $L_\gamma$ is the length of the loop as measured by the flat metric. Since $A_a(\vec{x})$ is Schwartz, we have, for arbitrarily large $N > 0$,
\[
|A_a(\vec{k})| < \frac{C_N}{k^N} \quad \text{for some } C_N > 0.
\] (71)

From (70) and (71)
\[
\int_{k > \Lambda} d^3k |X^a_{\gamma}(-\vec{k})A_a(\vec{k})| < \frac{C_{N,\gamma}}{\Lambda^{N-3}}, \quad C_{N,\gamma} = \frac{4\pi C_\gamma C_N}{N - 1}.
\] (72)
Thus, given $\delta > 0$, there exists $\Lambda(\gamma, \delta)$ such that
\[
\int_{k > \Lambda(\gamma, \delta)} d^3k |X^a_{\gamma}(-\vec{k})A_a(\vec{k})| < \delta.
\] (73)

Let $f(k) > 0$ be a smooth function such that
\[
f(k) = e^{\frac{k^2\rho^2}{2}} \quad \text{for } k < \Lambda(\gamma, \delta)
\]
\[
al \quad \text{for } \Lambda(\gamma, \delta) < k < 2\Lambda(\gamma, \delta)
\]
\[
= 1 \quad \text{for } k > 2\Lambda(\gamma, \delta)
\] (74)

Define $A^\delta_{a(\gamma)}(\vec{x})$ through its Fourier transform,
\[
A^\delta_{a(\gamma)}(\vec{k}) := f(k)A_a(\vec{k}).
\] (75)
Note that \( A^\delta_{\alpha(r)}(\vec{x}) \in \mathcal{A} \).

From (11), (75), and (73) it follows that

\[
| \int_{k > \Lambda(\gamma, \delta)} d^3 k X^a_{\gamma(r)}(-\vec{k}) A^\delta_{\alpha(r)}(\vec{k}) | < \delta. \tag{76}
\]

From (11) and (75)

\[
\int_{k < \Lambda(\gamma, \delta)} d^3 k X^a_{\gamma(r)}(-\vec{k}) A^\delta_{\alpha(r)}(\vec{k}) = \int_{k > \Lambda(\gamma, \delta)} d^3 k X^a_{\gamma(r)}(-\vec{k}) A^\delta_{\alpha(r)}(\vec{k}). \tag{77}
\]

Using (77)

\[
| H_{\gamma(r)}(A^\epsilon) - H_{\gamma}(A) | = | \exp \left( \int_{k > \Lambda(\gamma, \delta)} d^3 k X^a_{\gamma(r)}(-\vec{k}) A^\delta_{\alpha(r)}(\vec{k}) - X^a_{\gamma}(\vec{k}) A^\delta_{\alpha}(\vec{k}) \right). \tag{78}
\]

From (73), (76) and (78), for small enough \( \delta > 0 \), it can be seen that

\[
| H_{\gamma(r)}(A^\epsilon) - H_{\gamma}(A) | < 4\delta \tag{79}
\]

For the loops \( \gamma_i, i = 1..n \),

\[
\tilde{\Lambda}(\delta) := \max_i \Lambda(\delta, \gamma_i) \\
A^\delta_{\alpha(r)}(\vec{k}) := \tilde{f} A_{\alpha}(\vec{k}) \tag{80}
\]

with \( \tilde{f}(k) > 0 \) a smooth function such that

\[
\tilde{f}(k) = e^{\frac{k^2 \cdot 2}{2}}, \text{ for } k < \tilde{\Lambda}(\delta) \\
< e^{\frac{k^2 \cdot 2}{2}} \text{ for } \tilde{\Lambda}(\delta) < k < 2\tilde{\Lambda}(\delta) \\
= 1 \text{ for } k > 2\tilde{\Lambda}(\delta) \tag{81}
\]

Then, given \( \epsilon > 0 \), choose some \( \delta \leq \frac{\epsilon}{4} \) and set

\[
A^\epsilon_{\alpha}(\vec{k}) := A^\delta_{\alpha(r)}(\vec{k}). \tag{82}
\]

Then (69) holds.

**Lemma 2**: Given

(i) strongly independent loops \( \gamma_i, i = 1..n \), \( n \) finite, (ii) \( g_i \in U(1), i = 1..n \) and (iii) \( \epsilon > 0 \), there exists a connection \( A^\epsilon_{\alpha}(\vec{x}) \in \mathcal{A} \) such that

\[
| H_{\gamma(r)}(A^\epsilon) - g_i | < \epsilon \tag{83}
\]
for $i = 1..n$.

**Proof:** From [8], $A_a \in \mathcal{A}$ exists such that $H_{\gamma_i}(A) = g_i$, $i = 1..n$. Therefore it suffices to construct $A'_a$ such that $|H_{\gamma_i}(A') - H_{\gamma_i}(A)| < \epsilon$. But this is exactly the content of Lemma 1.

**Lemma 3:** Given $A_a(\vec{x}) \in \mathcal{A}$, there exists $A_{a(r)}(\vec{x}) \in \mathcal{A}$ such that

$$H_{\gamma(r)}(A) = H_{\gamma}(A_{(r)}). \quad (84)$$

for every $\gamma \in \mathcal{L}_{x_0}$.

**Proof:** From (11) and (12) it immediately follows that the required $A_{a(r)}(\vec{x})$ is determined by its Fourier transform via $A_{a(r)}(\vec{k}) = e^{-\frac{k^2\pi^2}{2r^2}} A_a(\vec{k})$.

**A2**

**Proposition:** The states $|\psi(\vec{x},\vec{v})i >:= \prod_{i=1}^{p} \hat{O}_{\vec{x},\vec{v}}|0 >\), $(p = 1, 2..)$ together with $|0 >$, span $\mathcal{D}_0$.

**Heuristic Proof:** The argument below is a bit formal, but we expect that it can be converted to a rigorous proof.

Define

$$|\psi, p >= \int d^3k_1..d^3k_p \psi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)(\prod_{i=1}^{p}\hat{a}_{a_i}(\vec{k}_i) + \hat{a}_{a_i}^\dagger(-\vec{k}_i))|0 >, \quad (85)$$

where $\psi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)$ has the same properties (a)-(c) (see section 4) as $\phi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)$.

$|\psi, p >$ along with $|0 >$ span $\mathcal{D}_0$. $|\psi, p >$ can be generated from $|\psi(\vec{x},\vec{v})i >$ as follows.

Note that from (40),

$$\frac{1}{2\pi^3} \int d^3x e^{-ik\cdot\vec{x}} \hat{O}_{\vec{x},\vec{n}} = \frac{e^{-\frac{k^2\pi^2}{2r^2}}}{\sqrt{2k}}(\hat{a}_{a}(\vec{k}) + \hat{a}_{a}^\dagger(-\vec{k})). \quad (86)$$

Define

$$g^{a_1..a_p}(\vec{k}_1..\vec{k}_p) = (\prod_{i=1}^{p} e^{k_i^2\pi^2/2r_2} \psi^{a_1..a_p}(\vec{k}_1..\vec{k}_p). \quad (87)$$

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Given $\vec{k}_i, i = 1..p$, it is possible to construct a triplet of vectors $\vec{u}_{a_i}(\vec{k}_i)$ ($a_i = 1, 2, 3$ for each $i$), such that

\[
\frac{(\vec{u}_b \times \vec{k}_i)^{a_i}}{k} = \delta_{b_i}^{a_i} - \frac{k^{a_i}k_{b_i}}{k^2}.
\]  
(88)

Then from (86), (87) and (88),

\[
|\psi, p> = \int (\prod_{l=1}^{p} d^3k_l) (\prod_{m=1}^{p} d^3x_m) \, \eta^{a_1..a_p}(\vec{k}_1..\vec{k}_p)(\prod_{i=1}^{p} \left( \frac{e^{-i\vec{k}_i \cdot \vec{u}_i}}{2\pi^2} \right))|\psi_{\tilde{\vec{x}}, \tilde{\vec{u}}_i}> \tag{89}
\]

It is in this formal sense that states of the type $|\psi_{\tilde{\vec{x}}, \tilde{\vec{u}}_i}>$ together with $|0>$ span $\mathcal{D}_0$.

**Lemma 2:** $\hat{O}_{\tilde{\vec{x}}, \tilde{\vec{u}}}$ is a densely defined, symmetric operator on the dense domain $\mathcal{D}_0$, which admits self adjoint extensions.

**Proof:** It is straightforward to check that

\[
\hat{O}_{\tilde{\vec{x}}, \tilde{\vec{u}}}|\phi, p> = \int (\prod_{i=1}^{p+1} d^3k_i) f^{a_{p+1}}(-\vec{k}_{p+1}) \phi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)(\prod_{j=1}^{p+1} \hat{a}^\dagger_{a_j}(\vec{k}_j))|0> \\
+ \, p \int (\prod_{i=1}^{p} d^3k_i) f_{a_1}(\vec{k}_1) \phi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)(\prod_{j=2}^{p+1} \hat{a}^\dagger_{a_j}(\vec{k}_j))|0> \tag{90}
\]

where

\[
f^a(\vec{k}) := \frac{i}{\sqrt{2\pi^2}} e^{i\vec{k} \cdot \hat{\vec{x}}}(\vec{n} \times \vec{k})^a \frac{e^{-k^2/2}}{\sqrt{2k}}. \tag{91}
\]

The ultraviolet behaviour of $\phi^{a_1..a_p}(\vec{k}_1..\vec{k}_p)$, $f^a(\vec{k})$ ensures that $||\hat{O}_{\tilde{\vec{x}}, \tilde{\vec{u}}}|\phi, p> ||$ is finite. Thus $\hat{O}_{\tilde{\vec{x}}, \tilde{\vec{u}}}$ is densely defined on $\mathcal{D}_0$. By inspection $\hat{O}_{\tilde{\vec{x}}, \tilde{\vec{u}}}$ is also symmetric on $\mathcal{D}_0$.

---

12 An explicit choice is as follows. Fix Cartesian coordinates $(x, y, z)$ and the corresponding unit vectors $(\hat{x}, \hat{y}, \hat{z})$. Then for $i = 1..p$, $\vec{u}_1 = \frac{\hat{x} \times \hat{z}}{k}$, $\vec{u}_2 = \frac{\hat{x} \times \hat{y}}{k}$, $\vec{u}_3 = \frac{\hat{y}}{k}$. 

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To show existence of its self adjoint extensions, it is sufficient to exhibit an anti-linear operator $\hat{C}$ on $\mathcal{F}$ with $\hat{C}^2 = 1$ which leaves $D_0$ invariant and commutes with $\hat{O}(\tilde{x},\tilde{n})$ [12, 13]. As in [13], take $\hat{C}$ to be the complex conjugation operator (in the standard Schrodinger representation) on $\mathcal{F} = L^2(S',d\mu)$ where $S'$ is the appropriate space of tempered distributions and $d\mu$ is the standard Gaussian measure, for free Maxwell theory. It can be seen that $\hat{C}a_a(\tilde{k}) = a_a(-\tilde{k})\hat{C}$ and $\hat{C}a^\dagger_a(\tilde{k}) = a^\dagger_a(-\tilde{k})\hat{C}$, and hence $\hat{O}(\tilde{x},\tilde{n})\hat{C} = \hat{C}\hat{O}(\tilde{x},\tilde{n})$.

**Lemma 3:**

$$\left\| \left( \frac{e^{i\hat{D}} - 1}{i\pi \epsilon_m^2} \right) \psi \right\| \rightarrow 0.$$ (92)
as $\epsilon_m \rightarrow 0$.

**Proof:**

$$i\hat{D} = i \int d^3kh^a(\tilde{k})(\hat{a}_a(\tilde{k}) + \hat{a}^\dagger_a(-\tilde{k}))$$ (93)

with

$$h^a(\tilde{k}) = \frac{e^{i\tilde{k}\cdot\tilde{x}}e^{-k^2/2}}{\sqrt{2\pi k^2}}(-\bar{\hat{a}}^a(\tilde{m},\tilde{x},\tilde{n}) - i\pi \epsilon_m^2 (\tilde{n} \times \tilde{k})^a)$$ (94)

A straightforward calculation, using [16], shows that

$$h^a(\tilde{k}) = \frac{ie^{i\tilde{k}\cdot\tilde{x}}e^{-k^2/2}}{\sqrt{2\pi k^2}}(\tilde{n} \times \tilde{k})^a(\pi \epsilon_m^2(\frac{2J_1(\alpha_k \epsilon_m)}{\alpha_k \epsilon_m} - 1))$$ (95)

with $\alpha_k := |\tilde{n} \times \tilde{k}|$.

Now, from (93) and (38),

$$e^{i\hat{D}} = e^{-\frac{1}{2} \int d^3k|\hat{a}(\tilde{k})\hat{a}(\tilde{k})|_e} e^{i \int d^3kh^a(\tilde{k})\hat{a}^\dagger_a(\tilde{k})} e^{i \int d^3kh^a(\tilde{k})\hat{a}_a(\tilde{k})}.$$ (96)

$$\Rightarrow <\phi, p | e^{i\hat{D}} | \phi, p > = e^{-\frac{1}{2} \int d^3k|\hat{a}(\tilde{k})\hat{a}(\tilde{k})|} \int (\prod_{i=1}^p d^3k_i \prod_{j=1}^p d^3l_j) \phi_{a_1..a_p}^{*} (\tilde{k}_1..\tilde{k}_p) \phi_{b_1..b_p}^{*} (\tilde{l}_1..\tilde{l}_p)$$

$$< 0 | (\prod_{j=1}^p \hat{a}_{b_j} (\tilde{l}_j) - ih_{b_j}^* (\tilde{l}_j)) (\prod_{i=1}^p \hat{a}^\dagger_{a_i} (\tilde{k}_i) + ih_{a_i} (\tilde{k}_i)) | 0 >$$ (97)
\[ \Rightarrow \langle \phi, p | e^{i\hat{H}} \hat{D} - 1 | \phi, p \rangle = n! (e^{-\int d^3k \hbar^a(k)\hbar_a(k)} - 1) \prod_{i=1}^p d^3k_i |\phi_{\alpha_1^a \cdots \alpha_p}(k_1, \cdots, k_p)|^2 \\
+ n! \hbar e^{-\int d^3k \hbar^a(k)\hbar_a(k)} \int (\prod_{i=2}^p d^3k_i) d^3k \hbar^a[i(k)\hbar^b_{a_1}(l)\phi_{\alpha_1 \cdots \alpha_p}(k, k_2, \cdots, k_p)\phi_{\alpha_2 \cdots \alpha_p}(l, k_2, \cdots, k_p) + O(h^4)]. \tag{98} \]

Since \( \frac{2J_1(\alpha_k \epsilon_m)}{\alpha_k \epsilon_m} - 1 \) is a bounded function, (95) implies that the \( O(h^4) \) terms do not contribute to (92) in the \( \epsilon_m \to 0 \) limit.

From Lemma 4 below and (95) the first term of (98) is of order \( \epsilon_m^\frac{n}{2} \) and the second is of order \( \epsilon_m^5 \). From this it is clear that \( ||(\frac{\alpha_k \epsilon_m}{\epsilon_m})^{-1} \psi|| \to 0 \) as \( \epsilon_m \to 0 \).

**Lemma 4:** Let \( n \) be a positive integer and \( g(\vec{k}) \) be a bounded function of rapid decrease (i.e. it falls to zero as \( k \to \infty \), faster than any inverse power of \( k \)). Then, as \( \epsilon \to 0 \),

\[ I := \int d^3k g(\vec{k}) \left( \frac{2J_1(\alpha_k \epsilon)}{\alpha_k \epsilon} - 1 \right)^n < C \epsilon^{-\frac{n}{2}} \tag{99} \]

for some positive constant \( C \) which depends on \( n \) and \( g \).

**Proof:**

\[ I \leq \int_{k \leq \epsilon^{-\frac{1}{2}}} d^3k |g(\vec{k})\left( \frac{2J_1(\alpha_k \epsilon)}{\alpha_k \epsilon} - 1 \right)^n| + \int_{k > \epsilon^{-\frac{1}{2}}} d^3k |g(\vec{k})\left( \frac{2J_1(\alpha_k \epsilon)}{\alpha_k \epsilon} - 1 \right)^n|. \tag{100} \]

In the first term the range of integration is such that \( \alpha_k \epsilon < \epsilon^{-\frac{1}{2}} \). A straightforward calculation shows that the small argument expansion of \( J_1(\alpha_k \epsilon) \) coupled with the rapid fall off property of \( g(\vec{k}) \) gives the bound

\[ \int_{k \leq \epsilon^{-\frac{1}{2}}} d^3k |g(\vec{k})\left( \frac{2J_1(\alpha_k \epsilon)}{\alpha_k \epsilon} - 1 \right)^n| \leq C_1(g,n) \epsilon^{-\frac{n}{2}}. \tag{101} \]

where \( C_1(g,n) \) is a positive constant dependent on both \( n \) and the properties of \( g \).

The rapid decrease property of \( g(\vec{k}) \) ensures that, for small enough \( \epsilon \), the second term of (100) falls off much faster than the first term. Hence, \( I < C \epsilon^{-\frac{n}{2}} \) where we have set \( C := 2C_1(g,n) \).

**References**


