Finite Action in $d_5$ Gauged Supergravity and Dilatonic Conformal Anomaly for Dual Quantum Field Theory

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ABSTRACT

Gauged supergravity (SG) with single scalar (dilaton) and arbitrary scalar potential is considered. Such dilatonic gravity describes special RG flows in extended SG where scalars lie in one-dimensional submanifold of total space. The surface counterterm and finite action for such gauged SG in three-, four- and five-dimensional asymptotically AdS space are derived. Using finite action and consistent gravitational stress tensor (local surface counterterm prescription) the regularized expressions for free energy, entropy and mass of $d_4$ dilatonic AdS black hole are found. The same calculation is done within standard reference background subtraction.

The dilaton-dependent conformal anomaly from $d_3$ and $d_5$ gauged SGs is calculated using AdS/CFT correspondence. Such anomaly should correspond to two- and four-dimensional dual quantum field theory which is classically (not exactly) conformally invariant, respectively. The candidate $c$-functions for $d_3$ and $d_5$ SGs are suggested. Their numerical investigation for particular example of exponential dilatonic potentials following from string or M-theory indicates to their monotonic behaviour in some regions.

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1 Introduction

AdS/CFT correspondence [1] may be realized in a sufficiently simple form as d5 gauged supergravity/boundary gauge theory correspondence. The reason is very simple: different versions of five-dimensional gauged SG (for example, \( N = 8 \) gauged SG [2] which contains 42 scalars and non-trivial scalar potential) could be obtained as compactification (reduction) of ten-dimensional IIB SG. Then, in practice it is enough to consider 5d gauged SG classical solutions (say, AdS-like backgrounds) in AdS/CFT set-up instead of the investigation of much more involved, non-linear equations of IIB SG. Moreover, such solutions describe RG flows in boundary gauge theory (for a very recent discussion of such flows see [3, 4, 32, 5, 6, 7, 8] and refs. therein). To simplify the situation in extended SG one can consider the symmetric (special) RG flows where scalars lie in one-dimensional submanifold of total space. Then, such theory is effectively described as d5 dilatonic gravity with non-trivial dilatonic potential. Nevertheless, it is still extremely difficult to make the explicit identification of deformed SG solution with the dual (non-conformal) gauge theory. As a rule [4, 7], only indirect arguments may be suggested in such identification\(^4\).

From another side, the fundamental holographic principle [9] in AdS/CFT form enriches the classical gravity itself (and here also classical gauged SG). Indeed, instead of the standard subtraction of reference background [10, 11] in making the gravitational action finite and the quasilocal stress tensor well-defined one introduces more elegant, local surface counterterm prescription [12]. Within it one adds the coordinate invariant functional of the intrinsic boundary geometry to gravitational action. Clearly, that does not modify the equations of motion. Moreover, this procedure has nice interpretation in terms of dual QFT as standard regularization. The specific choice of surface counterterm cancels the divergences of bulk gravitational action. As a by-product, it also defines the conformal anomaly of boundary QFT.

Local surface counterterm prescription has been successfully applied to

\(^4\)Such dual theory in massless case is, of course, classically conformally invariant and it has well-defined conformal anomaly. However, among the interacting theories only \( \mathcal{N} = 4 \) SYM is known to be exactly conformally invariant. Its conformal anomaly is not renormalized. For other, \( d^4 \) QFTs there is breaking of conformal invariance due to radiative corrections which give contribution also to conformal anomaly. Hence, one can call such theories as non-conformal ones.
construction of finite action and quasilocal stress tensor on asymptotically AdS space in Einstein gravity [12, 13, 14, 15, 16] and in higher derivative gravity [17]. Moreover, the generalization to asymptotically flat spaces is possible as it was first mentioned in ref.[18]. Surface counterterm has been found for domain-wall black holes in gauged SG in diverse dimensions [19]. However, actually only the case of asymptotically constant dilaton has been investigated there.

In the present paper we discuss the construction of finite action, consistent gravitational stress tensor and dilaton-dependent Weyl anomaly for boundary QFT (from bulk side) in three- and five-dimensional gauged supergravity with single scalar (dilaton) on asymptotically AdS background. Note that dilaton is not constant and the potential is chosen to be arbitrary. The implications of results for the study of RG flows in boundary QFT are presented, in particular, the candidate c-function is suggested.

The next section is devoted to the evaluation of Weyl anomaly from gauged supergravity with arbitrary dilatonic potential via AdS/CFT correspondence. We present explicit result for d3 and d5 gauged SGs. Such SG side conformal anomaly should correspond to dual QFT with broken conformal invariance in two and in four dimensions, respectively. The explicit form of d4 conformal anomaly takes few pages, so its lengthy dilaton-dependent coefficients are listed in Appendix. The comparison with similar AdS/CFT calculation of conformal anomaly in the same theory but with constant dilatonic potential is given. The candidates for c-function in two and four dimensions are proposed.

Section three is related with the study of the behaviour of candidate c-functions for explicit examples of exponential dilatonic potentials which follow from M-theory or from strings (sphere reduction). We consider d3 SG with one, two or three supersymmetries and d5 SG. The numerical investigation of c-function is done. The regions where it is monotonic are derived.

In section four we construct surface counterterms for d3 and d5 gauged SGs. As a result, the gravitational action in asymptotically AdS space is finite. On the same time, the gravitational stress tensor around such space is well defined. It is interesting that conformal anomaly defined in second section directly follows from the gravitational stress tensor with account of surface terms.

Section five is devoted to the application of finite gravitational action found in previous section in the calculation of thermodynamical quantities.
in dilatonic AdS black hole. The dilatonic AdS black hole is constructed approximately, using the perturbations around constant dilaton AdS black hole. The entropy, mass and free energy of such black hole are found using the local surface counterterm prescription to regularize these quantities. The comparison is done with the case when standard prescription: regularization with reference background is used. The explicit regularization dependence of the result is mentioned. Finally, in the Discussion the summary of results is presented and some open problems are mentioned.

2 Weyl anomaly for gauged supergravity with general dilaton potential

In the present section the derivation of dilaton-dependent Weyl anomaly from gauged SG will be given. As we note in section 4 this derivation can be made also from the definition of finite action in asymptotically AdS space.

We start from the bulk action of $d + 1$-dimensional dilatonic gravity with the potential $\Phi$

$$S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} + X(\phi)(\nabla\phi)^2 + Y(\phi)\Delta\phi + \Phi(\phi) + 4\lambda^2 \right\}.$$ (1)

Here $M_{d+1}$ is $d + 1$ dimensional manifold whose boundary is $d$ dimensional manifold $M_d$ and we choose $\Phi(0) = 0$. Such action corresponds to (bosonic sector) of gauged SG with single scalar (special RG flow). In other words, one considers RG flow in extended SG when scalars lie in one-dimensional submanifold of complete scalars space. Note also that classical vacuum stability restricts the form of dilaton potential [20]. As well-known, we also need to add the surface terms [10] to the bulk action in order to have well-defined variational principle. At the moment, for the purpose of calculation of Weyl anomaly (via AdS/CFT correspondence) the surface terms are irrelevant. The equations of motion given by variation of (1) with respect to $\phi$ and $G^{\mu\nu}$ are

$$0 = -\sqrt{-\hat{G}}\Phi'(\phi) - \sqrt{-\hat{G}}V'(\phi)\hat{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2\partial_\mu \left( \sqrt{-\hat{G}}\hat{G}^{\mu\nu}V(\phi)\partial_\nu\phi \right)$$ (2)
Here

\[ V(\phi) \equiv X(\phi) - Y'(\phi) . \]

We choose the metric \( \hat{G}_{\mu\nu} \) on \( M_{d+1} \) and the metric \( \hat{g}_{\mu\nu} \) on \( M_d \) in the following form

\[
d s^2 \equiv \hat{G}_{\mu\nu} d x^\mu d x^\nu = \frac{l^2}{4} \rho^{-2} d \rho d \rho + \sum_{i=1}^{d} \hat{g}_{ij} d x^i d x^j , \quad \hat{g}_{ij} = \rho^{-1} g_{ij} .
\]

Here \( l \) is related with \( \lambda^2 \) by \( 4\lambda^2 = d(d - 1)/l^2 \). If \( g_{ij} = \eta_{ij} \), the boundary of AdS lies at \( \rho = 0 \). We follow to method of calculation of conformal anomaly as it was done in refs.[21, 22] where dilatonic gravity with constant dilaton potential has been considered. Part of results of this section concerning Weyl anomaly with no dilaton derivatives has been presented already in letter [23].

The action (1) diverges in general since it contains the infinite volume integration on \( M_{d+1} \). The action is regularized by introducing the infrared cutoff \( \epsilon \) and replacing

\[
\int d^{d+1} x \rightarrow \int d^d x \int d \rho , \quad \int_{M_d} d^d x (\cdots) \rightarrow \int d^d x (\cdots) \big|_{\rho = \epsilon} .
\]

We also expand \( g_{ij} \) and \( \phi \) with respect to \( \rho \):

\[
g_{ij} = g_{(0)ij} + \rho g_{(1)ij} + \rho^2 g_{(2)ij} + \cdots , \quad \phi = \phi_{(0)} + \rho \phi_{(1)} + \rho^2 \phi_{(2)} + \cdots .
\]

Then the action is also expanded as a power series on \( \rho \). The subtraction of the terms proportional to the inverse power of \( \epsilon \) does not break the invariance under the scale transformation \( \delta g_{\mu\nu} = 2 \delta \sigma g_{\mu\nu} \) and \( \delta \epsilon = 2 \delta \sigma \epsilon \). When \( d \) is even, however, the term proportional to \( \ln \epsilon \) appears. This term is not invariant under the scale transformation and the subtraction of the \( \ln \epsilon \) term breaks the invariance. The variation of the \( \ln \epsilon \) term under the scale transformation is finite when \( \epsilon \rightarrow 0 \) and should be canceled by the variation of the finite term (which does not depend on \( \epsilon \)) in the action since the original action (1) is invariant under the scale transformation. Therefore the \( \ln \epsilon \) term \( S_{\ln} \) gives the Weyl anomaly \( T \) of the action renormalized by the subtraction of the terms which diverge when \( \epsilon \rightarrow 0 \) (d=4)

\[
S_{\ln} = -\frac{1}{2} \int d^d x \sqrt{-g} T .
\]
The conformal anomaly can be also obtained from the surface counterterms, what is discussed in Section 4.

First we consider the case of $d = 2$, i.e. three-dimensional gauged SG. The anomaly term $S_{\ln}$ proportional to $\ln \epsilon$ in the action is

\[
S_{\ln} = -\frac{1}{16\pi G} \int d^2 x \sqrt{-g(0)} \left\{ R(0) + X(\phi(0)) (\nabla \phi(0))^2 + Y(\phi(0)) \Delta \phi(0) \\
+ \phi(1) \Phi'(\phi(0)) + \frac{1}{2} g^{ij}_{(0)} g^{(1)ij} \Phi(\phi(0)) \right\}. \quad (9)
\]

The terms proportional to $\rho^0$ with $\mu, \nu = i, j$ in (3) lead to $g^{(1)ij}$ in terms of $g^{(0)ij}$ and $\phi(1)$.

\[
g^{(1)ij} = \left[ -R_{(0)ij} - V(\phi(0)) \partial_i \phi(0) \partial_j \phi(0) - g^{(0)ij} \Phi'(\phi(0)) \phi(1) \\
+ \frac{g_{(0)ij}}{l^2} \left\{ 2 \Phi'(\phi(0)) \phi(1) + R(0) + V(\phi(0)) g^{kl}_{(0)} \partial_k \phi(0) \partial_l \phi(0) \right\} \right.
\times \left( \Phi(\phi(0)) + \frac{3}{l^2} \right)^{-1} \\
\left. \times \Phi(\phi(0))^{(1)} \right] \times \left( \Phi(\phi(0)) + \frac{1}{l^2} \right)^{-1} \quad (10)
\]

In the equation (2), the terms proportional to $\rho^{-1}$ lead to $\phi(1)$ as following.

\[
\phi(1) = \left[ V'(\phi(0)) g^{ij}_{(0)} \partial_i \phi(0) \partial_j \phi(0) + 2 \frac{V(\phi(0))}{\sqrt{-g(0)}} \partial_i \left( \sqrt{-g(0)} g^{ij}_{(0)} \partial_j \phi(0) \right) \\
+ \frac{1}{2} \Phi'(\phi(0)) \left( \Phi(\phi(0)) + \frac{3}{l^2} \right)^{-1} \left\{ R(0) + V(\phi(0)) g^{ij}_{(0)} \partial_i \phi(0) \partial_j \phi(0) \right\} \right]
\times \left( \Phi''(\phi(0)) - \Phi'(\phi(0))^2 \left( \Phi(\phi(0)) + \frac{3}{l^2} \right)^{-1} \right)^{-1} \quad (11)
\]

Then anomaly term takes the following form using (10), (11)

\[
T = \frac{1}{8\pi G} \int d^2 x \sqrt{-g(0)} \left\{ R(0) + X(\phi(0)) (\nabla \phi(0))^2 + Y(\phi(0)) \Delta \phi(0) \\
+ \frac{1}{2} \left\{ \frac{3 \Phi'(\phi(0))}{l^2} \left( \Phi''(\phi(0)) \left( \Phi(\phi(0)) + \frac{3}{l^2} \right) - \Phi'(\phi(0))^2 \right)^{-1} - \Phi(\phi(0)) \right\} \right.
\times \left( R(0) + V(\phi(0)) g^{ij}_{(0)} \partial_i \phi(0) \partial_j \phi(0) \right) \left( \Phi(\phi(0)) + \frac{3}{l^2} \right)^{-1} \\
\left. + \frac{3 \Phi'(\phi(0))}{l^2} \left( \Phi''(\phi(0)) - \Phi'(\phi(0))^2 \left( \Phi(\phi(0)) + \frac{3}{l^2} \right)^{-1} \right)^{-1} \right]
\]

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\[ V_0 = \left( V'(\phi(0))g_{ij}^{(0)}\partial_i\phi(0)\partial_j\phi(0) + 2\frac{V(\phi(0))}{\sqrt{-g_{ij}^{(0)}}}\partial_i \left( \sqrt{-g_{ij}^{(0)}}g_{ij}^{(0)}\partial_j\phi(0) \right) \right). \] (12)

For \( \Phi(\phi) = 0 \) case, the central charge of two-dimensional conformal field theory is defined by the coefficient of \( R \). Then it might be natural to introduce \( c \)-function \( c \) for the case when the conformal symmetry is broken by the deformation in the following way:

\[ c = \frac{3}{2G} \left[ l + \frac{1}{2} \left\{ \frac{3\Phi'(\phi(0))}{l^2} - \left( \Phi''(\phi(0)) \Phi(\phi(0)) \right) \right. \right.
\]
\[ + \frac{3}{l^2} - \Phi'(\phi(0))^2 \right\} \left. \times \left( \Phi(\phi(0)) + \frac{3}{l^2} \right) \right]^1. \] (13)

Comparing this with radiatively-corrected \( c \)-function of boundary QFT \( (\text{AdS}_3/\text{CFT}_2) \) may help in correct bulk description of such theory. Note that numerical investigation of this \( c \)-function for few versions of d3 gauged SG is done in next section. Clearly, that in the regions where such candidate \( c \)-function is singular or not monotonic it cannot be the acceptable \( c \)-function. Presumably, the appearance of such regions indicates to the breaking of SG description.

Four-dimensional case is more interesting but also much more involved. The anomaly terms which proportional to \( \ln \epsilon \) are

\[ S_{\ln} = \frac{1}{16\pi G} \int d^4x \sqrt{-g_{(0)}} \left[ -\frac{1}{2l} g_{ij}^{(0)}g_{kl}^{(0)} \left( g_{(1)ij}g_{(1)kl} - g_{(1)ik}g_{(1)jl} \right) \right.
\]
\[ + \frac{l}{2} \left( R_{ij}^{(0)} - \frac{1}{2} g_{ij}^{(0)}R_{(0)} \right) g_{(1)ij} \right.
\]
\[ - \frac{2}{l} V'(\phi(0))\phi_{(1)}^2 + \frac{l}{2} V'(\phi(0))g_{ij}^{(0)}\partial_i\phi(0)\partial_j\phi(0) \right.
\]
\[ + lV(\phi(0))\phi_{(1)} \frac{1}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{ij}^{(0)}}g_{ij}^{(0)}\partial_j\phi(0) \right) \right.
\]
\[ + \frac{l}{2} V(\phi(0)) \left( g_{ij}^{(0)}g_{kl}^{(0)}g_{(1)ij} - \frac{1}{2} g_{ij}^{(0)}g_{(1)kl}g_{ij}^{(0)} \right) \partial_i\phi(0)\partial_j\phi(0) \right.
\]
\[ - \frac{l}{2} \left( \frac{1}{2} g_{ij}^{(0)}g_{(2)ij} - \frac{1}{4} g_{ij}^{(0)}g_{(0)}g_{(1)ik}g_{(1)jl} + \frac{1}{8} (g_{ij}^{(0)}g_{(1)ij})^2 \right) \Phi(\phi(0)) \right.
\]
\[ - \frac{l}{2} \left( \Phi'(\phi(0))\phi_{(2)} + \frac{1}{2} \Phi''(\phi(0))\phi_{(1)}^2 + \frac{1}{2} g_{ij}^{(0)}g_{(1)kl}g_{ij}^{(0)} \right) \Phi(\phi(0)) \phi_{(1)} \right] \right]. \] (14)
The terms proportional to $\rho^0$ with $\mu, \nu = i, j$ in the equation of the motion (3) lead to $g_{(1)ij}$ in terms of $g_{(0)ij}$ and $\phi_{(1)}$.

\[
g_{(1)ij} = \left[-R_{(0)ij} - V(\phi_{(0)}) \partial_i \phi_{(0)} \partial_j \phi_{(0)} - \frac{1}{3} g_{(0)ij} \Phi'(\phi_{(0)}) \phi_{(1)}
+ \frac{g_{(0)ij}}{l^2} \left\{ \frac{4}{3} \Phi'(\phi_{(0)}) \phi_{(1)} + R_{(0)} + V(\phi_{(0)}) g_{kl}^{(0)} \partial_k \phi_{(0)} \partial_l \phi_{(0)} \right\}
\times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \right] \times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1}.
\]

In the equation (2), the terms proportional to $\rho^{-2}$ lead to $\phi_{(1)}$ as follows:

\[
\phi_{(1)} = \left[V'(\phi_{(0)}) g_{ij}^{(0)} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + 2 \frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{ij}^{(0)} \partial_j \phi_{(0)} \right)
+ \frac{1}{2} \Phi'(\phi_{(0)}) \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \left\{ R_{(0)} + V(\phi_{(0)}) g_{ij}^{(0)} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \right\}
\times \left( \frac{8V(\phi_{(0)})}{l^2} + \Phi''(\phi_{(0)}) - \frac{2}{3} \Phi'(\phi_{(0)})^2 \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \right)^{-1} \right] (16)
\]

In the equation (3), the terms proportional to $\rho^1$ with $\mu, \nu = i, j$ lead to $g_{(2)ij}$.

\[
g_{(2)ij} = \left[-\frac{1}{3} \left\{ g_{(1)ij} \Phi'(\phi_{(0)}) \phi_{(1)} + g_{(0)ij} (\Phi'(\phi_{(0)}) \phi_{(2)} + \frac{1}{2} \Phi''(\phi_{(0)}) \phi_{(1)}^2) \right\}
- \frac{2}{l^2} g_{kl}^{(0)} g_{(1)kl} g_{ij}^{(0)} + \frac{1}{l^2} g_{mn}^{(0)} g_{(0)mn} g_{(1)kl} g_{(0)ij}
- \frac{2}{l^2} g_{(0)ij} \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{8}{l^2} \right)^{-1} \times \left\{ \frac{2}{l^2} g_{kl}^{(0)} g_{(0)kl} g_{(1)mn} g_{(1)mn} \right\}
- \frac{4}{3} \left( \Phi'(\phi_{(0)}) \phi_{(2)} + \frac{1}{2} \Phi''(\phi_{(0)}) \phi_{(1)}^2 \right) - \frac{1}{3} g_{ij}^{(0)} g_{(1)ij} \Phi'(\phi_{(0)}) \phi_{(1)}
+ V'(\phi_{(0)}) g_{ij}^{(0)} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + \frac{2V(\phi_{(0)})}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{ij}^{(0)} \partial_j \phi_{(0)} \right)
\right] \times \left( \frac{1}{3} \Phi(\phi_{(0)}) \right)^{-1}.
\]
And the terms proportional to $\rho^{-1}$ in the equation (2), lead to $\phi_{(2)}$ as follows:

$$
\phi_{(2)} = [V''(\phi(0))\phi(1)g^{ij}_0(0)\partial_i\phi(0)\partial_j\phi(0) + V'(\phi(0))\left(g^{ik}_0g^{jl}_0 - \frac{1}{2}g^{ij}_0g^{kl}_0\right)g_{(1)kl}\partial_i\phi(0)\partial_j\phi(0) + \frac{2V''(\phi(0))\phi(1)}{\sqrt{-g(0)}}\partial_i\left(\sqrt{-g(0)}g^{ij}_0\partial_j\phi(0)\right) - \frac{4}{l^2}V(\phi(0))\phi^2(1) - \frac{1}{2}\Phi''(\phi(0))\phi^2(1) - \frac{1}{2}g^{kl}_{(0)}g_{(1)kl}\Phi''(\phi(0))\phi(1) - \left(-\frac{1}{4}g^{ij}_0g^{kl}_0g_{(1)lk}g_{(1)ji} + \frac{1}{8}(g^{ij}_0g^{kl}_0)^2\right)\phi'(\phi(0)) - \frac{1}{2}\Phi'(\phi(0))\left(\frac{1}{3}\Phi(\phi(0)) + \frac{8}{l^2}\right)^{-1} \times \left\{ \frac{2}{l^2}g^{mn}_{(0)}g_{(1)km}g_{(1)ln} - \frac{2}{3}\Phi''(\phi(0))\phi^2(1) - \frac{1}{3}g^{ij}_{(0)}g_{(1)ij}\phi'(\phi(0))\phi(1) + V'(\phi(0))g^{ij}_0(0)\partial_i\phi(0)\partial_j\phi(0) + \frac{2V(\phi(0))\phi(1)}{\sqrt{-g(0)}}\partial_i\left(\sqrt{-g(0)}g^{ij}_0\partial_j\phi(0)\right)\right\}]
\times \left(\Phi''(\phi(0)) - \frac{2}{3}\Phi'(\phi(0))^2 \left(\frac{1}{3}\Phi(\phi(0)) + \frac{8}{l^2}\right)^{-1}\right)^{-1}.
$$

Then we can get the anomaly (14) in terms of $g_{(0)ij}$ and $\phi(0)$, which are boundary values of metric and dilaton respectively by using (15), (16), (17), (18). In the following, we choose $l = 1$, denote $\Phi(\phi(0))$ by $\Phi$ and abbreviate the index (0) for the simplicity. Then substituting (16) into (15), we obtain

$$
g_{(1)ij} = \tilde{c}_1R_{ij} + \tilde{c}_2g_{ij}R + \tilde{c}_3g_{ij}g^{kl}\partial_k\phi\partial_l\phi + \frac{\tilde{c}_4g_{ij}}{\sqrt{-g}}\left(\sqrt{-g}g^{kl}\partial_l\phi\right) + \tilde{c}_5\partial_i\phi\partial_j\phi.
$$

The explicit form of $\tilde{c}_1, \tilde{c}_2, \cdots \tilde{c}_5$ is given in Appendix A. Further, substituting (16) and (19) into (18), one gets

$$
\phi_{(2)} = d_1R^2 + d_2Rg^{ij}\partial_i\phi\partial_j\phi + d_3Rg^{ij}\partial_i\phi\partial_j\phi + d_4Rg^{ij}\partial_i\phi\partial_j\phi + d_5Rg^{ij}\partial_i\phi\partial_j\phi + d_6(g^{ij}\partial_i\phi\partial_j\phi)^2 + d_7\left(\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)\right)^2.
$$
Here, the explicit form of $d_1, \cdots d_8$ is given in Appendix A. Substituting (16), (19) and (20) into (17), one gets

\[
g^{ij}g^{(2)ij} = f_1 R^2 + f_2 R_{ij} R^{ij} + f_3 R^{ij} \partial_i \phi \partial_j \phi \\
+ f_4 R g^{ij} \partial_i \phi \partial_j \phi + f_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
+ f_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + f_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
+ f_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi). \tag{21}
\]

Again, the explicit form of very complicated functions $f_1, \cdots f_8$ is given in Appendix A. Finally substituting (16), (19), (20) and (21) into the expression for the anomaly (14), we obtain,

\[
T = -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} + h_3 R^{ij} \partial_i \phi \partial_j \phi \\
+ h_4 R g^{ij} \partial_i \phi \partial_j \phi + h_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
+ h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + h_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
+ h_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right]. \tag{22}
\]

Here

\[
h_1 = 3 \left\{ (24 - 10 \Phi) \Phi^6 \\
+ (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' (\Phi'' + 8 \Phi^2) \Phi'' + 72 (-8 + 14 \Phi + \Phi^2) \Phi^2 \\
+ 2 \Phi^4 \left\{ (108 + 162 \Phi + 7 \Phi^2) \Phi'' + 72 (-8 + 14 \Phi + \Phi^2) \Phi^2 \\
- 2 \Phi^2 \left\{ (6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi'' \\
+ 4 (11232 + 6156 \Phi + 552 \Phi^2 + 13 \Phi^3) \Phi'' \Phi \\
+ 32 (-2592 + 468 \Phi + 96 \Phi^2 + 5 \Phi^3) \Phi V \right\} \right\}
\]

\]
We also give the explicit forms of $h_3, \cdots h_8$ in Appendix A. Thus, we found the complete Weyl anomaly from bulk side. This expression which should describe dual d4 QFT of QCD type, with broken SUSY looks really complicated. The interesting remark is that Weyl anomaly is not integrable in general. In other words, it is impossible to construct the anomaly induced action. This is not strange, as it is usual situation for conformal anomaly when radiative corrections are taken into account.

In case of the dilaton gravity in [21] corresponding to $\Phi = 0$ (or more generally in case that the axion is included [24] as in [22]), we have the following expression:

$$T = \frac{l^3}{8\pi G} \int d^4x \sqrt{-g(0)} \left[ \frac{1}{8} R(0)_{ij} R^i_j - \frac{1}{24} R^2(0) - \frac{1}{2} R^{ij}(0) \partial_i \varphi(0) \partial_j \varphi(0) + \frac{1}{6} R(0) g^{ij}(0) \partial_i \varphi(0) \partial_j \varphi(0) \right. $$

$$\left. + \frac{1}{4} \left( \frac{1}{\sqrt{-g(0)}} \partial_i \left( \sqrt{-g(0)} g^{ij}(0) \partial_j \varphi(0) \right) \right)^2 + \frac{1}{3} \left( g^{ij}(0) \partial_i \varphi(0) \partial_j \varphi(0) \right)^2 \right] \tag{24}$$

Here $\varphi$ can be regarded as dilaton. In the limit of $\Phi \to 0$, we obtain

$$h_1 \to \frac{3 \cdot 62208 \Phi''(8V)^2}{16 \cdot 6^2 \cdot 24 \cdot 18 \Phi''(8V)^2} = \frac{1}{24}$$

$$h_2 \to -\frac{3 \cdot 288 \Phi''}{8 \cdot 6^2 \cdot 24 \Phi''} = -\frac{1}{8}$$

$$h_3 \to -\frac{3 \cdot 288 (\Phi''V - \Phi'V')}{4 \cdot 6^2 \cdot 24 \Phi''} = -\frac{1}{4} \frac{(\Phi''V - \Phi'V')}{\Phi''}$$

$$h_4 \to \frac{3 \cdot 62208 \Phi''V(8V)^2 + 6\Phi' \cdot 384 \cdot (-5184) \cdot V^2 V'}{8 \cdot 6^2 \cdot 24 \Phi'' \cdot (18 \cdot 8V)^2} = \frac{1}{12} \frac{(\Phi''V - \Phi'V')}{\Phi''}$$

$$h_5 \to \frac{\Phi' \cdot 64V \cdot (373248V^3 - 139968V'^2)}{8 \cdot 6^2 \cdot 24 \Phi''}$$

$$h_6 \to \left\{-\Phi'' \cdot 64V \cdot (373248V^3 - 139968V'^2) \right\}$$
\[
\begin{align*}
&\frac{1}{16} \cdot 6^2 \cdot 24 \Phi'' \cdot (18 \cdot 8V)^2 \\
&= \left\{ -\Phi'' V \cdot \left( V^3 - \frac{3}{8} V'^2 \right) + 2 \Phi' V' \cdot \left( V^3 + \frac{2}{16} V'^2 - \frac{3}{8} V'' \right) \right\}
\end{align*}
\]

\[
h_7 \to \frac{V \cdot 8 \cdot 18^2 \Phi'' V \cdot 2 \cdot 12V}{24 \Phi'' \cdot (18 \cdot 8V)^2} = \frac{V}{8}
\]

\[
h_8 \to \frac{32 \cdot 18^2 \Phi'' V \cdot 2 \cdot 12 \cdot V''}{4 \cdot 24 \Phi'' (18 \cdot 8V)^2} = \frac{V'}{8V}.
\]

Especially if we choose

\[ V = -2, \]

we obtain,

\[
h_1 \to \frac{1}{24}, \quad h_2 \to -\frac{1}{8}, \quad h_3 \to \frac{1}{2}, \quad h_4 \to -\frac{1}{6}
\]

\[
h_5 \to 0, \quad h_6 \to \frac{1}{3}, \quad h_7 \to -\frac{1}{4}, \quad h_8 \to 0
\]

and we find that the standard result (conformal anomaly of $\mathcal{N} = 4$ super YM theory covariantly coupled with $\mathcal{N} = 4$ conformal supergravity [25]) in (24) is reproduced [21, 26].

We should also note that the expression (22) cannot be rewritten as a sum of the Gauss-Bonnet invariant $G$ and the square of the Weyl tensor $F$, which are given as

\[
\begin{align*}
G &= R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl} \\
F &= \frac{1}{3} R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl},
\end{align*}
\]

This is the signal that the conformal symmetry is broken.

When $\phi$ is constant, only two terms corresponding to $h_1$ and $h_2$ survive in (22):

\[
\begin{align*}
T &= -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} \right] \\
&= -\frac{1}{8\pi G} \left[ \left( h_1 + \frac{1}{3} h_2 \right) R^2 + \frac{1}{2} h_2 (F - G) \right].
\end{align*}
\]
As $h_1$ depends on $V$, we may compare the result with the conformal anomaly from, say, scalar or spinor QED, or QCD in the phase where there are no background scalars and (or) spinors. The structure of the conformal anomaly in such a theory has the following form

$$T = \hat{a}G + \hat{b}F + \hat{c}R^2.$$ (30)

where

$$\hat{a} = \text{constant} + a_1 e^2, \quad \hat{b} = \text{constant} + a_2 e^2, \quad \hat{c} = a_3 e^2.$$ (31)

Here $e^2$ is the electric charge (or $g^2$ in case of QCD). Imagine that one can identify $e$ with the exponential of the constant dilaton (using holographic RG [27, 28]). $a_1$, $a_2$ and $a_3$ are some numbers. Comparing (29) and (30), we obtain

$$\hat{a} = \hat{b} = \frac{h_2}{16\pi G}, \quad \hat{c} = -\frac{1}{8\pi G} \left( h_1 + \frac{1}{3} h_2 \right).$$ (32)

When $\Phi$ is small, one gets

$$h_1 = \frac{1}{24} \left[ 1 - \frac{1}{8} \Phi + \frac{1}{8} \left( \frac{\Phi'}{\Phi''} \right)^2 + \frac{25}{2592} \Phi^2 - \frac{17}{216} \frac{(\Phi')^2}{\Phi''} + \frac{1}{576} \frac{(\Phi')^2}{V} + \frac{1}{96} \frac{(\Phi')^4}{(\Phi'')^2} + O \left( \Phi^3 \right) \right],$$

$$h_2 = -\frac{1}{8} \left[ 1 - \frac{1}{8} \Phi + \frac{1}{8} \left( \frac{\Phi'}{\Phi''} \right)^2 + \frac{5}{576} \Phi^2 - \frac{3}{64} \frac{(\Phi')^2}{\Phi''} + \frac{1}{96} \frac{(\Phi')^4}{(\Phi'')^2} + O \left( \Phi^3 \right) \right].$$ (33)

If one assumes

$$\Phi(\phi) = ae^{b\phi}, \quad (|a| \ll 1),$$ (34)

then

$$h_2 = -\frac{1}{8} \left[ 1 - \frac{a^2}{36} e^{2b\phi} + O \left( a^3 \right) \right],$$

$$h_1 + \frac{1}{3} h_2 = \frac{a^2}{24} \left( -\frac{5}{162} + \frac{b^2}{576V} \right) e^{2b\phi} + O \left( a^3 \right).$$ (35)
Comparing (35) with (31) and (32) and assuming

$$e^2 = e^{2b\phi},$$

we find

$$a_1 = -a_2 = \frac{1}{16\pi G} \cdot \frac{1}{8} \cdot \frac{a^2}{36},$$

$$a_3 = -\frac{1}{8\pi G} \cdot \frac{a^2}{24} \left( -\frac{5}{162} + \frac{b^2}{576V} \right).$$

(37)

Here $V$ should be arbitrary but constant. We should note $\Phi(0) \neq 0$. One can absorb the difference into the redefinition of $l$ since we need not to assume $\Phi(0) = 0$ in deriving the form of $h_1$ and $h_2$ in (23). Hence, this simple example suggests the way of comparison between SG side and QFT descriptions of non-conformal boundary theory.

In order that the region near the boundary at $\rho = 0$ is asymptotically AdS, we need to require $\Phi \to 0$ and $\Phi' \to 0$ when $\rho \to 0$. One can also confirm that $h_1 \to \frac{1}{24}$ and $h_2 \to -\frac{1}{8}$ in the limit of $\Phi \to 0$ and $\Phi' \to 0$ even if $\Phi'' \neq 0$ and $\Phi''' \neq 0$. In the AdS/CFT correspondence, $h_1$ and $h_2$ are related with the central charge $c$ of the conformal field theory (or its analog for non-conformal theory). Since we have two functions $h_1$ and $h_2$, there are two ways to define the $c$-function when the conformal field theory is deformed:

$$c_1 = \frac{24\pi h_1}{G}, \quad c_2 = \frac{-8\pi h_2}{G}.$$  

(38)

If we put $V(\phi) = 4\lambda^2 + \Phi(\phi)$, then $l = \left( \frac{12}{V(0)} \right)^{\frac{1}{2}}$. One should note that it is chosen $l = 1$ in (38). We can restore $l$ by changing $h \to l^3 h$ and $k \to l^4 k$ and $\Phi' \to l\Phi'$, $\Phi'' \to l^2 \Phi''$ and $\Phi''' \to l^3 \Phi'''$ in (22). Then in the limit of $\Phi \to 0$, one gets

$$c_1, \quad c_2 \to \frac{\pi}{G} \left( \frac{12}{V(0)} \right)^{\frac{3}{2}},$$

(39)

what agrees with the definition of the previous work [29] in the limit. The $c$-function $c_1$ or $c_2$ in (38) is, of course, more general definition. It is interesting to study the behaviour of candidate $c$-function for explicit values of dilatonic potential at different limits. It also could be interesting to see what is the analogue of our dilaton-dependent $c$-function in non-commutative YM theory (without dilaton, see [30]).
3 Examples of Dilatonic Potential from M-theory and c-Function

In this section, we consider some examples of c-function for different choices of dilatonic potential. In ref.[23] we investigated the behaviour of proposed candidate c-function for two dilatonic potentials numerically. It has been shown that they indeed may serve as c-function but only near AdS space, i.e. in asymptotic. The reason is that candidate c-functions are monotonic only near AdS limit. Far away of conformity (of AdS limit) they are not monotonic and even may be singular. That indicates to breaking of supergravity description.

In [31], several examples of the potentials in gauged supergravity are given. They appeared as a result of sphere reduction in M-theory or string theory, down to three or five dimensions. Their properties are described in detail in refs.[31]. The potentials have the following form:

\[ 4\lambda^2 + \Phi(\phi) = \frac{d(d-1)}{a_1^2 - \frac{1}{a_1a_2}} \left( \frac{1}{a_1^2} e^{a_1\phi} - \frac{1}{a_1a_2} e^{a_2\phi} \right). \]  

(40)

Here \( a_1 \) and \( a_2 \) are constant parameters depending on the model. We also normalize the potential so that \( 4\lambda^2 + \Phi(\phi) \to d(d-1) \) when \( \phi \to 0 \). For simplicity, we choose \( G = l = 1 \) in this section.

For \( \mathcal{N} = 1 \) model in \( D = d + 1 = 3 \) dimensions

\[ a_1 = 2\sqrt{2}, \quad a_2 = \sqrt{2}. \]  

(41)

Then the c-function in (13) has the following form:

\[ c = -\left( 3 \left( -8 - 3 \sqrt{2} + 8 e^{\sqrt{2} \phi} 
+ 3 \sqrt{2} e^{\sqrt{2} \phi} + 36 e^2 \sqrt{2} \phi + 8 e^3 \sqrt{2} \phi - 8 e^4 \sqrt{2} \phi \right) \right)/
\left( 8 \left( 1 + 2 e^{\sqrt{2} \phi} - 12 e^2 \sqrt{2} \phi - 4 e^3 \sqrt{2} \phi + 4 e^4 \sqrt{2} \phi \right) \right). \]  

(42)

For \( D = 3, \mathcal{N} = 2 \), one gets

\[ a_1 = \sqrt{6}, \quad a_2 = 2 \sqrt{\frac{2}{3}}. \]  

(43)
\[ c = \left( 3 \left( 16 + 3 \sqrt{6} - 24 e^{\sqrt{2} \phi} - 3 \sqrt{6} e^{\sqrt{2} \phi} + 24 e^{2 \sqrt{2} \phi} \right) \\
+ 24 e^4 \sqrt{2} \phi - 24 e^5 \sqrt{2} \phi - 68 e^{\sqrt{6} \phi} + 16 e^2 \sqrt{6} \phi \right) \right) / \\
\left( 8 \left( -2 + 3 e^{\sqrt{2} \phi} + 2 e^{\sqrt{6} \phi} \right) \right) \left( -1 + 6 e^2 \sqrt{2} \phi + 4 e^{\sqrt{6} \phi} \right) \right), \] (44)

For \( D = 3, \mathcal{N} = 3 \) model, we have

\[ a_1 = \frac{4}{\sqrt{3}}, \quad a_2 = \sqrt{3}, \] (45)

and

\[ c = \left( 3 \left( 12 + 3 \sqrt{3} - 16 e^{\sqrt{3} \phi} - 3 \sqrt{3} e^{\sqrt{3} \phi} \\
- 58 e^{\sqrt{5} \phi} + 24 e^{2 \sqrt{3} \phi} - 16 e^{7 \phi} + 12 e^{8 \phi} + 24 e^{10 \phi} \right) \right) / \\
\left( 4 \left( -3 + 4 e^{\sqrt{3} \phi} + 2 e^{4 \phi} \right) \right) \left( -1 + 6 e^{\sqrt{3} \phi} - 8 e^{10 \phi} \right) \right), \] (46)

On the other hand, for \( D = d + 1 = 5, \mathcal{N} = 1 \) model, \( a_1 \) and \( a_2 \) are

\[ a_1 = 2 \sqrt{\frac{5}{3}}, \quad a_2 = \frac{4}{\sqrt{15}}. \] (47)

Then the c-functions in (38) are given by

\[ c_1 = -\left( 3 \left( -54 + 135 e^2 \sqrt{5} \phi - 708 e^2 \sqrt{7} \phi - 342 e^4 \sqrt{2} \phi + 300 e^{8 \phi} \\
- 200 e^{9 \phi} + 435 e^{14 \phi} + 300 e^{16 \phi} + 80 e^{20 \phi} \right) \right) / \\
\left( 8 \left( 3 + 4 e^2 \sqrt{5} \phi - 10 e^{\sqrt{15} \phi} \right)^2 \right) \left( -6 + 15 e^2 \sqrt{5} \phi - 11 e^2 \sqrt{7} \phi + 10 e^{\sqrt{15} \phi} + 10 e^{10 \phi} \right), \] (48)
\[ c_2 = \left( 3 \left( -1458 + 3645 e^{2 \sqrt{3} \phi} ight) -113400 e^4 \sqrt{3} \phi -122472 e^6 \sqrt{3} \phi +964604 e^8 \sqrt{3} \phi \\
-433680 e^{12} \sqrt{3} \phi +8142320 e^{14} \sqrt{3} \phi -7571168 e^{16} \sqrt{3} \phi +3650880 e^{18} \sqrt{3} \phi +64000 e^{22} \sqrt{3} \phi -84078 e^{2} \sqrt{3} \phi \\
-5000328 e^{12} \sqrt{3} \phi +595593 e^{14} \sqrt{3} \phi -109512 e^{8} \sqrt{3} \phi +826240 e^{4} \sqrt{3} \phi +226710 e^{16} \sqrt{3} \phi -2517840 e^{22} \sqrt{3} \phi +4132080 e^{26} \sqrt{3} \phi \\
+7350984 e^{28} \sqrt{3} \phi -15584040 e^{32} \sqrt{3} \phi -8724096 e^{36} \sqrt{3} \phi +24740364 e^{38} \sqrt{3} \phi -18805608 e^{42} \sqrt{3} \phi -1102320 e^{46} \sqrt{3} \phi \\
-1010400 e^{50} \sqrt{3} \phi +191896 e^{2} \sqrt{3} \phi -826240 e^{4} \sqrt{3} \phi \right) / \]
\[ \left( 16 \left( 3 + 4 e^2 \sqrt{3} \phi - 10 e^{4 \sqrt{3} \phi} \right)^2 \right) \]
\[ \left( 9 + 18 e^2 \sqrt{3} \phi + 40 e^4 \sqrt{3} \phi + 18 e^{4 \sqrt{3} \phi} + 40 e^{8 \sqrt{3} \phi} - 44 e^{10 \sqrt{3} \phi} \right)^2 \]
\[ \left( -6 + 15 e^2 \sqrt{3} \phi - 11 e^2 \sqrt{3} \phi + 10 e^{4 \sqrt{3} \phi} + 10 e^{10 \sqrt{3} \phi} \right) \right) . \]  

Here we have chosen

\[ V = -2 . \]  

The behaviors of \( c \) in (42) for \( D = 3, N = 1 \) model and \( c_1 \) in (48) and \( c_2 \) in (49) for \( D = 5, N = 1 \) model are drawn in Figs.1.

For any \( c \)-function, there appear singularities at finite negative and positive values of \( \phi \). The \( c \)-function \( c \) in Fig.1 diverges negatively for finite negative \( \phi \) and diverges positively for finite positive \( \phi \). Between the singularities, \( c \) is the monotonically increasing function of \( \phi \), i.e. it is realistic. The \( c \)-functions \( c_1 \) in Fig.2 and \( c_2 \) in Fig.3 are almost identical with each other but their behavior is rather different from that of \( c \) in Fig.1. Both of \( c_1 \) and \( c_2 \) have maxima at \( \phi = 0 \) and they diverge negatively at finite negative and positive values of \( \phi \). The \( c \)-functions \( c_1 \) and \( c_2 \) decrease monotonically with respect to the absolute value of \( \phi \) between the singularities. Then, they indeed may serve as \( c \)-function in this region. In the solutions in [31], the
Figure 1: \( \phi \) (horizontal axis) versus \( c \) (vertical axis) for \( D = 3, \mathcal{N} = 1 \) model.

Figure 2: \( \phi \) (horizontal axis) versus \( \frac{c_1}{24\pi} = h_1 \) (vertical axis) for \( D = 5, \mathcal{N} = 1 \) model.
Figure 3: $\phi$ (horizontal axis) versus $\frac{c_2}{8\pi} = -h_2$ (vertical axis) for $D = 5$, $\mathcal{N} = 1$ model.

Cauchy horizon corresponds to $\phi = 0$. Therefore if the energy scale increases, the absolute value $|\phi|$ of $\phi$ increases. Therefore the $c$-functions should be decreasing functions with respect to the absolute value $|\phi|$ when $\phi$ is small. The $c$-functions $c_1$ and $c_2$ in Figs.2 and 3 seems to be consistent. In the solutions in [31], there always appear two separated branches corresponding to $\phi > 0$ or $\phi < 0$. The $c$-function $c$ in Fig.1 seems to be consistent only for the branch corresponding to $\phi < 0$.

Hence, we discussed the typical behaviour of candidate $c$-functions. However, it is not clear which role should play dilaton in above expressions as holographic RG coupling constant in dual QFT. It could be mass, quantum fields or coupling constants, but the explicit rule with what it should be identified is absent. The big number of usual RG parameters in dual QFT suggests also that there should be considered gauged SG with few scalars. However, the corresponding expression for anomaly looks awkful already in single scalar case.

4 Surface Counterterms and Finite Action

As well-known, we need to add the surface terms to the bulk action in order to have the well-defined variational principle. Under the variation of the
metric $\hat{G}^{\mu\nu}$ and the scalar field $\phi$, the variation of the action (1) is given by

$$\delta S = \delta S_{M_{d+1}} + \delta S_{M_d}$$

$$\delta S_{M_{d+1}} = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1} x \sqrt{-G} \left[ \delta \hat{G}^{\zeta\xi} \left\{ -\frac{1}{2} G_{\zeta\xi} \{ \hat{R} \\
+ (X(\phi) - Y'(\phi)) (\nabla\phi)^2 + \Phi(\phi) + 4A^2 \} + \hat{R}_{\zeta\xi} + (X(\phi) - Y''(\phi)) \partial_\zeta \phi \partial_\xi \phi \right\} + \delta \phi \{ (X'(\phi) - Y''(\phi)) (\nabla\phi)^2 + \Phi'(\phi) \\
- \frac{1}{\sqrt{-G}} \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} (X(\phi) - Y'(\phi)) \partial_\nu \phi \right) \left\} \right] .$$

$$\delta S_{M_d} = \frac{1}{16\pi G} \int_{M_d} d^dx \sqrt{-\hat{g}} n_\mu \left[ \partial^{\mu} \left( \hat{G}_{\xi\nu} \delta \hat{G}^{\xi\nu} \right) - D_\nu \left( \delta \hat{G}^{\mu\nu} \right) + Y(\phi) \partial^{\mu} (\delta \phi) \right] .$$

Here $\hat{g}_{\mu\nu}$ is the metric induced from $\hat{G}_{\mu\nu}$ and $n_\mu$ is the unit vector normal to $M_d$. The surface term $\delta S_{M_d}$ of the variation contains $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$ and $n^\xi \partial_\mu (\delta \phi)$, what makes the variational principle ill-defined. In order that the variational principle is well-defined on the boundary, the variation of the action should be written as

$$\delta S_{M_d} = \lim_{\rho \to 0} \int_{M_d} d^dx \sqrt{-\hat{g}} \left[ \delta \hat{G}^{\xi\nu} \{ \cdots \} + \delta \phi \{ \cdots \} \right]$$

after using the partial integration. If we put $\{ \cdots \} = 0$ for $\{ \cdots \}$ in (52), one could obtain the boundary condition. If the variation of the action on the boundary contains $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$ or $n^\xi \partial_\mu (\delta \phi)$, however, we cannot partially integrate it on the boundary in order to rewrite the variation in the form of (52) since $n_\mu$ expresses the direction perpendicular to the boundary. Therefore the “minimum” of the action is ambiguous. Such a problem was well studied in [10] for the Einstein gravity and the boundary term was added to the action. It cancels the term containing $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$. We need to cancel also the term containing $n^\mu \partial_\mu (\delta \phi)$. Then one finds the boundary term [21]

$$S_{b}^{(1)} = -\frac{1}{8\pi G} \int_{M_d} d^dx \sqrt{-\hat{g}} \left[ D_\mu n^\mu + Y(\phi) n_\mu \partial^\mu \phi \right] .$$

We also need to add surface counterterm $S_{b}^{(2)}$ which cancels the divergence coming from the infinite volume of the bulk space, say AdS. In order to
investigate the divergence, we choose the metric in the form (5). In the parametrization (5), \( n^\mu \) and the curvature \( R \) are given by

\[
\begin{align*}
n^\mu &= \left( \frac{2\rho}{l}, 0, \ldots, 0 \right) \\
R &= \rho R + \frac{3\rho^2}{l^2} \hat{g}^{ij} \hat{g}^{kl} \hat{g}_{ik} \hat{g}_{jl} - \frac{4\rho^2}{l^2} \hat{g}^{ij} \hat{g}_{ij} - \frac{\rho^2}{l^2} \hat{g}^{ij} \hat{g}^{kl} \hat{g}_{ij} \hat{g}_{kl}.
\end{align*}
\]

Expanding \( g_{ij} \) and \( \phi \) with respect to \( \rho \) as in (7), we find the following expression for \( S + S_b^{(1)} \):

\[
S + S_b^{(1)} = \frac{1}{16\pi G} \lim_{\rho \to 0} \int d^d x l \rho^\frac{d}{2} \sqrt{-\hat{g}} \left[ \frac{2 - 2d}{l^2} - \frac{1}{d} \Phi(\phi_0) \\
+ \rho \left\{ -\frac{1}{d-2} R(0) - \frac{1}{l^2} \hat{g}_{ij}(0) \hat{g}_{ij} \\
- \frac{1}{d-2} \left( X(\phi(0)) \left( \nabla(0) \phi(0) \right)^2 + Y(\phi(0)) \Delta g(0) \\
+ \Phi'(\phi(0)) \phi(1) \right) \right\} + O(\rho^2) \right].
\]

Then for \( d = 2 \)

\[
S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \sqrt{-\hat{g}} \left[ \frac{2}{l} + \frac{1}{2} \Phi(\phi) \right]
\]

and for \( d = 3, 4, \)

\[
S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \left[ \sqrt{-\hat{g}} \left\{ \frac{2d - 2}{l} \Phi(\phi) + \frac{l}{d-2} R + \frac{2l}{d(d-2)} \Phi(\phi) \\
+ \frac{l}{d-2} \left( X(\phi) \left( \nabla(0) \phi(0) \right)^2 + Y(\phi) \Delta g(0) \right) \right\} - \frac{l^2}{d(d-2)} n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \Phi(\phi) \right) \right].
\]

Here we used

\[
\sqrt{-\hat{g}} \Phi(\phi) = \rho^\frac{d}{2} \sqrt{-g(0)} \left\{ \Phi(\phi(0)) \\
+ \rho \left\{ \frac{1}{2} \hat{g}_{ij}(0) \hat{g}_{ij}(1) \Phi(\phi(0)) + \Phi'(\phi(0)) \phi(1) \right\} + O(\rho^2) \right\}
\]

\[
n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \Phi(\phi) \right) = \frac{2}{l} \rho^\frac{d}{2} \sqrt{-g(0)} \left\{ -\frac{d}{2} \Phi(\phi(0)) \\
+ \rho \left( 1 - \frac{d}{2} \right) \left\{ \frac{1}{2} \hat{g}_{ij}(0) \hat{g}_{ij}(1) \Phi(\phi(0)) + \Phi'(\phi(0)) \phi(1) \right\} + O(\rho^2) \right\}.
\]
Note that $S_b^{(2)}$ in (56) or (57) is only given in terms of the boundary quantities except the last term in (57). The last term is necessary to cancel the divergence of the bulk action and it is, of course, the total derivative in the bulk theory:

$$\int d^d x n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \Phi(\phi) \right) = \int d^{d+1} x \sqrt{-\hat{G}} \Box \Phi(\phi) \ . \quad (59)$$

Thus we got the boundary counterterm action for gauged SG. Using these local surface counterterms as part of complete action one can show explicitly that bosonic sector of gauged SG in dimensions under discussion gives finite action in asymptotically AdS space. The corresponding example will be given in next section.

Recently the surface counterterms for the action with the dilaton (scalar) potential are discussed in [19]. Their counterterms seem to correspond to the terms cancelling the leading divergence when $\rho \to 0$ in (55). However, they seem to have only considered the case where the dilaton becomes asymptotically constant $\phi \to \phi_0$. If we choose $\phi_0 = 0$, the total dilaton potential including the cosmological term $V_{\text{dilaton}}(\phi) \equiv 4\lambda^2 + \Phi(\phi)$ approaches to $V_{\text{dilaton}}(\phi) \to 4\lambda^2 = d(d-1)/l^2$. Then if we only consider the leading $\rho$ behavior and the asymptotically constant dilaton, the counterterm action in (56) and/or (57) has the following form

$$S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \sqrt{-\hat{g}} \left( \frac{2d - 2}{l} \right) , \quad (60)$$

which coincides with the result in [19] when the spacetime is asymptotically AdS.

Let us turn now to the discussion of deep connection between surface counterterms and holographic conformal anomaly. It is enough to mention only $d = 4$. In order to control the logarithmically divergent terms in the bulk action $S$, we choose $d - 4 = \epsilon < 0$. Then

$$S + S_b = \frac{1}{\epsilon} S_{\text{in}} + \text{finite terms} \ . \quad (61)$$

Here $S_{\text{in}}$ is given in (14). We also find

$$g^{ij}_{(0)} \frac{\delta}{\delta g_{(0)}^j} S_{\text{in}} = -\frac{\epsilon}{2} L_{\text{in}} + \mathcal{O} \left( \epsilon^2 \right) \ . \quad (62)$$
Here $\mathcal{L}_{in}$ is the Lagrangian density corresponding to $S_{in}: S_{in} = \int d^{d+1}L_{in}$. Then combining (61) and (62), we obtain the trace anomaly:

$$T = \lim_{\epsilon \to 0} \frac{2g^{ij}}{\sqrt{-g(0)}} \frac{\delta(S + S_b)}{\delta g_{(0)}^{ij}} = -\frac{1}{2}L_{in}, \quad (63)$$

which is identical with the result found in (8). If we use the equations of motion (15), (16), (17) and (18), we finally obtain the expression (22) or (103). Hence, we found the finite gravitational action (for asymptotically AdS spaces) in 5 dimensions by adding the local surface counterterm. This action correctly reproduces holographic trace anomaly for dual (gauge) theory. In principle, one can also generalize all results for higher dimensions, say, d6, etc. With the growth of dimension, the technical problems become more and more complicated as the number of structures in boundary term is increasing.

5 Dilatonic AdS Black Hole and its Mass

Let us consider the black hole or “throat” type solution for the equations of the motion (2) and (3) when $d = 4$. The surface term (57) may be used for calculation of the finite black hole mass and/or other thermodynamical quantities.

For simplicity, we choose

$$X(\phi) = \alpha \text{ (constant)} , \quad Y(\phi) = 0 \quad (64)$$

and we assume the spacetime metric in the following form:

$$ds^2 = -e^{2\rho} dt^2 + e^{2\sigma} dr^2 + r^2 \sum_{i=1}^{d-1} (dx^i)^2 \quad (65)$$

and $\rho$, $\sigma$ and $\phi$ depend only on $r$. The equations (2) and (3) can be rewritten in the following form:

$$0 = e^{\rho+\sigma} \Phi'(\phi) - 2\alpha \left(e^{\rho-\sigma} \phi'\right)'$$

$$0 = -\frac{1}{3} e^{2\rho} \left(\Phi(\phi) + \frac{12}{l^2}\right) + \left(\rho'' + (\rho')^2 - \rho' \sigma' + \frac{3\rho'}{r}\right) e^{2\rho-2\sigma} \quad (67)$$

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\begin{align}
0 &= \frac{1}{3}e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) - \rho'' - (\rho')^2 + \rho'\sigma' + \frac{3\sigma'}{r} + \alpha (\phi')^2 
\label{eq:68}
0 &= \frac{1}{3}e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) r^2 + k + \{ r (\sigma' - \rho') - 2 \} e^{-2\sigma} .
\label{eq:69}
\end{align}

Here $' \equiv \frac{d}{dr}$. If one defines new variables $U$ and $V$ by

$$U = e^{\sigma} \, , \quad V = r^2 e^{\phi - \sigma} ,$$

we obtain the following equations from (66-69):

\begin{align}
0 &= r^3 \Phi'(\phi) - 2\alpha (rV \phi')' \label{eq:71}
0 &= \frac{1}{3}e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) r^3 U + kr - V' \label{eq:72}
0 &= \frac{3U'}{rU} + \alpha (\phi')' . \label{eq:73}
\end{align}

We should note that only three equations in (66-69) are independent. There is practical problem in the construction of AdS BH with non-trivial dilaton, especially for arbitrary dilatonic potential. That is why we use below the approximate technique which was developed in ref.[32] for constant dilatonic potential.

When $\Phi(0) = \Phi'(0) = \phi = 0$, a solution corresponding to the throat limit of D3-brane is given by

$$U = 1 \, , \quad V = V_0 \equiv \frac{r^4}{l^2} - \mu .$$

In the following, we use large $r$ expansion and consider the perturbation around (74). It is assumed

$$\Phi(\phi) = \bar{\mu} \phi^2 + \mathcal{O} (\phi^3) . \label{eq:75}$$

Then one can neglect the higher order terms in (75). We obtain from (71)

$$0 \sim \bar{\mu} r^3 \phi + \alpha \left( \frac{r^5}{l^2} \phi' \right)' . \label{eq:76}$$

The solution of eq.(76) is given by

$$\phi = cr^{-\beta} , \ (c \text{ is a constant}) , \ \beta = 2 \pm \sqrt{4 - \frac{\bar{\mu}l^2}{\alpha}} . \label{eq:77}$$
Consider \( r \) is large or \( c \) is small, and write \( U \) and \( V \) in the following form:

\[
U = 1 + c^2 u, \quad V = V_0 + c^2 v.
\] (78)

Then from (72) and (73), one gets

\[
u = u_0 + \frac{\alpha \beta}{6} r^{-2 \beta}, \quad v = v_0 - \frac{\bar{\mu}(\beta - 6)}{6(\beta - 4)(\beta - 2)} r^{-2 \beta + 4}.
\] (79)

Here \( u_0 \) and \( v_0 \) are constants of the integration. Here we choose

\[
v_0 = u_0 = 0.
\] (80)

The horizon which is defined by

\[
V = 0
\] (81)

lies at

\[
r = r_h \equiv l^2 \mu^{\frac{1}{2}} + c^2 \frac{\bar{\mu}(\beta - 6)l^{\frac{2}{\beta}} - \beta \mu^{\frac{1}{2} - \frac{\beta}{2}}}{24(\beta - 4)(\beta - 2)}.
\] (82)

And the Hawking temperature is

\[
T = \frac{1}{4\pi} \left[ \frac{1}{r^2} \frac{dV}{dr} \right]_{r=r_h} = \frac{1}{4\pi} \left\{ 4l^{-\frac{3}{2}} \mu^{\frac{1}{2}} + c^2 \frac{\bar{\mu}(\beta - 6)(2\beta - 3)l^{\frac{2}{\beta} - \beta} - \beta \mu^{\frac{1}{2} - \frac{\beta}{2}}}{6(\beta - 4)(\beta - 2)} \right\}.
\] (83)

We now evaluate the free energy of the black hole within the standard prescription [33, 34]. The free energy \( F \) can be obtained by substituting the classical solution into the action \( S \):

\[
F = TS.
\] (84)

Here \( T \) is the Hawking temperature. Using the equations of motion in (2) \((X = \alpha, Y = 0, 4\lambda^2 = \frac{12}{l^2})\), we obtain

\[
0 = \frac{5}{3} \left( \Phi(\phi) + \frac{12}{l^2} \right) + \hat{R} + \alpha (\nabla \phi)^2.
\] (85)
Substituting (85) into the action (1) after Wick-rotating it to the Euclid signature

$$S = \frac{1}{16\pi G} \cdot \frac{2}{3} \int_{M_5} d^5\sqrt{G} \left( \Phi(\phi) + \frac{12}{T^2} \right) = \frac{1}{16\pi G} \cdot \frac{2V(3)}{3} \int_{r_h}^{\infty} d\ell^3 U \left( \Phi(\phi) + \frac{12}{T^2} \right). \quad (86)$$

Here $V(3)$ is the volume of the 3d space ($\int d^3x \cdots = \beta V(3) \int d\ell^3 \cdots$) and $\beta$ is the period of time, which can be regarded as the inverse of the temperature $T (\frac{1}{T})$. The expression (86) contains the divergence. We regularize the divergence by replacing

$$\int_{r_h}^{\infty} d\ell \to \int_{r_{\text{max}}}^{r_{\text{max}}} d\ell \quad (87)$$

and subtract the contribution from a zero temperature solution, where we choose $\mu = c = 0$, and the solution corresponds to the vacuum or pure AdS:

$$S_0 = \frac{1}{16\pi G} \cdot \frac{2}{3} \cdot \frac{12V(3)}{T^2} \sqrt{\frac{G_{tt}(r = r_{\text{max}}, \mu = c = 0)}{G_{tt}(r = r_{\text{max}})}} \int_{r_h}^{r_{\text{max}}} d\ell^3 r. \quad (88)$$

The factor $\sqrt{\frac{G_{tt}(r = r_{\text{max}}, \mu = c = 0)}{G_{tt}(r = r_{\text{max}})}}$ is chosen so that the proper length of the circles which correspond to the period $\frac{1}{T}$ in the Euclid time at $r_{\text{max}}$ coincides with each other in the two solutions. Then we find the following expression for the free energy,

$$F = \lim_{r_{\text{max}} \to \infty} T (S - S_0) = \frac{V(3)}{2\pi G T^2} \left[ -\frac{T^2 \mu}{8} + c^2 \mu^{1 - \frac{2}{\beta}} \left\{ \frac{(\beta - 1)}{12\beta(\beta - 4)(\beta - 2)} \right\} + \cdots \right]. \quad (89)$$

Here we assume $\beta > 2$ or the expression $S - S_0$ still contains the divergences and we cannot get finite results. However, the inequality $\beta > 2$ is not always satisfied in the gauged supergravity models. In that case the expression in (89) would not be valid. One can express the free energy $F$ in (89) in terms of the temperature $T$ instead of $\mu$:

$$F = \frac{V(3)}{16\pi G} \left[ -\pi T^4 l^6 + c^2 l^{8-4\beta} T^{4-2\beta} \tilde{\mu} \left( \frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{6\beta(\beta - 4)(\beta - 2)} \right) + \cdots \right]. \quad (90)$$
Then the entropy $S$ and the energy (mass) $E$ is given by

\[
S = V(3) \left[ \frac{4\pi T^3 l^6}{16\pi G} + c^2 l^8 T^3 \mu \left( \frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{3\beta(\beta - 4)} \right) + \cdots \right]
\]

\[
E = F + TS = V(3) \left[ \frac{3\pi T^4 l^6}{16\pi G} + c^2 l^8 \left( \pi T^4 \right)^{1/2} \mu \left( \frac{(2\beta - 3)(2\beta^3 - 15\beta^2 + 22\beta - 4)}{6\beta(\beta - 4)(\beta - 2)} \right) + \cdots \right]
\]

We now evaluate the mass using the surface term of the action in (57), i.e. within local surface counterterm method. The surface energy momentum tensor $T_{ij}$ is now defined by ($d = 4$)

\[
\delta S_b^{(2)} = \sqrt{-\delta g} \delta \phi T_{ij}
\]

\[
= \frac{1}{16\pi G} \left[ \sqrt{-\delta g} \delta g^{ij} \left\{ -\frac{1}{2} g_{ij} \left( \frac{6}{l} + \frac{l}{2} \hat{R} + \frac{l}{4} \Phi(\phi) \right) \right\} \right.
\]

\[
+ \frac{l^2}{4} n^\mu \partial_\mu \left\{ \sqrt{-\delta g} \delta g^{ij} \hat{g}_{ij} \Phi(\phi) \right\} .
\]

Note that the energy-momentum tensor is still not well-defined due to the term containing $n^\mu \partial_\mu$. If we assume $\delta \hat{g}^{ij} \sim O(\rho^a)$ for large $\rho$ when we choose the coordinate system (5), then

\[
n^\mu \partial_\mu \left( \delta \hat{g}^{ij} \right) \sim \frac{2}{l} \delta \hat{g}^{ij} (a_1 + \partial_\rho) (\cdot) .
\]

Or if $\delta \hat{g}^{ij} \sim O(r^{a_2})$ for large $r$ when we choose the coordinate system (65), then

\[
n^\mu \partial_\mu \left( \delta \hat{g}^{ij} \right) \sim \delta \hat{g}^{ij} e^\sigma \left( \frac{a_2}{r} + \partial_r \right) (\cdot) .
\]

As we consider the black hole-like object in this section, one chooses the coordinate system (65) and assumes Eq.(94). Then mass $E$ of the black hole like object is given by

\[
E = \int d^{d-1}x \sqrt{\sigma} N \delta T_{tt} \left( u^t \right)^2 .
\]
Here $\delta T_{tt}$ is the difference of the $(t, t)$ component of the energy-momentum tensor in the spacetime with black hole like object from that in the reference spacetime, which we choose to be AdS, and $u^i$ is the $t$ component of the unit time-like vector normal to the hypersurface given by $t =$constant. By using the solution in (78) and (79), the $(t, t)$ component of the energy-momentum tensor in (92) has the following form:

$$T_{tt} = \frac{3}{16\pi G} r^2 \left[ 1 - \frac{\ell^2 \mu}{r^4} + \ell^2 \bar{\mu} e^2 \left( \frac{1}{12} - \frac{1}{6\beta(\beta - 6)} \right) - \frac{\beta - 6}{6(\beta - 4)(\beta - 2)} - \frac{(3 - \beta)(1 + a_2)}{12} \right] r^{-2\beta} + \ldots . \quad (96)$$

If we assume the mass is finite, $\beta$ should satisfy the inequality $\beta > 2$, as in the case of the free energy in (89) since $\sqrt{\sigma N} (u^i)^2 = hr^2$ for the reference AdS space. Then the $\beta$-dependent term in (96) does not contribute to the mass and one gets

$$E = \frac{3\mu V_{(3)}}{16\pi G}. \quad (97)$$

Using (83)

$$E = \frac{3\ell^6 V_{(3)} \pi T^4}{16\pi G} \left( 1 - c^2 \tilde{\mu} l^2^{-4\beta} \left( \pi T^4 \right)^{-\frac{\beta}{\rho}} \frac{(\beta - 6)(2\beta - 3)}{(\beta - 4)(\beta - 2)} \right), \quad (98)$$

which does not agree with the result in (91). This might express the ambiguity in the choice of the regularization to make the finite action. A possible origin of it might be following. We assumed $\phi$ can be expanded in the (integer) power series of $\rho$ in (7) when deriving the surface terms in (57). However, this assumption seems to conflict with the classical solution in (77), where the fractional power seems to appear since $r^2 \sim \frac{1}{\rho}$. In any case, in QFT there is no problem in regularization dependence of the results. In many cases (see example in ref.[17]) the explicit choice of free parameters of regularization leads to coincidence of the answers which look different in different regularizations. As usually happens in QFT the renormalization is more universal as the same answers for beta-functions may be obtained while using different regularizations. That suggests that holographic renormalization group should be developed and the predictions of above calculations should be tested in it.

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6 Discussion

In summary, we constructed surface counterterm for gauged supergravity with single scalar and arbitrary scalar potential in three and five dimensions. As a result, the finite gravitational action and consistent stress tensor in asymptotically AdS space is found. Using this action, the regularized expressions for free energy, entropy and mass are derived for d5 dilatonic AdS black hole. From another side, finite action may be used to get the holographic conformal anomaly of boundary QFT with broken conformal invariance. Such conformal anomaly is calculated from d5 and d3 gauged SG with arbitrary dilatonic potential with the use of AdS/CFT correspondence. Due to dilaton dependence it takes extremely complicated form. Within holographic RG where identification of dilaton with some coupling constant is made, we suggested the candidate c-function for d2 and d4 boundary QFT. The numerical behaviour of c-function for a number of explicit dilatonic potentials following from M-theory is given.

We expect that our results may be very useful in explicit identification of supergravity description (special RG flow) with the particular boundary gauge theory (or its phase) what is very non-trivial task in AdS/CFT correspondence. We show that on the example of constant dilaton and special form of dilatonic potential where qualitative agreement of holographic conformal anomaly and QFT conformal anomaly (with the account of radiative corrections) from QED-like theory with single coupling constant may be achieved.

Our work may be extended in various directions. First of all, we can consider large number of scalars, say 42 as in $N = 8$ d5 SG, and construct the corresponding Weyl anomaly from the bulk side. However, this is technically very complicated problem as even in case of single scalar the complete answer for d4 anomaly takes few pages. The calculation of surface counterterm in d5 gauged SG with many scalars is slightly easier task. However, again the application of surface counterterm for the derivation of regularized thermodynamical quantities in multi-scalar AdS black holes (when they will be constructed) is complicated. Second, the generalization of surface counterterm for higher dimensions (say, $d = 7, 9$) is possible. Third, in general the extension of AdS/CFT set-up to non-conformal boundary theories is challenging problem. In this respect, better investigation of candidate c-functions from bulk and from boundary is required as well as their comparison in all
detail. The related question is bulk calculation of Casimir effect in the presence of dilaton and comparison of it with QFT result, including radiative corrections.

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A Coefficients of conformal anomaly

In this appendix, we give the explicit values of the coefficients appeared in the calculation of $d = 4$ conformal anomaly.

Substituting (16) into (15), we obtain

$$g_{(1)ij} = \tilde{c}_1 R_{ij} + \tilde{c}_2 g_{ij} R + \tilde{c}_3 g_{ij} g^{kl} \partial_k \phi \partial_l \phi$$

$$+ \tilde{c}_4 g_{ij} \frac{\partial_k}{\sqrt{-g}} \left( \sqrt{-g} g^{kl} \partial_l \phi \right) + \tilde{c}_5 \partial_i \phi \partial_j \phi$$

(99)

$$\tilde{c}_1 = -\frac{3}{6 + \Phi}$$

$$\tilde{c}_2 = -\frac{3 \{ \Phi'^2 - 6 (\Phi'' + 8 V) \}}{2 (6 + \Phi) \{-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V)\}}$$

$$\tilde{c}_3 = \frac{-3 \Phi'^2 V + 18 V (\Phi'' + 8 V) - 2 (6 + \Phi) \Phi' V'}{2 (6 + \Phi) \{-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V)\}}$$

$$\tilde{c}_4 = -\frac{2 \Phi' V}{-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V)}$$

$$\tilde{c}_5 = -\frac{V}{2 + \frac{2}{3}}.$$  

(100)

Further, substituting (16) and (99) into (18), we obtain

$$\phi_{(2)} = d_1 R^2 + d_2 R_{ij} R^{ij} + d_3 R^{ij} \partial_i \phi \partial_j \phi$$

$$+ d_4 R g^{ij} \partial_i \phi \partial_j \phi + d_5 R \frac{1}{\sqrt{-g}} \partial_l \left( \sqrt{-g} g^{ij} \partial_l \phi \right)$$

$$+ d_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + d_7 \left( \frac{1}{\sqrt{-g}} \partial_l \left( \sqrt{-g} g^{ij} \partial_l \phi \right) \right)^2$$

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\[ +d_k g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}g^{ij} \partial_j \phi) \]  

(101)

\[ d_1 = - \left[ 9 \Phi' \left\{ 2 (12 + \Phi) \Phi'^4 - ( - 864 + 36 \Phi + 24 \Phi^2 + \Phi^3) \Phi'' \right. \right. \]
\[ + 192 (12 + \Phi)^2 \Phi'' V + 64 (2592 + 612 \Phi + 48 \Phi^2 + \Phi^3) V^2 \]
\[ - 2 \Phi^2 \left( (216 + 30 \Phi + \Phi^2) \Phi'' + 144 (10 + \Phi) V \right) \]
\[ \left. + (6 + \Phi)^2 (24 + \Phi) \Phi' (\Phi'' + 8 V') \right\} \] / \[ 8 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \]
\[ \times \left\{ -2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \]

\[ d_2 = \frac{9 (12 + \Phi) \Phi'}{4 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \}

\[ d_3 = \frac{3 (3 (12 + \Phi) \Phi' V - 2 (144 + 30 \Phi + \Phi^2) V'')}{2 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \}

\[ d_4 = \frac{(3 (-6 (12 + \Phi) \Phi + 6 (108 + 24 \Phi + \Phi^2) \Phi'^4 V'' \right. \right. \]
\[ + 4 (2592 + 684 \Phi + 48 \Phi^2 + \Phi^3) (\Phi'' + 8 V) (9 + \Phi) \Phi'' \]
\[ + 4 (12 + \Phi) V') V'' - (6 + \Phi) \Phi'^2 (3 (144 + 30 \Phi + \Phi^2) \Phi'' V \]
\[ + (1980 \Phi'' + 216 \Phi \Phi'' + 5 \Phi^2 \Phi'' + 27360 V + 4176 \Phi V \]
\[ + 128 \Phi^2 V) V') + 2 \Phi'^3 (3 (216 + 30 \Phi + \Phi^2) \Phi'' V \]
\[ - 2 \left( -2160 V^2 - 216 \Phi V^2 + 864 V'' + 324 \Phi V'' \right. \]
\[ + 36 \Phi^2 V'' + \Phi^3 V'')) + \Phi' (3 (-864 + 36 \Phi + 24 \Phi^2 + \Phi^3) \Phi'' V \]
\[ + 2 \Phi'' \left( -41472 V^2 - 6912 \Phi V^2 - 288 \Phi^2 V^2 + 15552 V'' \right. \]
\[ + 6696 \Phi V + 972 \Phi^2 V'' + 54 \Phi^3 V'' + \Phi^4 V'' \right. \]
\[ - 2 (248832 V^3 + 58752 \Phi V^3 + 4608 \Phi^2 V^3 \]
\[ + 96 \Phi^3 V^3 + 15552 \Phi'' V' + 6696 \Phi \Phi'' V'' + 972 \Phi^2 \Phi'' V'' \]
\[ + 54 \Phi^3 \Phi'' V' + 4 \Phi^4 \Phi'' V'' + 124416 V'^2 + 53568 \Phi V'^2 \]
\[ + 7776 \Phi^2 V'^2 + 432 \Phi^3 V'^2 + 8 \Phi^4 V'^2 - 124416 V V'' \]
\[ - 53568 \Phi V V'' - 7776 \Phi^2 V V'' - 432 \Phi^3 V V'' \]
\[ - 8 \Phi^4 V V'')) \left(4 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} (-2 \Phi'^2 \right. \]
\[ \left. + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \) \]
\[ d_5 = \frac{-3 \Phi^4 V + 2 (432 + 42 \Phi + \Phi^2) \Phi V (\Phi' + 8 V) + \Phi^2 V (6 + \Phi) \Phi'' - 8 (162 + 7 \Phi) V - 4 (24 + \Phi) \Phi^3 V'}{-2 (432 + 42 \Phi + \Phi^2) \Phi' (\Phi'' V - \Phi'' V')) / (2 \Phi^2 - (24 + \Phi) \Phi'') (-2 \Phi^2 + (18 + \Phi) (\Phi'' + 8 V)) \] 

\[ d_6 = \frac{-(54 (12 + \Phi) \Phi V^2 + 12 (828 + 168 \Phi + 5 \Phi^2) \Phi^4 V V') + 4 (2592 + 684 \Phi + 48 \Phi^2 + \Phi^3) V' (54 \Phi'' V + 4608 V^3 + 192 \Phi V^3 + 108 \Phi'' V' + 24 \Phi \Phi'' V') + \Phi^2 \Phi'' V' + 864 V^2 + 192 \Phi V^2 + 8 \Phi^2 V^2 - 1728 V V'' - 384 \Phi V V'' - 16 \Phi^2 V V'' + 2 \Phi'' (504 V^2 + 12 \Phi V^2 - 108 V'' - 24 \Phi V'' - \Phi^2 V'')) + (6 + \Phi) \Phi^2 (9 (144 + 30 \Phi + \Phi^2) \Phi'' V^2 - 2 V'' (14796 \Phi'' V + 1368 \Phi'' V + 33 \Phi'' V) + 88992 V^2 + 4680 \Phi V^2 + 36 \Phi^2 V^2 - 20736 V'' - 5472 \Phi V'' - 384 \Phi^2 V'' - 8 \Phi^3 V'')) + 2 \Phi^3 (27 (216 + 30 \Phi + \Phi^2) \Phi'' V^2 + 4 (12312 V^3 + 1836 \Phi V^3 + 72 \Phi^2 V^3 + 2376 V^2 + 864 \Phi V^2 + 90 \Phi^2 V^2 + 2 \Phi^3 V^2 + 2592 V V'' + 972 \Phi V V'' + 108 \Phi^2 V V'' + 3 \Phi V V'')) - \Phi' (27 (2304 + 516 \Phi + 40 \Phi^2 + \Phi^3) \Phi'' V^2 + 4 \Phi'' (217728 V^3 + 44064 \Phi V^3 + 3024 \Phi^2 V^3 + 72 \Phi^3 V^3 + 81648 V^2 + 34992 \Phi V^2 + 5040 \Phi^2 V^2 + 276 \Phi^3 V^2 + 5 \Phi^4 V^2 + 46656 V V'' + 20088 \Phi V V'' + 2916 \Phi^2 V V'' + 162 \Phi^3 V V'' + 3 \Phi^4 V V'') + 4 V (746496 V^3 + 129600 \Phi V^3 + 6912 \Phi^2 V^3 + 144 \Phi^3 V^3 - 46656 \Phi'' V' - 20088 \Phi \Phi'' V' - 2916 \Phi^2 \Phi'' V' - 162 \Phi^3 \Phi'' V' - 3 \Phi^4 \Phi'' V' - 404352 V^2 - 177984 \Phi V^2 - 26784 \Phi^2 V^2 - 1584 \Phi^3 V^2 - 32 \Phi^4 V^2 + 373248 V V'' + 160704 \Phi V V'' + 23328 \Phi^2 V V'' + 1296 \Phi^3 V V'' + 24 \Phi^4 V V'')) / (8 (6 + \Phi)^2 (-2 \Phi^2 + (24 + \Phi) \Phi'') (-2 \Phi^2 + (18 + \Phi) (\Phi'' + 8 V)) \] 

\[ d_7 = (2 V (36 \Phi^3 V - 3 (18 + \Phi) \Phi') V ((26 + \Phi) \Phi'' - 8 (18 + \Phi) V) + 4 (432 + 42 \Phi + \Phi^2) \Phi^2 V') \]
\[ \begin{align*}
&\frac{+(18 + \Phi)^2 \,(24 + \Phi) \,(\Phi'' \,V - 2 \,(\Phi' + 4 \,V) \,V')}{{(2 \,\Phi'^2 - (24 + \Phi) \,\Phi'') \,- \,2 \,\Phi'^2 + (18 + \Phi) \,(\Phi'' + 8 \,V))^2}})
\end{align*} \]
\[
d_8 = -(6 \,\Phi^4 \,V^2 - 4 \,(156 + 5 \,\Phi) \,\Phi^3 \,V \,V' - 2 \,(18 + \Phi) \,\Phi' \,V
\]
\[
(3 \,(24 + \Phi) \,\Phi'' \,V + (-276 \,\Phi'' - 11 \,\Phi' \,\Phi'' + 480 \,V + 32 \,\Phi \,V) \,V')
\]
\[
+2 \,(432 + 42 \,\Phi + \Phi^2) \,(3 \,\Phi'^2 \,V^2
\]
\[
+2 \,(18 + \Phi) \,V \,-\,(\Phi''' \,V' + 8 \,V \,V''
\]
\[
+2 \,\Phi'' \,(12 \,V^3 + 18 \,V'^2 + \Phi \,V'^2 + 18 \,V \,V'' + \Phi \,V \,V'')
\]
\[
+\Phi'^2 \,(3 \,(6 + \Phi) \,\Phi'' \,V^2 - 8 \,(486 \,V^3 + 21 \,\Phi \,V^3 + 432 \,V'^2
\]
\[
+42 \,\Phi \,V'^2 + \Phi^2 \,V'^2 + 432 \,V \,V'' + 42 \,\Phi \,V \,V'' + \Phi^2 \,V \,V'')
\]
\[
)/ \,(2 \,(2 \,\Phi'^2 - (24 + \Phi) \,\Phi'') \,-\,(2 \,\Phi'^2 + (18 + \Phi) \,(\Phi'' + 8 \,V))
\]
\]
\]
Substituting (16), (99) and (101) into (17), one gets
\[ \begin{align*}
g^{ij}g_{(2)ij} &= f_1 R^2 + f_2 R_{ij}R^{ij} + f_3 R^{ij} \partial_i \Phi \partial_j \Phi \\
&+ f_4 g^{ij} \partial_i \phi \partial_j \phi + f_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
&+ f_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + f_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
&+ f_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
&+ f_9 \left( 2 \,(\Phi^6 - 72 \,(12 + \Phi) \,\Phi'' \,(\Phi'' + 8 \,V)^2 \\
&-2 \,\Phi^4 \,(24 + \Phi) \,\Phi'' + 8 \,(18 + \Phi) \,V) \\
&+\Phi'^2 \,(324 + 12 \,\Phi - \Phi^2) \,\Phi'^2 \\
&+8 \,(540 + 48 \,\Phi + \Phi^2) \,\Phi'' \,V + 64 \,(180 + 24 \,\Phi + \Phi^2) \,V'^2 \\
&+(6 + \Phi)^2 \,\Phi^3 \,(\Phi'' + 8 \,V')) \right) / \\
&\left[ 2 \,(6 + \Phi)^2 \left\{ -2 \,\Phi'^2 + (24 + \Phi) \,\Phi' \right\} \\
&\times \left\{ -2 \,\Phi'^2 + (18 + \Phi) \,(\Phi'' + 8 \,V) \right\}^2 \right] \\
f_1 &= \frac{9 \,(\Phi'^2 - 6 \,\Phi'')}{(6 + \Phi)^2 \,\{ -2 \,\Phi'^2 + (24 + \Phi) \,\Phi'' \}} \\
f_2 &= \frac{6 \,(-3 \,\Phi'^2 \,V + 18 \,\Phi'' \,V + 2 \,(6 + \Phi) \,\Phi' \,V'')}{(6 + \Phi)^2 \,(-2 \,\Phi'^2 + (24 + \Phi) \,\Phi'')} \\
f_3 &= \frac{0}{(6 + \Phi)^2 \,(-2 \,\Phi'^2 + (24 + \Phi) \,\Phi'')} \\
&\end{align*} \]
33
\[ f_4 = -(3 \ ( -12 \Phi^6 \ V + 432 \ (12 + \Phi) \ \Phi'' \ V \ (\Phi'' + 8 \ V) V' + 8 \ (6 + \Phi) \ \Phi^5 \ V' + (6 + \Phi) \ \Phi' ((1044 + 168 \ \Phi + 7 \ \Phi^2) \ \Phi'' + 8 \ (1476 + 192 \ \Phi + 7 \ \Phi^2) \ \Phi'' V + 256 \ (216 + 30 \ \Phi + \Phi^2) \ V^2) V' + 2 \ (6 + \Phi) \ \Phi^4 \ (3 \ (6 + \Phi) \ \Phi''' V + 6 \ (\Phi'' + 8 \ V) V' + 192 \ \Phi \ \Phi''' V' + 912 \ V + 88 \ \Phi \ V) V' + +(6 \ \Phi'' + 3 \ \Phi \ \Phi'' + 192 \ \Phi \ V + 36 \ \Phi'' V + 2 \ (216 \ V^2 - 12 \ \Phi \ V^2 + 36 \ \Phi'' V + 30 \ \Phi^2 \ \Phi''' V + 3 \ (6 + \Phi) \ \Phi' (\Phi''' V - \Phi'' V')) \]
\[
(4608 V^3 + 192 \Phi V^3 + 108 \Phi'' V')
+24 \Phi \Phi'' V' + \Phi^2 \Phi'' V' + 864 V'^2
+192 \Phi V'^2 + 8 \Phi^2 V'^2 - 1728 V V'' - 384 \Phi V V'' - 16 \Phi^2 V V''\)
+6 \Phi^4 (9 (24 + \Phi) \Phi'' V'^2 + 4 (324 V^3 + 18 \Phi V^3
+36 V'^2 + 12 \Phi V'^2 + \Phi^2 V'^2 + 36 V V'' + 12 \Phi V V'' + \Phi^2 V V''))
-\Phi^2 (27 (396 + 36 \Phi + \Phi^2) \Phi'' V'^2 + 4 \Phi'' (29160 V^3
+2592 \Phi V^3 + 54 \Phi^2 V^3 + 4104 V'^2 + 1620 \Phi V V'^2 + 198 \Phi^2 V'^2
+7 \Phi^3 V'^2 + 1944 V V'' + 756 \Phi V V'' + 90 \Phi^2 V V'' + 3 \Phi^3 V V'')
+4 V (67392 V^3 + 6912 \Phi V^3 + 144 \Phi^2 V^3
-1944 \Phi'' V' - 756 \Phi \Phi'' V'
-90 \Phi^2 \Phi'' V' - 3 \Phi^3 \Phi'' V' - 5184 V'^2 - 2016 \Phi V'^2 - 240 \Phi^2 V'^2
-8 \Phi^3 V'^2 + 15552 V V'' + 6048 \Phi V V''
+720 \Phi^2 V V'' + 24 \Phi^3 V V''))/\\)
(2 (6 + \Phi)^2 (-2 \Phi^2 + (24 + \Phi) \Phi'') (-2 \Phi^2
+(18 + \Phi) (\Phi'' + 8 V))^2)
\]

\[
f_7 = -4 V (4 \Phi^4 V - 2 (78 + 5 \Phi) \Phi^2 \Phi'' V
+(18 + \Phi)^2 \Phi'' V (\Phi'' + 24 V)
+8 (18 + \Phi) \Phi^3 V'' + 2 \Phi^2 (\Phi'' V - 2 (\Phi'' + 4 V) V'))/\\)
((2 \Phi^2 - (24 + \Phi) \Phi'') (-2 \Phi^2 + (18 + \Phi) (\Phi'' + 8 V))^2)
\]

\[
f_8 = ( - 56 \Phi^4 V V' - 4 (18 + \Phi)^2 \Phi'' V (\Phi'' + 24 V) V'
-4 \Phi^2 V (3 (18 + \Phi) \Phi'' V
-(24 \Phi'' + 15 \Phi \Phi'' + 288 V + 16 \Phi V) V')
+2 \Phi^3 (9 \Phi'' V^2 - 8 (6 V^3 + 18 V'^2 + \Phi V'^2
+18 V V'' + \Phi V V'')) + \Phi'
(9 (10 + \Phi) \Phi'' V^2
+8 (18 + \Phi)^2 V (-\Phi'' V' + 8 V V'') - 8 \Phi'' (126 V^3 + 3 \Phi V^3
-324 V'^2 - 36 \Phi V'^2 - \Phi^2 V'^2
-324 V V'' - 36 \Phi V V'' - \Phi^2 V V'')))/\\)
((2 \Phi^2 - (24 + \Phi) \Phi'') (-2 \Phi^2 + (18 + \Phi) (\Phi'' + 8 V))^2) .
\]

Finally substituting (16), (99), (101) and (102) into the expression for the
anomaly (14), we obtain,

\[
T = -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} + h_3 R^i_{\ j} \partial_i \phi \partial_j \phi \\
+ h_4 R g^{ij} \partial_i \phi \partial_j \phi + h_5 R \frac{\partial_i}{\sqrt{-g}} \left( g^{ij} \partial_j \phi \right) \\
+ h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + h_7 \left( \frac{\partial_i}{\sqrt{-g}} \left( g^{ij} \partial_j \phi \right) \right)^2 \\
+ h_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i \left( g^{ij} \partial_j \phi \right) \right]
\]

(103)

\[
h_1 = \left[ 3 \left\{ (24 - 10 \Phi) \Phi^6 \\
+ (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' (\Phi'' + 8 \Phi V)^2 \\
+ 2 \Phi'^4 \left\{ (108 + 162 \Phi + 7 \Phi^2) \Phi'' + 72 (-8 + 14 \Phi + \Phi^2) \Phi V \right\} \\
- 2 \Phi'^2 \left\{ (6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi'' \Phi \\
+ 4 (11232 + 6156 \Phi + 552 \Phi^2 + 13 \Phi^3) \Phi'' \Phi V \right\} \\
+ 32 (-2592 + 468 \Phi + 96 \Phi^2 + 5 \Phi^3) \Phi'' \Phi^2 \right\} \right] / \\
\left[ 16 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \right]
\]

\[
h_2 = -3 \left\{ (12 - 5 \Phi) \Phi'^2 (288 + 72 \Phi + \Phi^2) \Phi'' \right\} / \\
8 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\}
\]

\[
h_3 = -(3 ((12 - 5 \Phi) \Phi'^2 V + (288 + 72 \Phi + \Phi^2) \Phi'' V \\
+ 2 (-144 - 18 \Phi + \Phi^2) \Phi' V'))/ \\
(4 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') \right)
\]

\[
h_4 = (-6 (-12 + 5 \Phi) \Phi^6 V \\
+ 3 (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' V (\Phi'' + 8 \Phi V)^2 \\
+ 2 (-684 - 48 \Phi + 11 \Phi^2) \Phi^6 V' \\
+ (6 + \Phi) \Phi' ((-31104 - 2772 \Phi + 120 \Phi^2 + 13 \Phi^3) \Phi'' \Phi \\
+ 8 (-62208 - 7092 \Phi - 132 \Phi^2 + 7 \Phi^3) \Phi'' V \right)
\]

36
\[-(6 + \Phi) \Phi'^3 (9 (-144 - 18 \Phi + \Phi^2)) \Phi'' V
\]
\[+ ( -3492 \Phi'' + 252 \Phi \Phi'' + 19 \Phi^2 \Phi'' -71712 V - 4944 \Phi V + 208 \Phi^2 V) V' \]
\[+ 6 \Phi'^4 ((108 + 162 \Phi + 7 \Phi^2) \Phi'' V + 2 (-288 \Phi^2 V^2
\[+ 504 \Phi V^2 + 36 \Phi^2 V^2 + 864 \Phi'' V + 252 \Phi V'' + 12 \Phi^2 V'' - \Phi^3 V''))
\]
\[-6 \Phi^2 ((6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi'' V
\[-82944 V^3 + 14976 \Phi V^3
\[+ 3072 \Phi^2 V^3 + 160 \Phi^3 V^2 - 15552 \Phi'' V'')
\[-5400 \Phi \Phi'' V' - 468 \Phi^2 \Phi'' V''
\[+ 6 \Phi^3 \Phi'' V' + \Phi^4 \Phi'' V' - 124416 V^2
\[-43200 \Phi V^2 - 3744 \Phi^2 V^2
\[+ 48 \Phi^3 V^2 + 8 \Phi^4 V^2 + 124416 V V''
\[+ 43200 \Phi V V'' + 3744 \Phi^2 V V''
\[-48 \Phi^3 V V'' - 8 \Phi^4 V V''
\[+ \Phi'' (44928 V^2 + 24624 \Phi V^2 + 2208 \Phi^2 V^2
\[+ 52 \Phi^3 V^2 + 15552 \Phi'' V + 5400 \Phi V''
\[+ 468 \Phi^2 V'' - 6 \Phi^3 V'' - \Phi^4 V'')) /
\[8 (6 + \Phi)^2 ( -2 \Phi^2 + (24 + \Phi) \Phi'') ( -2 \Phi^2
\[+ (18 + \Phi) (\Phi'' + 8 V)) (2)
\]
\[h_5 = (\Phi' (-10 \Phi^3 + 8 V + \Phi^2 V') (-426 + \Phi) \Phi'' V - 8 (270 + \Phi) V)
\[+ \Phi \Phi'' V (-7 (6 + \Phi) \Phi'' + 8 (174 + 5 \Phi) V) + 12 (-24 + \Phi) \Phi'^3 V'
\[+ 6 (-432 - 6 \Phi + \Phi^2) \Phi' (\Phi'' V - \Phi'' V')) /
\[4 (2 \Phi^2 - (24 + \Phi) \Phi'') ( -2 \Phi^2 + (18 + \Phi) (\Phi'' + 8 V)^2)
\]
\[h_6 = (18 (-12 + 5 \Phi) \Phi^6 V^2 + 4 (2772 + 384 \Phi - 13 \Phi^2) \Phi^5 V V'
\[- \Phi'' (3 (124416 + 44928 \Phi
\[+ 4212 \Phi^2 + 144 \Phi^3 + \Phi^4) \Phi'' V^2
\[+ 48 \Phi'' (124416 V^3 + 44928 \Phi V^3 + 4212 \Phi^2 V^3 + 144 \Phi^3 V^3 + \Phi^4 V^3
\[- 23328 V^2 - 10368 \Phi V^2 - 1584 \Phi^2 V^2 - 96 \Phi^3 V^2 - 2 \Phi^4 V^2)
\[+ 64 V (373248 V^3 + 134784 \Phi V^3
\[+ 12636 \Phi^2 V^3 + 432 \Phi^3 V^3 + 3 \Phi^4 V^3 - 139968 V^2
\]
\[37]
\[-50544 \phi \phi'^2 - 4320 \phi^2 \phi'^2 + 216 \phi^3 \phi'^2 + 36 \phi^4 \phi'^2 + \phi^5 \phi'^2)\]
\[\phi^6 (9 (144 - 18 \phi + \phi^2)) \phi'' \phi'^2\]
\[-2 \phi' (17244 \phi'' \phi - 540 \phi \phi'' \phi + 29 \phi^2 \phi'' \phi - 99360 \phi^2 + 1992 \phi \phi'^2 + 212 \phi^2 \phi'^2 + 20736 \phi'' \phi'' + 3774 \phi \phi'' \phi'' - 8 \phi^3 \phi'' \phi')\]
\[\phi^6 (2 (6 + \phi) \phi' \phi'^2 ((6208 + 3708 \phi - 24 \phi^2 + \phi^3)) \phi'' \phi'^2 \phi' - 4 \phi'' \phi' (248832 \phi^2 - 11736 \phi \phi'^2 + 840 \phi^2 \phi'^2 + 34 \phi^3 \phi'^2 + 46656 \phi'' \phi'' + 11016 \phi \phi'' \phi'' + 468 \phi^2 \phi'' \phi'' - 18 \phi^3 \phi'' \phi' - \phi' \phi'' \phi')\]
\[-2 (432 - 6 \phi + \phi^2) (4608 \phi^3 + 192 \phi \phi'^3 + 108 \phi'' \phi' \phi'' \phi' + 24 \phi \phi'' \phi' \phi'' \phi' + 864 \phi^2 \phi'' \phi' - 384 \phi \phi \phi' \phi'' \phi' - 16 \phi^2 \phi' \phi'' \phi' + 192 \phi \phi'^2 \phi'' \phi' - 1728 \phi \phi'' \phi' + 384 \phi \phi' \phi'' \phi' + 4 (4752 \phi^3 + 1116 \phi \phi^3 + 66 \phi^2 \phi^3 - 3240 \phi \phi^2 \phi^3 - 1008 \phi \phi^2 \phi^2 \phi^2 - 66 \phi^2 \phi^2 \phi^2 + 2 \phi^3 \phi\phi' \phi'' \phi' + 2592 \phi \phi \phi'' \phi' - 756 \phi \phi \phi'' \phi' - 36 \phi \phi \phi'' \phi' + 3 \phi^3 \phi \phi'' \phi' + \phi'' \phi' (88128 \phi^3 + 67608 \phi \phi^3 + 5040 \phi^2 \phi^3 + 90 \phi^3 \phi^3 + 125712 \phi^2 + 46656 \phi \phi \phi'' \phi' + 1404 \phi \phi \phi'' \phi' + 18 \phi^3 \phi \phi'' \phi' + 3 \phi^4 \phi \phi'' \phi') + (3 \phi + 1197 \phi + 84 \phi^2 + 2 \phi^3) \phi'' \phi^2 \phi' + 4 \phi^2 \phi'' \phi' (16 \phi^4 + 38144 \phi + 30528 \phi^3 + 80 \phi^3 + 15552 \phi'' \phi') + 5400 \phi \phi'' \phi' \phi'' \phi' + 468 \phi^2 \phi'' \phi' \phi'' \phi' - 6 \phi^3 \phi'' \phi' \phi'' \phi' - 4 \phi^4 \phi'' \phi' \phi'' \phi' + 72576 \phi^2 \phi^2 \phi^2 + 28224 \phi \phi \phi' \phi'' \phi' + 3360 \phi^2 \phi \phi'^2 \phi' + 112 \phi^3 \phi \phi'^2 \phi' - 124416 \phi \phi \phi'' \phi' - 43200 \phi \phi \phi'' \phi' - 3744 \phi^2 \phi \phi'' \phi' - 48 \phi^3 \phi \phi'' \phi' + 8 \phi^4 \phi \phi'' \phi' + (16 (6 + \phi)^2 (-2 \phi^2 + (24 + \phi) \phi'')) (2 \phi^2 + (18 + \phi) \phi'' \phi' + 8 \phi)^2)\]
\[h_7 = -(V (84 \phi^4 \phi - 8 (18 + \phi)^2 \phi'' \phi (3 \phi' + 2 (12 + \phi) \phi + 3 \phi' + 2 (12 + \phi) \phi) + \phi^2 \phi (3 (1876 - 40 \phi + \phi^2) \phi'' + 8 (18 + \phi)^2 \phi) - 4 (432 - 6 \phi + \phi^2) \phi'' \phi' - (24 + \phi) (18 + \phi)^2 \phi' (\phi'' \phi - 2 (\phi'' + 4 \phi) \phi'')) / ((2 \phi^2 - (24 + \phi) \phi'' (2 \phi' + (18 + \phi) (\phi'' + 8 \phi))^2)\]

38
\[ h_8 = \left( -10 \Phi^5 V^2 + 4 (-204 + 5 \Phi) \Phi^4 V V' \
+32 (18 + \Phi)^2 \Phi'' V (-3 \Phi'' + 2 (-12 + \Phi) V) V' \
+2 \Phi^2 V (3 (-432 - 6 \Phi + \Phi^2) \Phi''' V \
+(7416 \Phi'' + 270 \Phi \Phi''' - 11 \Phi^2 \Phi'' \
+1728 V - 480 \Phi V - 32 \Phi^2 V) V'\right) \
+ \Phi^6 ((426 + \Phi) \Phi'' V^2 - 8 (270 V^3 + \Phi V^3 \
+432 V/2 + 6 \Phi V^2 - \Phi^2 V'^2 + 432 V V'' + 6 \Phi V V'' - \Phi^2 V V'')) \
+ \Phi' (-6 \Phi (7 \Phi''^2 V^2 - 232 \Phi'' V^3 + 360 \Phi''' V V' \
-360 \Phi'' V^2 - 360 \Phi'' V V'' - 2880 V^2 V'') \
+4 \Phi^3 (\Phi''' V V' - \Phi'' V'^2 - \Phi'' V V'' - 8 V^2 V'') \
+31104 (-\Phi''^2 V V' + \Phi''' V'^2 + \Phi'' V V'' + 8 V^2 V'') \
-\Phi^2 (7 \Phi''^2 V^2 - 40 \Phi'' V^3 - 48 \Phi''' V V'' + 48 \Phi'' V'^2 \
+48 \Phi''' V V'' + 384 V^2 V'')) / \
(4 (2 \Phi^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))}\].

References


