On electrodynamics of rapidly moving sources

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Abstract

Rapidly moving sources create pairs in the vacuum and lose energy. In consequence of this, the velocity of a charged body cannot approach the speed of light closer than a certain limit which depends only on the coupling constant. The vacuum back-reaction secures the observance of the conservation laws. A source can lose up to 50% of energy and charge because of the vacuum instability.
1 Introduction

The present article is an extended account of the work reported in Ref. [1]. We consider a classical system of electric charges which make a source of the electromagnetic field and move in the self field. However, we take into account that this source is immersed in the real vacuum, and the field that it generates excites the vacuum charges. The problem is figuring out the vacuum back-reaction on the motion of the source.

The electromagnetic field generated by a source is a solution of the expectation-value equations in the in-vacuum state. For this state to exist [2], the source is assumed asymptotically static in the past. In consequence of this assumption, the solution always contains a contribution of the static vacuum polarization whose principal effect is screening of the monopole moment of the source. The polarization occurs in the whole of space and increases infinitely in the vicinity of the source thereby causing the ultraviolet disaster. However, we show that this infinity does not affect the motion of the source. After the self-action of a pointlike charge is properly eliminated, the force exerted by the source on itself is finite. By choosing the spatial scale of the system exceeding the Compton size of the vacuum particle, we abstract ourselves from the static polarization altogether.

The effect that we are concerned about is the vacuum instability caused by a nonstationarity of the source [2,3]. A nonstationary electromagnetic field is capable of creating in the vacuum real particles having the electric charge. When the frequency of the source exceeds the threshold of pair creation, it emits a flux of energy and charge carried by the created particles. The main question is how much energy can be extracted from a source by means of this mechanism? An attempt to answer this question without taking into account the back-reaction of the vacuum on the motion of the source leads to a contradiction with the energy conservation law [3]. The radiation rate grows unboundedly with the energy of the source, and, at a sufficiently high energy, the source appears to give more than it has.

The problem of self-consistent motion is solved below for the simplest model of a pair-creating source. The model is a charged spherical shell expanding in the self field. This choice is made to avoid the complications connected with an emission of the electromagnetic waves. A high-frequency source will generally emit both the electromagnetic
waves by the classical mechanism and charged particles by the quantum mechanism. The two radiations overlap in a nontrivial way: the energy of the vacuum of charged particles goes partially into the electromagnetic radiation and amplifies it [4]. Accounting for the vacuum back-reaction is then necessary already for a removal of the infrared disaster, and the problem of restoration of the energy conservation law concerns both components of radiation [4]. By choosing the source spherically symmetric, we exclude the emission of waves, and, thereby, put off the solution of this more complicated problem.

The solution in the case of the spherical shell is in the fact that there emerges a new kinematic bound on the velocity of the source. Raising the energy of the source results in an increase of its acceleration, which causes an intensification of the vacuum particle production, which entails a reinforcement of its back-reaction, which results in a deceleration of the source. As a result, however high the energy may be, the velocity of a charged body cannot approach the speed of light closer than a certain limit. Within a given type of coupling, this limit is universal. It does not depend on the parameters of the source, only on the coupling constant. The back-reaction effect is nonanalytic in the coupling constant and restores completely the conservation laws. Up to 50% of energy and charge can be extracted from the source by raising its initial energy.

The effect of vacuum instability is of significance for rapidly moving, or high-frequency sources. The high-frequency approximation [3,4] is the only approximation made in the solution. The phenomenological, or axiomatic theory of the vacuum [5] is used in which the expectation-value equations are specified by a set of operator form factors. In the high-frequency approximation, only the polarization operator is involved in the calculation of the induced charge\(^1\). Its relevant properties are postulated in Section 2.

In Section 3, a closed set of equations is obtained for the motion of the source in the self electromagnetic field. The technique needed for dealing with the nonlocal expectation-value equations is presented in Section 4. In Section 5, the static solution is considered, valid outside some future light cone. The solution for a moving source is obtained in Section 6, and its ultraviolet behaviour is studied in Sections 7 and 8. In Sections 9-11, the force of the vacuum back-reaction is calculated, and the equation of motion of the

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\(^1\)For the calculation of the induced energy, one needs also the gravitational form factors [3,4].
source is solved in the high-frequency approximation. The rate of emission of charge is calculated in Section 12.
Electrically charged source coupled to the vacuum charges

Our starting point is the action for the electromagnetic field generated by a source

\[ S = S_{\text{cl}} + S_{\text{vac}} + S_{\text{source}} \]  \hspace{1cm} (2.1)

in which the source is a set of particles with masses \( M_i \) and charges \( e_i \):

\[ S_{\text{source}} = \sum_i \int ds \left( \frac{M_i}{2} g_{\mu\nu}(x_i(s)) \frac{dx_i^\mu}{ds} \frac{dx_i^\nu}{ds} + e_i A_\mu(x_i(s)) \frac{dx_i^\mu}{ds} \right). \]  \hspace{1cm} (2.2)

Here \( g_{\mu\nu} \) is the flat metric

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \]  \hspace{1cm} (2.3)

\( A_\mu(x) \) is the electromagnetic potential, and \( s \) is the proper time of the \( i \)-th particle. The quantities

\[ M = \sum_i M_i, \quad e = \sum_i e_i \]  \hspace{1cm} (2.4)

are the full mass and charge of the source \((c = 1)\). For definiteness, the charge \( e \) will be considered positive.

In Eq. (2.1), \( S_{\text{cl}} \) is the classical action of the electromagnetic field

\[ S_{\text{cl}} = -\frac{1}{16\pi} \int dx \, g^{1/2} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  \hspace{1cm} (2.5)

and \( S_{\text{vac}} \) is the effective action of the vacuum charges. The phenomenological, or axiomatic theory of the vacuum [5] will be used in which \( S_{\text{vac}} \) is taken as the general expansion over the basis of nonlocal invariants [6]:

\[ S_{\text{vac}} = -\frac{1}{16\pi} \int dx \, g^{1/2} F_{\mu\nu} f(-\square) F^{\mu\nu} + O(F \times F \times F \ldots). \]  \hspace{1cm} (2.6)

The higher-order terms of this expansion are of the form

\[ \int dx \, g^{1/2} f(\square_1, \square_2, \square_3, \ldots \square_{1+2}, \square_{1+3}, \ldots) F_1 \times F_2 \times F_3 \ldots \]  \hspace{1cm} (2.7)

where \( f \)'s are functions of the D'Alembert operators (the form factors). In (2.7), the operator \( \square_n \) acts on \( F_n \), and the operator \( \square_{m+n} \) acts on the product \( F_m F_n \). All form factors are assumed to admit spectral representations through the resolvents \( 1/(\mu^2 - \square) \).
The expectation-value equations of the electromagnetic field, associated with the action (2.1) are the following equations for the current $j^\alpha$:

$$\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha, \quad (2.8)$$

$$j^\alpha + f(-\Box)j^\alpha + O(F \times F \ldots) = j^\alpha_{\text{bare}} \quad (2.9)$$

with the retarded resolvents [5] for $f(-\Box)$ and the higher-order form factors. The current $j^\alpha_{\text{bare}}$ as given by the action (2.2) is of the form

$$j^\alpha_{\text{bare}}(x) = \sum_i \int ds \, e_i \frac{1}{g^{1/2}(x)} \delta^{(4)}(x - x_i(s)) \frac{dx_i^\alpha(s)}{ds}. \quad (2.10)$$

The full set of exact form factors provides a complete phenomenological description of the vacuum of particles having a given type of charge (here the electric charge). On the other hand, the form factors are to be calculated from some quantum-field model, and, even within a given model, this calculation can never be complete let alone the fact that one never knows the ultimate model of the vacuum. The virtue of the axiomatic approach is in the fact that, for the physical questions of interest, the detailed form of the form factors is unimportant. Only some of their properties are important. These properties should be postulated, and the expectation-value problem solved with the form factors which are otherwise arbitrary. The axiomatic approach is model-independent, and at the same time it is a tool for testing various models and approximations therein. [5]

The present paper gives an example of this approach.

A possibility of truncating the series (2.6) depends on the expectation-value problem in question which, in its turn, is specified by the properties of the source [3]. Our concern in the present paper is a high-frequency source creating pairs. Let $\nu$ be its typical fre-

quency. In the problem of particle creation by a nonstationary external or mean field, this field is considered high-frequency if the energy $h\nu$ dominates both the rest energy of the vacuum particle and its static (Coulomb) energy in this field [3,4]. The high-frequency approximation is the condition of validity of the expansion (2.6) [3].

\footnote{For the expectation-value equations there is no direct least-action principle but they differ from the variational equations of the Feynman effective action only by the boundary conditions for the resolvents [7].}

\footnote{We assume that there are no incoming electromagnetic waves.}
The linear expectation-value equations obtained by truncating Eq. (2.9) are solved by the ansatz
\[ j^\alpha = j_{\text{bare}}^\alpha - \gamma(-\Box) j_{\text{bare}}^\alpha \] (2.11)
in which \( \gamma(-\Box) \) is some retarded form factor. Its relevant properties are to be postulated. We assume that the function \( \gamma(-\Box) \) is analytic in the complex plane of \(-\Box\) except at the real negative half-axis where it has a cut:
\[ \frac{1}{2\pi i} [\gamma(-\mu^2 - i0) - \gamma(-\mu^2 + i0)] = \Delta(\mu^2). \] (2.12)
The properties of the spectral-mass function \( \Delta(\mu^2) \) that need to be specified are (i) positivity
\[ \Delta(\mu^2) \geq 0, \] (2.13)
(ii) the presence of a lower bound in the spectrum
\[ \Delta(\mu^2) \propto \theta(\mu^2 - 4m^2), \quad m \neq 0, \] (2.14)
and (iii) finiteness at large spectral mass
\[ \Delta(\mu^2)|_{\mu^2 \to \infty} = \frac{\kappa^2}{24\pi} \neq 0. \] (2.15)
Eq. (2.14) introduces the mass of the vacuum particles \( m \), and Eq. (2.15) introduces the coupling constant \( \kappa^2 \). Finally, the function \( \gamma(-\Box) \) must satisfy the normalization condition
\[ \gamma(0) = 0. \] (2.16)

Redefining the spectral-mass function as
\[ \Delta(\mu^2) = \frac{\kappa^2}{24\pi} \Gamma(\mu^2), \quad \mu^2 \geq 4m^2, \] (2.17)
we obtain from Eqs. (2.12)-(2.16)
\[ \gamma(-\Box) = \frac{\kappa^2}{24\pi} \int_{4m^2}^{\infty} d\mu^2 \Gamma(\mu^2) \left( \frac{1}{\mu^2 - \Box} - \frac{1}{\mu^2} \right). \] (2.18)
To summarize, we assume that the expectation-value equations (2.11) hold with the form factor (2.18) in which the resolvent \( 1/(\mu^2 - \Box) \) is retarded \([5,7]\), and \( \Gamma(\mu^2) \) satisfies the conditions
\[ \Gamma(\mu^2) \geq 0, \quad \Gamma(\infty) = 1. \] (2.19)
In the underlying quantum field theory, Eqs. (2.13)-(2.15) assume (i) positivity of the metric of the physical Hilbert space, (ii) the presence of an energy threshold for pair creation, and (iii) a logarithmic divergence of the charge renormalization. Eq. (2.16) is a condition that $\kappa^2$ is the renormalized coupling constant. For example, in the case of the electron-positron vacuum, the equations above hold with

$$\Gamma(\mu^2) = \left(1 - \frac{4m^2}{\mu^2}\right)^{1/2}\left(1 + \frac{2m^2}{\mu^2}\right) + \text{multi-loop contributions}$$

(2.20)

and

$$\kappa^2 = 8\alpha + O(\alpha^2)$$

(2.21)

where $\alpha$ is the fine-structure constant. In the case where $S_{\text{vac}}$ is the standard loop [3] with the abelian commutator curvature

$$\hat{R}_{\mu\nu} = \hat{\Omega} F_{\mu\nu} ,$$

(2.22)

the coupling constant is defined by the matrix $\hat{\Omega}$:

$$\kappa^2 = -\text{tr} \hat{\Omega}^2 ,$$

(2.23)

and the spectral-mass function is

$$\Gamma(\mu^2) = \left(1 - \frac{4m^2}{\mu^2}\right)^{3/2} .$$

(2.24)

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4For the standard loop with arbitrary metric, connection, and potential the calculations can be carried out in the general form, and the results tabulated. The one-loop action for any model is then obtained by combining the standard loops. (See Refs. [3,4] and references therein.)
3 The charged shell expanding in the self field

To avoid complications connected with an emission of the electromagnetic waves [4], the source will be chosen spherically symmetric, and, moreover, the particles in the action (2.2) will be assumed to pack a thin spherical shell. This amounts to choosing the solution of the form

\[ x_i^\mu(s) = \{t(s), r(s), \theta_i, \varphi_i\}, \quad \frac{d\theta_i}{ds} = 0, \quad \frac{d\varphi_i}{ds} = 0 \]  

(3.1)

with \( t(s) \) and \( r(s) \) independent of \( i \), and identifying \( i \) with the set \( \{\theta_i, \varphi_i\} \). Then the motion of the source boils down to a radial motion in an electric field of a single particle with mass \( M \) and charge \( e \). The electric field is the self field of the shell. An important fact is that the electric field is discontinuous on a charged surface, and the force exerted by the shell on itself is determined by one half of the sum of the electric fields on both sides of the shell [8]. Writing the law of motion of the shell in the form

\[ r = \rho(t) \]  

(3.2)

one obtains

\[ M \frac{d}{dt} \left( \frac{\dot{r}}{\sqrt{1 - \dot{r}^2}} \right) = e \frac{E_+ + E_-}{2} \]  

(3.3)

where \( E_+ \) and \( E_- \) are the electric fields outside and inside the shell.

Any spherically symmetric electromagnetic field is determined by a single function \( e(t, r) \) which is the charge contained at the time instant \( t \) inside the sphere of area \( 4\pi r^2 \):

\[ e(t, r) = \int d\bar{x} \, \delta^1/2(r - \bar{r}) \delta(t - \bar{t}) \nabla_\mu \tilde{j}^\mu(\bar{x}). \]  

(3.4)

In terms of this function the solution of the conservation equation \( \nabla_j j^\alpha = 0 \) is

\[ -4\pi r^2 j^\mu = \left( \nabla^\mu t \frac{\partial}{\partial r} + \nabla_\mu r \frac{\partial}{\partial t} \right) e(t, r), \]  

(3.5)

and the solution of the Maxwell equations (2.8) is

\[ F_{\mu\nu} = (\nabla_\mu r \nabla_\nu t - \nabla_\mu t \nabla_\nu r) E \]  

(3.6)

with the electric field

\[ E = \frac{e(t, r)}{r^2}. \]  

(3.7)
The function \( e(t, r) \) must satisfy the condition of regularity of the electric field at \( r = 0 \)

\[
e(t, 0) = 0 \tag{3.8}
\]

and the normalization condition

\[
e(t, \infty) = e. \tag{3.9}
\]

Since \( j_{\text{bare}}^\alpha \) in Eq. (2.10) is conserved, it is also of the form (3.5):

\[
-4\pi r^2 j_{\text{bare}}^\mu = \left( \nabla^\mu t \frac{\partial}{\partial r} + \nabla^\mu r \frac{\partial}{\partial t} \right) e_{\text{bare}}(t, r), \tag{3.10}
\]

and, for \( e_{\text{bare}}(t, r) \), Eq. (2.10) yields the obvious result

\[
e_{\text{bare}}(t, r) = e(\theta(r - \rho(t))). \tag{3.11}
\]

Thus, owing to the conservation of the current \( j^\alpha \), which is a corollary of the expectation-value equations (2.11), only one of these equations is independent. Finally, on account of Eq. (3.7), the equation of motion of the shell (3.3) takes the form

\[
M \frac{d}{dt} \left( \frac{\dot{\rho}}{\sqrt{1 - \rho^2}} \right) = e \frac{e_+(t) + e_-(t)}{2\rho^2} \tag{3.12}
\]

where

\[
e_\pm(t) = e(t, \rho(t) \pm 0). \tag{3.13}
\]

Eqs. (2.11), (3.5), and (3.10)-(3.13) make a closed set of equations for \( e(t, r) \) and \( \rho(t) \).

The setting of the problem with the in-vacuum of quantum fields implies that the external or mean fields generated by the source are asymptotically static in the past [2]. Accordingly, it will be assumed that, before some time instant \( t = t_{\text{start}} \), the shell was kept at some constant value of \( r, r = r_{\text{min}} \), and next was let go. Eq. (3.12) will thus be solved with the initial conditions

\[
\rho \bigg|_{t_{\text{start}}} = r_{\text{min}}, \quad \dot{\rho} \bigg|_{t_{\text{start}}} = 0. \tag{3.14}
\]

The energy of this initial state is already affected by the static vacuum polarization. However, for

\[
mr_{\text{min}} \simeq 1 \tag{3.15}
\]
this effect is negligible (see Eq. (5.18) below), and it does not make sense to consider \( r_{\text{min}} \) smaller than the Compton size of the vacuum particle. Then, up to a small correction, the energy of the shell (with the rest energy subtracted) retains its classical value

\[
E = \frac{e^2}{2r_{\text{min}}},
\]  

(3.16)

and so does the acceleration of the shell at \( t = t_{\text{start}} \)

\[
\dot{\rho}\big|_{t_{\text{start}}} = \frac{E}{M} \frac{1}{r_{\text{min}}}.
\]  

(3.17)

Since the shell moves with acceleration, it creates particles from the vacuum provided that its typical frequency exceeds the threshold of pair creation: \( h \nu > 2mc^2 \). At the high-frequency limit \( h \nu \gg mc^2 \) its vacuum radiation stops depending on the mass \( m \) [4]. As seen from Eq. (3.17), the typical frequency \( \nu \) is proportional to \( E/M \).

\[
\nu = \frac{E}{M} \frac{1}{r_{\text{min}}}.
\]  

(3.18)

The bigger the ratio \( E/M \), the bigger is the acceleration at \( t_{\text{start}} \), and the more violent is the creation of particles. Therefore, it is interesting to consider the ultrarelativistic shell \( (E/M) \gg 1 \). The latter condition can be enhanced to provide for the high-frequency regime:

\[
\frac{E}{M} \gg mr_{\text{min}}.
\]  

(3.19)

At the same time, under condition (3.15) the shell does not probe the small scales where the present description may break down. Assuming both Eqs. (3.15) and (3.19) one switches over from the consideration of the static vacuum polarization to studying the vacuum reaction on a rapidly moving source creating pairs. This is the purpose of the present work.

Without predetermining the law of motion \( \rho(t) \) one may assume that, beginning with \( t = t_{\text{start}} \), the shell expands monotonically with an increasing velocity \( \dot{\rho}(t) \) which at \( t = \infty \) reaches some finite value \( \dot{\rho}(\infty) \). Then \( \dot{\rho}(\infty) \) may serve as a measure at late time of the acceleration at \( t_{\text{start}} \). The world line of the shell is shown in Fig. 1. As \( (E/M) \to \infty \), the velocity \( \dot{\rho}(t) \) approaches 1 at all \( t \) except in a small sector near \( t = t_{\text{start}} \). The world line
of the shell approaches then the broken line $N$ in Fig. 1. These assumptions are valid for the classical motion of the shell

$$\frac{M}{\sqrt{1 - \rho^2}} + \frac{1}{2} \frac{e^2}{\rho} = M + \mathcal{E}, \quad (3.20)$$

and they cannot be invalidated by the quantum corrections if the coupling constant $\kappa^2$ is small.
4 The retarded resolvent

As in Refs. [3,4], it is convenient to express the resolvent in the expectation-value equations through the operator

$$\mathcal{H}_q = \sqrt{2q} K_1(\sqrt{2q\Box}) , \quad q < 0$$

(4.1)

depending on the parameter $q$, whose retarded kernel is of the form

$$\mathcal{H}_q X(x) = \frac{1}{4\pi} \int_{\text{past of } x} d\bar{x} \bar{g}^{1/2} \delta(\sigma(x, \bar{x}) - q) X(\bar{x}) .$$

(4.2)

Here $K_1$ is the order-1 Macdonald function, and $\sigma(x, \bar{x})$ is the world function: one half of the square of the geodetic distance between the points $x$ and $\bar{x}$ [9]. The integration in Eq. (4.2) is over the past sheet of the hyperboloid of equal geodetic distance $\sqrt{-2q}$ from the observation point $x$.

The needed expression is provided by the formula

$$\frac{1}{\mu^2 - \Box} = \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) K_0(\sqrt{2q\Box})$$

(4.3)

involving the Macdonald and Bessel functions, and the result is the following expression for the kernel of the retarded resolvent:

$$\frac{1}{\mu^2 - \Box} X = \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) \frac{d}{dq} \mathcal{H}_q X$$

(4.4)

with $\mathcal{H}_q X$ in Eq. (4.2). If the test function is a tensor

$$X = X^{\mu_1 \ldots \mu_n} ,$$

(4.5)

Eq. (4.2) is an abbreviation of

$$\mathcal{H}_q X^{\mu_1 \ldots \mu_n}(x) = \frac{1}{4\pi} \int_{\text{past of } x} d\bar{x} \bar{g}^{1/2} \delta(\sigma(x, \bar{x}) - q) g_{\mu_1}(x, \bar{x}) \ldots g_{\mu_n}(x, \bar{x}) X^{\mu_1 \ldots \mu_n}(\bar{x})$$

(4.6)

where $g_{\mu}(x, \bar{x})$ is the propagator of the geodetic parallel transport [9].

As an example of the use of Eq. (4.4) one may derive the retarded kernel of the massless operator $1/\Box$. For a test function $X$ asymptotically static in the past, the operator $\mathcal{H}_q$ decreases as $q \rightarrow -\infty$ [3]:

$$\mathcal{H}_q X \bigg|_{q \rightarrow -\infty} \propto \frac{1}{\sqrt{-q}} .$$

(4.7)
Thus one obtains
\[-\frac{1}{\Box} X(x) = \mathcal{H}_q X \big|_{q=0} = \frac{1}{4\pi} \int_{\text{past of } x} d\vec{x} \, \tilde{g}^{1/2} \delta(\sigma(x, \vec{x})) \, X(\vec{x}). \tag{4.8}\]

When \( X \) is a spherically symmetric scalar \( X = X(t, r) \), and the coordinates (2.3) are used, one has
\[\mathcal{H}_q X(x) = \frac{1}{2} \int_{\text{past of } x} d\tilde{t} \, d\tilde{r} \tilde{r}^2 \int_{-1}^{1} d(\cos \omega) \, \delta(\sigma - q) X(\tilde{t}, \tilde{r}), \tag{4.9}\]
\[2\sigma = -(t - \tilde{t})^2 + (r - \tilde{r})^2 + 2r\tilde{r}(1 - \cos \omega) \tag{4.10}\]
where \( \omega \) is the arc length between the points \((\theta, \varphi)\) and \((\tilde{\theta}, \tilde{\varphi})\) on the unit two-sphere. Denote \( \dot{\sigma} \) the world function of the two-dimensional Lorentzian section:
\[\dot{\sigma} = -\frac{1}{2}(t - \tilde{t})^2 + \frac{1}{2}(r - \tilde{r})^2. \tag{4.11}\]
It follows from Eq. (4.10) that, on the hyperboloid \( \sigma = q \), the range \(-1 < \cos \omega < 1\) is equivalent to the following range of variation of \( \dot{\sigma} \):
\[q - 2r\tilde{r} < \dot{\sigma} < q. \tag{4.12}\]
Therefore, the result of the angle integration in Eq. (4.9) is
\[\mathcal{H}_q X(x) = \frac{1}{2r} \int_{\text{past of } x} d\tilde{t} \, d\tilde{r} \tilde{r} X(\tilde{t}, \tilde{r}) \theta(q - \dot{\sigma}) \theta(\dot{\sigma} + 2r\tilde{r} - q), \tag{4.13}\]
\[\frac{d}{dq} \mathcal{H}_q X(x) = \frac{1}{2r} \int_{\text{past of } x} d\tilde{t} \, d\tilde{r} \tilde{r} X(\tilde{t}, \tilde{r}) \left[ \delta(\dot{\sigma} - q) - \delta(\dot{\sigma} + 2r\tilde{r} - q) \right]. \tag{4.14}\]
In the past of the observation point \( x \), the boundaries specified by the \( \theta \)-functions in Eq. (4.13) are of the form
\[\dot{\sigma} - q = 0 : \quad \tilde{t} = t - \sqrt{(r - \tilde{r})^2 - 2q}, \tag{4.15}\]
\[\dot{\sigma} + 2r\tilde{r} - q = 0 : \quad \tilde{t} = t - \sqrt{(r + \tilde{r})^2 - 2q}, \tag{4.16}\]
and Eq. (4.13) can be rewritten as
\[\mathcal{H}_q X = \frac{1}{2r} \int_{0}^{\infty} d\tilde{r} \tilde{r} \int_{t - \sqrt{(r + \tilde{r})^2 - 2q}}^{t - \sqrt{(r - \tilde{r})^2 - 2q}} d\tilde{t} \, X(\tilde{t}, \tilde{r}). \tag{4.17}\]
Eq. (4.17) yields a simple expression in the case where the source $X$ is static: $X(t, r) = X(r)$. In this case one obtains

$$\mathcal{H}_q X = \frac{1}{2r} \int_0^\infty d\tilde{r} \tilde{r} X(\tilde{r}) \left( \sqrt{(r + \tilde{r})^2 - 2q} - \sqrt{(r - \tilde{r})^2 - 2q} \right)$$

(4.18)

and

$$\frac{1}{\mu^2 - \Box} X = \frac{1}{2\mu r} \int_0^\infty d\tilde{r} \tilde{r} X(\tilde{r}) \left[ \exp(-\mu|\tilde{r} - r|) - \exp(-\mu(r + \tilde{r})) \right]$$

(4.19)

where use is made of the integral

$$\int_{-\infty}^0 dq \frac{J_0(\mu\sqrt{2q})}{\sqrt{\alpha^2 - 2q}} = \frac{1}{\mu} \exp(-\mu|\alpha|).$$

(4.20)

Of course, for a static $X$, expression (4.18) could be obtained simpler by integrating in Eq. (4.9) first over $\tilde{t}$

$$\mathcal{H}_q X = \frac{1}{2} \int_0^\infty d\tilde{r} \tilde{r}^2 \int_{-1}^1 d(\cos \omega) \frac{X(\tilde{r})}{\sqrt{(r - \tilde{r})^2 + 2r\tilde{r}(1 - \cos \omega) - 2q}}$$

(4.21)

and next over $\cos \omega$. 
The static vacuum polarization

The propagator of parallel transport $g^\mu_\mu(x, \bar{x})$ for the metric (2.3) projects on the basis vectors as follows:

$$\nabla_\mu t \, g^\mu_\mu(x, \bar{x}) = \nabla_\mu \bar{t},$$  \hspace{1cm} (5.1)

$$\nabla_\mu r \, g^\mu_\mu(x, \bar{x}) = \cos \omega \, \nabla_\mu \bar{r} + \bar{r} \, \nabla_\mu \cos \omega.$$  \hspace{1cm} (5.2)

One can choose any of the two projections to convert Eq. (2.11) into a scalar equation. Specifically, one can use the fact that, owing to Eq. (5.1), the operation of projecting on $\nabla t$ commutes with the action of any nonlocal form factor. Hence

$$\nabla_\alpha t \, j^\alpha = \nabla_\alpha t \, j^\alpha_\text{bare} - \gamma(\Box) \nabla_\alpha t \, j^\alpha_\text{bare},$$  \hspace{1cm} (5.3)

and by Eqs. (3.5) and (3.11)

$$\nabla_\alpha t \, j^\alpha_\text{bare} = \frac{1}{4\pi r^2} \frac{\partial}{\partial r} \text{e}(t, r),$$  \hspace{1cm} (5.4)

$$\nabla_\alpha t \, j^\alpha_\text{bare} = \frac{e}{4\pi r^2} \delta(r - \rho(t)).$$  \hspace{1cm} (5.5)

Strictly outside and inside the shell, the function (5.5) vanishes. Therefore, in each of these regions, the local terms on the right-hand side of Eq. (5.3), i.e., the terms in which the current $j^\alpha_\text{bare}$ appears at the observation point can be omitted. Specifically, the subtraction term in the spectral formula (2.18) can be omitted. As a result, one obtains the equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \text{e}(t, r) = -\frac{\kappa^2}{6} \int_0^\infty d\mu^2 \Gamma(\mu^2) \frac{1}{\mu^2 - \Box} (\nabla_\alpha t \, j^\alpha_\text{bare})$$  \hspace{1cm} (5.6)

which is valid separately in two regions for the point $(r, t)$: outside and inside the shell. Below, the notation $\varepsilon$ is used for the function

$$\varepsilon(t, r) = \theta(r - \rho(t)) - \theta(\rho(t) - r).$$  \hspace{1cm} (5.7)

The broken lines in Fig. 1 bound the future light cone of the point of start. Denote $P$ (for past) the exterior of this cone. By causality, the region $P$ can be affected only by the static sector of the evolution of the shell. Therefore, when calculating $\text{e}(t, r)$ for the
point \((r, t)\) in \(P\), the law of motion of the shell can be taken \(\rho(t) = r_{\text{min}}\). With this law, Eqs. (4.19) and (5.5) yield straight away

\[
\frac{1}{\mu^2 - \Box} (\nabla_a j^a_{\text{bare}}) = \frac{e}{r} \frac{1}{8\pi \mu r_{\text{min}}} \left[ \exp\left(-\mu |r - r_{\text{min}}|\right) - \exp\left(-\mu (r + r_{\text{min}})\right) \right], \quad (5.8)
\]

\((r, t) \in P\)

and one obtains

\[
\frac{\partial}{\partial r} e(t, r) = -e \frac{r}{r_{\text{min}}} \frac{\kappa^2}{24\pi} \int_{2m}^{\infty} d\mu \Gamma(\mu^2) \left[ \exp\left(-\mu |r - r_{\text{min}}|\right) - \exp\left(-\mu (r + r_{\text{min}})\right) \right], \quad (5.9)
\]

\((r, t) \in P\).

Here the expression in the square brackets is positive. Therefore, in view of the condition \(\Gamma(\mu^2) \geq 0\), one has

\[
\frac{1}{e} \frac{\partial}{\partial r} e(t, r) < 0 \quad (5.10)
\]

both outside and inside the shell.

Eqs. (3.8) and (3.9) appear now in the role of boundary conditions for the regions inside and outside the shell respectively, and in both regions they fix the solution of Eq. (5.9). The solution for \((r, t) \in P\) is

\[
e(t, r) = e \left[ 1 + \frac{\varepsilon}{2} \right] + \frac{e}{r_{\text{min}}} \frac{\kappa^2}{24\pi} \int_{2m}^{\infty} d\mu \frac{\mu^2}{\mu^2} \Gamma(\mu^2) \left[ (1 + \varepsilon \mu r) \exp\left(-\mu \varepsilon (r - r_{\text{min}})\right) - (1 + \mu r) \exp\left(-\mu (r + r_{\text{min}})\right) \right] \quad (5.11)
\]

and is, of course, static.

As shown below, a number of features of the static solution above persists also beyond \(P\), i.e., for a moving shell. These features are summarized in Fig. 2. First of all, inside the shell there is charge, and this charge is negative. This is a consequence of the positivity of the spectral-mass function and the nonlocal nature of the expectation-value equations. The retarded nonlocal form factor collects the bare charge from the whole interior of the past light cone of the observation point. Since the world line of the shell crosses this cone at any location of the observation point, the vacuum inside the shell gets polarized.

On the other hand, the total charge inside every sphere surrounding the shell is positive. This occurs owing to a jump of \(e(t, r)\) across the shell, i.e., owing to the positive
charge of the shell itself. The jump is, however, infinite, and so are the values of \( e(t, r) \) on both sides of the shell. This infinity, of different signs inside and outside, develops in the Compton neighbourhood of the shell. At a large distance from the shell, the polarization falls off exponentially owing to the presence of the threshold \( \mu \geq 2m \) in the spectral integral. Qualitatively, at each given \( t \), the \( e(t, r) \) for a moving shell has a similar shape.

The mechanism by which the singularity on the shell’s surface emerges is noteworthy since it appears repeatedly in the consideration below. This mechanism is connected with the convergence of the spectral integral in Eq. (5.11) at the upper limit. The integrand in Eq. (5.11) provides an exponential cut off at large spectral mass but only for \( r \neq r_{\text{min}} \). At \( r = r_{\text{min}} \), in view of the condition \( \Gamma(\infty) = 1 \), the integral becomes logarithmically divergent. This is none other than the ultraviolet divergence of the charge renormalization. For an observer at infinity, the shell appears as an electric monopole screened by the vacuum. With \( e(t, \infty) \) normalized as in Eq. (3.9), the unscreened monopole

\[
e_+ = e(t, r_{\text{min}} + 0) \quad (t \leq t_{\text{start}})
\]

should be infinite.

However, in the present case the source is not a pointlike object. The total charge inside the shell

\[
e_- = e(t, r_{\text{min}} - 0) \quad (t \leq t_{\text{start}})
\]

is also infinite and has the opposite sign. Owing to this fact, the force moving the shell is finite. Indeed, with Eq. (5.11) one is able to calculate the acceleration of the shell at \( t = t_{\text{start}} \)

\[
\ddot{\rho}\big|_{t_{\text{start}}} = \frac{e}{Mr_{\text{min}}^2} \frac{e_+ + e_-}{2}. \tag{5.14}
\]

Making the sum \( e_+ + e_- \) in the spectral integral one obtains unambiguously

\[
\frac{e_+ + e_-}{2} = e \frac{\kappa^2}{r_{\text{min}} 24\pi} \int_0^\infty \frac{d\mu}{\mu^2} \Gamma(\mu^2) \left[ 1 - (1 + \mu r_{\text{min}}) \exp(-2\mu r_{\text{min}}) \right]. \tag{5.15}
\]

This way of subtracting infinities is physically equivalent to giving the shell a Compton width (see Section 9 for a refinement of this point).
The function in the square brackets in Eq. (5.15)

\[ f(\mu r_{\text{min}}) = 1 - (1 + \mu r_{\text{min}}) \exp(-2\mu r_{\text{min}}) \]  

(5.16)
is positive since

\[ \frac{d}{dx} f(x) > 0 \quad \text{for} \quad x > 0 \]  

(5.17)
and \( f(0) = 0 \). Therefore, the force in Eq. (5.14) is in all cases repulsive. In the case \( \mu r_{\text{min}} \ll 1 \) it even acquires an extra amplifying factor \( |\log(\mu r_{\text{min}})| \). However, under condition (3.15) this force differs negligibly from its classical value:

\[ \frac{e^+ + e^-}{2} = \frac{e}{2} + e \frac{\kappa^2}{24\pi} \left[ \frac{1}{\mu r_{\text{min}}} \int_0^\infty \frac{dx}{x^2} \Gamma(m^2 x^2) + O(\exp(-2\mu r_{\text{min}})) \right]. \]  

(5.18)

Also, the charge inside the shell is then concentrated almost entirely in the Compton neighbourhood of the shell, and so is the excess of charge over \( e \) outside the shell. In this way the correspondence principle is fulfilled. On the other hand, no large scales or low energies can save one from the development of the singularity within the Compton neighbourhood of the shell. Its appearance may be understood as a signal that a charge cannot be localized more accurately than within a Compton neighbourhood. The charges of the shell immersed in the real vacuum are always annihilated and created anew in a slightly different place. As a result, the shell gets smeared to a Compton width. In this way the quantum uncertainty manifests itself.

For the sake of comparison consider also a pointlike source. This is the limiting case of the charged ball

\[ e_{\text{bare}}(t, r) = \begin{cases} \frac{e r^3}{r_0^3}, & r < r_0 \\ e, & r > r_0 \end{cases} \]  

(5.19)
as \( r_0 \to 0 \). In this case, assuming that the observation point is outside the ball, Eqs. (4.19) and (3.10) yield

\[ \frac{1}{\mu^2 - \Box} (\nabla_\alpha j^\alpha_{\text{bare}}) = \frac{e}{4\pi r} \exp(-\mu r). \]  

(5.20)

With the boundary condition (3.9) one then obtains from Eq. (5.6)

\[ e(t, r) = e + e \frac{\kappa^2}{12\pi} \int_0^\infty \frac{d\mu}{\mu} \Gamma(\mu^2) (1 + \mu r) \exp(-\mu r). \]  

(5.21)
The electric field \( (3.7) \) with this \( e(t, r) \) can be written down as

\[
E = -\frac{\partial}{\partial r} U, \tag{5.22}
\]

\[
U = \frac{e}{r} \left( 1 + \frac{\kappa^2}{12\pi} \int_{2m}^{\infty} \frac{d\mu}{\mu} \Gamma(\mu^2) \exp(-\mu r) \right). \tag{5.23}
\]

With the spectral-mass function in Eq. (2.20) and \( \kappa^2 \) in Eq. (2.21), this reproduces the textbook result for the "modified Coulomb law". In fact, the Coulomb law is not modified as seen from Eq. (3.7). What gets modified is the charge distribution. The pointlike charge induces the same infinite screening as the charged shell does.
6 Solution for the moving shell

Consider Eq. (4.13) in which $X(t, r)$ is identified with the charge density (5.5). The lines (4.15) and (4.16) on the $\bar{r}, \bar{t}$ plane bound the mapping on this plane of the past (sheet of the) hyperboloid $\sigma(x, \bar{x}) = q$ of the observation point $x$. The observation point has the coordinates $r, t$ and is shown in Fig. 3 along with the two boundaries of the mapping of its past hyperboloid (the bold lines). At $q = 0$ the past hyperboloid becomes the past light cone whose mapping on the $\bar{r}, \bar{t}$ plane is bounded by the light lines in Fig. 3. The world line of the shell $\bar{r} = \rho(\bar{t})$ which is also shown in Fig. 3 crosses the upper boundary of the past hyperboloid at some point $r_+, t_+$ and the lower boundary at some point $r_-, t_-$. These points are determined by the equations

$$
\begin{aligned}
&\begin{cases}
  r_+ &= \rho(t_+) , \\
  t_+ &= t - \sqrt{(r-r_+)^2 - 2q ,}
\end{cases} \\
&\begin{cases}
  r_- &= \rho(t_-) , \\
  t_- &= t - \sqrt{(r+r_-)^2 - 2q ,}
\end{cases}
\end{aligned}
$$

(6.1)

(6.2)

and their locations on the world line of the shell depend on the location of the observation point $r, t$ and on the value of $q$. At $q = 0$ the coordinates solving Eqs. (6.1) and (6.2) will be denoted

$$
r_0^+, t_0^+, r_0^-, t_0^-
$$

(6.3)

respectively.

With the notation above, Eqs. (4.13) and (5.5) yield

$$
\mathcal{H}_q (\nabla_\alpha t^\alpha j^\alpha_{bare}) = \frac{e}{8\pi r} \int_{t_-}^{t_+} \frac{dt}{\rho(t)}.
$$

(6.4)

Using Eqs. (6.1), (6.2) one can calculate

$$
\frac{\partial t_{\pm}}{\partial q} \equiv \frac{1}{A_{\pm}} ,
$$

(6.5)

$$
A_+ = (t - t_+) - (r - r_+)\dot{\rho}(t_+),
$$

(6.6)

$$
A_- = (t - t_-) + (r + r_-)\dot{\rho}(t_-),
$$

(6.7)
and hence
\[ \frac{d}{dq} \mathcal{H}_d (\nabla \alpha t \bar{j}_{\text{bare}}) = \frac{e}{8\pi r} \left( \frac{1}{r_+ A_+} - \frac{1}{r_- A_-} \right) . \] (6.8)

With this expression, Eq. (4.4) yields
\[ \frac{1}{\mu^2 - \Box} (\nabla \alpha t j_{\text{bare}}^\alpha) = \frac{e}{8\pi r} \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) \left( \frac{1}{r_+ A_+} - \frac{1}{r_- A_-} \right) . \] (6.9)

and then from Eq. (5.6) one obtains
\[ \frac{\partial}{\partial r} e(t, r) = -r \frac{e \kappa^2}{48 \pi} \int \mu^2 \Gamma(\mu^2) \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) \left( \frac{1}{r_+ A_+} - \frac{1}{r_- A_-} \right) . \] (6.10)

Eq. (6.10) should now be integrated over \( r \) along the line \( t = \text{const.} \) with the boundary conditions (3.8) and (3.9) inside and outside the shell respectively. This integration can be done explicitly, and the final result is
\[ e(t, r) = e \left( \frac{1 + \varepsilon}{2} + e \frac{\kappa^2}{24 \pi} \frac{24}{4m^2} \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) F(q, t, r) \right) , \] (6.11)

\[ w(\mu, t, r) = \frac{1}{\mu^2} \frac{1 + \varepsilon}{2} + \int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) F(q, t, r) \] (6.12)

with \( \varepsilon \) in Eq. (5.7), and
\[ F(q, t, r) = \frac{1}{2} \left[ \int_{t_+}^{t_-} dt \rho(t) \log((t - t_-) - (r + r_-)) - \log((t - t_+) + (r - r_+)) \right] . \] (6.13)

It follows from Eqs. (6.1), (6.2) that the arguments of both log’s in Eq. (6.13) are nonnegative.

For the proof of the result above, first use Eqs. (6.1), (6.2) to show that the derivative of the function (6.13) is
\[ \frac{\partial}{\partial r} F(q, t, r) = -r \frac{1}{2} \left( \frac{1}{r_+ A_+} - \frac{1}{r_- A_-} \right) , \] (6.14)

and thereby expression (6.11) satisfies the equation (6.10). Next note that at \( r = 0 \) the points \( r_+, t_+ \) and \( r_-, t_- \) coincide. Hence
\[ F(q, t, 0) = 0 , \] (6.15)
and thereby expression (6.11) with $\varepsilon = -1$ satisfies the boundary condition (3.8).

Finally, consider expression (6.11) with $\varepsilon = +1$ at the limit where the observation point $r, t$ moves to spatial infinity: $r \to \infty$ at a fixed $t$. At any $t$ and a sufficiently large $r$, the point $r, t$ will enter the region $P$ which is affected only by the static sector of the evolution of the shell. Indeed, as the observation point moves to spatial infinity, both points $r_+, t_+$ and $r_-, t_-$ shift to the past along the world line of the shell and turn out to be on its static sector. Therefore,

$$F(q, t, r)\bigg|_{r \to \infty} = F_{\text{stat}}(q, r)\bigg|_{r \to \infty}$$

where

$$F_{\text{stat}}(q, r) = \frac{1}{2} \left[ \frac{1}{r_{\min}} \left( \sqrt{(r + r_{\min})^2 - 2q} - \sqrt{(r - r_{\min})^2 - 2q} \right) + \log \left( \frac{\sqrt{(r + r_{\min})^2 - 2q} - (r + r_{\min})}{\sqrt{(r - r_{\min})^2 - 2q} - (r - r_{\min})} \right) \right].$$

With this expression the integral in Eq. (6.12) can be calculated:

$$\int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) F_{\text{stat}}(q, r) = -\frac{1}{\mu^2} \frac{1 + \varepsilon_0}{2} + \frac{1}{2\mu^3 r_{\min}} \times \left[ (1 + \varepsilon_0 \mu r) \exp(-\mu \varepsilon_0 (r - r_{\min})) - (1 + \mu r) \exp(-\mu (r + r_{\min})) \right],$$

$$\varepsilon_0 = \theta(r - r_{\min}) - \theta(r_{\min} - r),$$

and it is seen why the explicit term in $1/\mu^2$ is introduced in Eq. (6.12). Here use is made of the integrals

$$\int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) \log \left( \frac{\sqrt{a^2 - 2q} \pm |a|}{\sqrt{-2q}} \right) = \pm \frac{1}{\mu^2} \left[ 1 - \exp(-\mu |a|) \right],$$

and the result agrees with Eq. (5.11). Thus one obtains from Eqs. (6.11) and (6.16)

$$e(t, r)\bigg|_{r \to \infty} = e + \frac{e}{r_{\min}} \frac{\kappa^2}{24\pi} \int_{2\mu}^{\infty} d\mu \frac{\mu^2}{\Gamma(\mu^2)} \left( \exp(\mu r_{\min}) - \exp(-\mu r_{\min}) \right) \times (1 + \mu r) \exp(-\mu r)\bigg|_{r \to \infty}$$

(6.22)
whence

\[ e(t, r) \bigg|_{r \to \infty} = e + O\left(\exp(-2mr)\right) \quad (6.23) \]

and thereby the boundary condition (3.9) is satisfied.
7 Convergence of the spectral integral

The integral (6.12) in $q$ always converges. Indeed, when the observation point $r, t$ is fixed, and $q \to -\infty$, the points $r_+, t_+$ and $r_-, t_-$ again shift to the past and turn out to be on the static sector of the world line of the shell. Therefore,

$$F(q, t, r)\big|_{q=\infty} = F_{\text{stat}}(q, r)\big|_{q=\infty}$$  \hspace{1cm} (7.1)

whence

$$F(q, t, r)\big|_{q=\infty} = \frac{rr_{\text{min}}^2}{(-2q)^{3/2}}.$$  \hspace{1cm} (7.2)

Since also

$$F(q, t, r)\big|_{q=0} = O\left(\log(-2q)\right)$$  \hspace{1cm} (7.3)

as discussed below, the integral in Eq. (6.12) converges even at $\mu = 0$ and at any location of the observation point $r, t$.

The behaviour (7.3) is in all cases calculable directly from Eqs. (6.13) and (6.1), (6.2) but its coefficient is different for different locations of the observation point. For the observation point outside the shell one obtains

$$F(q, t, r)\big|_{r>\rho(t)} = \frac{1}{2} \log(-2q) + \frac{1}{2} \sum_{n=0}^{\infty} a_n(t, r)(-2q)^n, \quad q \to 0$$  \hspace{1cm} (7.4)

with

$$a_0(t, r) = -\log[4(r - r_0^0)(r + r_0^0)] + \int_{r_0^0}^{r_{\text{max}}} \frac{dt}{\rho(t)}$$  \hspace{1cm} (7.5)

whereas, for the observation point inside the shell,

$$F(q, t, r)\big|_{r<\rho(t)} = \frac{1}{2} \sum_{n=0}^{\infty} b_n(t, r)(-2q)^n, \quad q \to 0$$  \hspace{1cm} (7.6)

with

$$b_0(t, r) = \log\left(\frac{r_0^0 - r}{r_0^- + r}\right) + \int_{r_0^-}^{r_{\text{max}}} \frac{dt}{\rho(t)}.$$  \hspace{1cm} (7.7)

The result for the observation point on the shell is different from both Eqs. (7.4) and (7.6):

$$F(q, t, r)\big|_{r=\rho(t)} = \frac{1}{4} \log(-2q) + \frac{1}{2} \sum_{n=0}^{\infty} c_n(t)(-2q)^{n/2}, \quad q \to 0.$$  \hspace{1cm} (7.8)
Here the expansion is in half-integer powers with

\[ c_0(t) = -\log 2[\rho(t) + \rho(t_0^t)] + \log \left( \frac{1}{1 + \dot{\rho}(t)} \right) + \int_{\rho(t_0^t)}^{t} \frac{d\bar{t}}{\rho(\bar{t})}, \]

(7.9)

\[ c_1(t) = \frac{1}{\sqrt{1 - \rho^2(t)}} \left( \frac{1}{\rho(t)} + \frac{1}{2} \frac{\dot{\rho}(t)}{1 - \rho^2(t)} \right), \]

(7.10)

and \( t_0^t \) taken at \( r = \rho(t) \). The cause of this discontinuity is in the fact that at \( q = 0 \) the upper boundary of the hyperboloid acquires a conic singularity (Fig. 3). When the observation point crosses the shell, the point \( r_0^t, t_0^t \) passes through the vertex of the cone.

The behaviour of \( F(q, t, r) \) at \( q \to 0 \) determines the leading (power) behaviour of the integral (6.12) at \( \mu \to \infty \). The latter behaviour can be obtained by integrating by parts with the aid of the relation

\[ J_0(\mu \sqrt{-2q}) = \frac{2}{\mu^2} \frac{\partial}{\partial q} q \frac{\partial}{\partial q} J_0(\mu \sqrt{-2q}). \]

(7.11)

In the case (7.4) or (7.6) one may write

\[
\begin{align*}
\int_{-\infty}^{0} dq \, J_0(\mu \sqrt{-2q}) F(q, t, r) &= \left( q - \frac{2}{\mu^2} q \frac{\partial}{\partial q} \right) \sum_{n=0}^{N} \left( \frac{2}{\mu^2} \right)^n \left( \frac{\partial}{\partial q} q \frac{\partial}{\partial q} \right)^n F(q, t, r) \bigg|_{q=0} \\
&+ \left( \frac{2}{\mu^2} \right)^{N+1} \int_{-\infty}^{0} dq \, J_0(\mu \sqrt{-2q}) \left( \frac{\partial}{\partial q} q \frac{\partial}{\partial q} \right)^{N+1} F(q, t, r) 
\end{align*}
\]

(7.12)

where use is made of Eq. (7.2). Since, for any \( N \), the integral on the right-hand side of Eq. (7.12) converges and decreases as \( \mu \to \infty \), one obtains

\[
\int_{-\infty}^{0} dq \, J_0(\mu \sqrt{-2q}) F(q, t, r) \bigg|_{r > \rho(t)} = -\frac{1}{\mu^2} + O\left( \frac{1}{\mu^{2N}} \right), \quad \mu \to \infty \]

(7.13)

\[
\int_{-\infty}^{0} dq \, J_0(\mu \sqrt{-2q}) F(q, t, r) \bigg|_{r < \rho(t)} = O\left( \frac{1}{\mu^{2N}} \right), \quad \mu \to \infty \]

(7.14)

where the remainder decreases faster than any power of \( 1/\mu^2 \). (It decreases exponentially, see below.)

In the case (7.8) the integration by parts as above can be done only once. One may write

\[ \frac{\partial}{\partial q} q \frac{\partial}{\partial q} F(q, t, \rho(t)) = \frac{1}{\sqrt{-2q}} \Phi(\sqrt{-2q}) \]

(7.15)
where $\Phi(x)$ is analytic at $x = 0$, and

$$
\int_{-\infty}^{0} dq J_{0}(\mu \sqrt{-2q}) \frac{\partial}{\partial q} q \frac{\partial}{\partial q} F(q, t, \rho(t)) = \frac{1}{\mu} \int dx J_{0}(x) \Phi\left(\frac{x}{\mu}\right) = \frac{1}{\mu} \left(\Phi(0) + O\right),
$$

(7.16)

$O \to 0$, $\mu \to \infty$

where

$$
\Phi(0) = -\frac{1}{4} c_1(t)
$$

(7.17)

by Eq. (7.8). Hence one obtains

$$
\int_{-\infty}^{0} dq J_{0}(\mu \sqrt{-2q}) F(q, t, r)_{r=\rho(t)} = -\frac{1}{2\mu^2} - \frac{c_1(t)}{2\mu^3} + O, \quad \mu \to \infty
$$

(7.18)

with $c_1(t)$ in Eq. (7.10).

As a result, in both regions outside and inside the shell, the behaviour of the function $w(\mu, t, r)$ in Eq. (6.12) is

$$
w(\mu, t, r)_{r \neq \rho(t)} = O\left(\frac{1}{\mu^{2N}}\right), \quad \forall N, \mu \to \infty
$$

(7.19)

whereas, on the shell,

$$
w(\mu, t, \rho(t) \pm 0) = \pm \frac{1}{2\mu^2} + O\left(\frac{1}{\mu^3}\right), \quad \mu \to \infty.
$$

(7.20)

Recalling the condition $\Gamma(\infty) = 1$, one infers that the spectral-mass integral in Eq. (6.11) converges and defines the function $e(t, r)$ in all cases except in the case where the point $r, t$ is on the world line of the shell. In the latter case the spectral integral diverges logarithmically, the divergent terms having the same coefficients but different signs on the two sides of the shell. It follows that the distribution $e(t, r)$ is singular on the shell’s surface, and the next task is obtaining the form of this singularity.
Singularity of the electric field on the shell’s surface

For obtaining the behaviour of \( e(t, r) \) on the shell’s surface, the behaviour of the function \( w(\mu, t, r) \) at \( \mu \to \infty \) should be known including the exponentially decreasing terms. These are determined by the singularities of the function \( F(q, t, r) \) in the complex plane of the variable \( z = \sqrt{-2q} \).

The function \( F(q, t, r) \) will now be considered only off the shell. It is convenient first to integrate by parts

\[
\int_{-\infty}^{0} dq J_0(\mu \sqrt{-2q}) F(q, t, r) = -\frac{2}{\mu} \int_{0}^{\infty} d\sqrt{-2q} J_1(\mu \sqrt{-2q}) q \frac{\partial}{\partial q} F(q, t, r) \tag{8.1}
\]

and next use the analyticity of \( q(\partial F/\partial q) \) at \( q = 0 \) to write

\[
\int_{0}^{\infty} dx J_1(\mu x) \left( q \frac{\partial}{\partial q} F \right) \bigg|_{2q=-x^2} = \frac{1}{\mu} \left( q \frac{\partial}{\partial q} F \right) \bigg|_{q=0} + \frac{1}{2} \text{Re} \int_{-\infty}^{\infty} dx H_1^{(1)}(\mu x + i\epsilon) \left( q \frac{\partial}{\partial q} F \right) \bigg|_{2q=-x^2} \tag{8.2}
\]

where \( H_1^{(1)} \) is the Hankel function. For the function \( w(\mu, t, r) \) both outside and inside the shell this gives

\[
w(\mu, t, r) = -\frac{1}{\mu} \text{Re} \int_{\mathcal{C}} dz H_1^{(1)}(\mu z) \left[ q \frac{\partial}{\partial q} F(q, t, r) \right] \bigg|_{2q=-z^2} \tag{8.3}
\]

where the contour \( \mathcal{C} \) passes above the real axis and closes counter-clockwise in the upper half-plane. Here

\[
q \frac{\partial}{\partial q} F(q, t, r) = \frac{1}{4A_+} \left[ \frac{2q}{r_+} - (r_+ - r) \left( 1 - \rho^2(t_+) \right) \right] - \frac{1}{4} \dot{\rho}(t_+)
- \frac{1}{4A_-} \left[ \frac{2q}{r_-} - (r_- + r) \left( 1 - \rho^2(t_-) \right) \right] + \frac{1}{4} \dot{\rho}(t_-) \tag{8.4}
\]

with \( A_{\pm} \) in Eqs. (6.6) and (6.7).

Of all singularities of the function (8.4) in the variable \( z = \sqrt{-2q} \), we are presently interested in the ones that have the least \( |\text{Im} \ z| \) as the point \( r, t \) approaches the shell. These are easily identified with the solutions of the equation \( A_+ = 0 \). Indeed, as \( r \to \rho(t) \), these solutions shift to \( q = 0 \) and, thereby, to \( \text{Im} \ z = 0 \) whereas the remaining singularities stay
at $\text{Im } z \neq 0$. This can be seen from the fact that, apart from the factor $1/A_+$, expression (8.4) with $\text{Im } z = 0$, i.e., with real $q \leq 0$ is nonsingular including at $r = \rho(t)$.

The equation

$$A_+ = 0$$

(8.5)

along with Eq. (6.1) determines both $q$ and the point $r_+, t_+$. Denote $q^*$ the solution for $q$, and $r^*, t^*$ the solution for $r_+, t_+$. The solution for $r_+, t_+$ proves to be real. Indeed, the point $r^*, t^*$ is defined by the equations

$$t - t^* = (r - r^*)\dot{\rho}(t^*),$$

(8.6)

$$r^* = \rho(t^*)$$

(8.7)

and is thus a point at which the world line of the shell crosses the line specified by Eq. (8.6). The latter line is shown in Fig. 4 (line $L$). It passes through the observation point $r, t$ and, at least in some neighbourhood of this point, is spacelike. This can be checked by calculating along $L$

$$\frac{dr^*}{dt^*} = 1 + \frac{(r - r^*)\ddot{\rho}(t^*)}{\dot{\rho}(t^*)}. $$

(8.8)

It follows that, at least when the observation point is sufficiently close to the shell, the intersection at $r^*, t^*$ exists and is unique (Fig. 4). The solution for $q$ is then real and positive:

$$2q^* = \eta^2, \quad \eta = |r - r^*|\sqrt{1 - \dot{\rho}^2(t^*)}$$

(8.9)

whence for $z$ one obtains two complex conjugate solutions

$$z = \pm i\eta.$$  

(8.10)

Introducing a notation for the coefficient of $1/(4A_+)$ in Eq. (8.4), one has

$$q \frac{\partial}{\partial q} F(q, t, r) \bigg|_{A_+ = 0} = \frac{1}{4A_+} \left( \beta \bigg|_{A_+ = 0} \right),$$

(8.11)

and one may calculate

$$\frac{\partial A_+^2}{\partial q} = -2\alpha$$

(8.12)

with

$$\alpha = 1 - \dot{\rho}^2(t_+) - (r_+ - r)\dot{\rho}(t_+),$$

(8.13)

$$\beta = \frac{2q}{r_+} - (r_+ - r) \left( 1 - \dot{\rho}^2(t_+) \right).$$

(8.14)
Both $\alpha$ and $\beta$ are finite and nonvanishing at $A_+ = 0$, and, moreover,

$$
\alpha \bigg|_{A_+ = 0} > 0 \quad (8.15)
$$

at least when the observation point is sufficiently close to the shell. It follows that the solutions (8.10) are branch points of the function (8.4):

$$
A_+^2 \bigg|_{q=q^*} = -2\alpha \bigg|_{q=q^*} (q - q^*) + \ldots
= \alpha \bigg|_{q=q^*} (z^2 + \eta^2) + \ldots , \quad (8.16)
$$

$$
q \frac{\partial}{\partial q} F(q, t, r) \bigg|_{A_+ = 0} = \left( \frac{\beta}{4\sqrt{\alpha} A_+ = 0} \right) \frac{1}{\sqrt{z^2 + \eta^2}} . \quad (8.17)
$$

Of the two branch points, the integral (8.3) picks up the one in the upper half-plane: $z = +i\eta$, and its contribution at large $\mu$ is

$$
w(\mu, t, r) \to \frac{1}{\mu^2 \eta} \left( \frac{\beta}{2\sqrt{\alpha} A_+ = 0} \right) \exp(-\mu \eta) . \quad (8.18)
$$

The contribution (8.18) is the leading one as the observation point $r, t$ approaches the shell. Summarizing the calculation above, one obtains

$$
w(\mu, t, r) = \left[ \frac{\varepsilon}{2\mu^2} \frac{r}{r^*} \frac{\sqrt{1 - \dot{\rho}^2(t^*)}}{\sqrt{1 - \dot{\rho}^2(t^*) - (r^* - r)\dot{\rho}(t^*)}} + O\left( \frac{1}{\mu^3} \right) \right] \exp \left( -\mu |r^* - r| \sqrt{1 - \dot{\rho}^2(t^*)} \right) ,
\quad r \to \rho(t) , \mu \to \infty \quad (8.19)
$$

with $r^*, t^*$ the solution of the equations (8.6), (8.7). It follows immediately from these equations that, when the observation point $r, t$ is on the shell, the point $r^*, t^*$ coincides with $r, t$ (see Fig. 4). Therefore, as $r \to \rho(t)$, one may expand

$$
\rho(t^*) = \rho(t) + \dot{\rho}(t)(t^* - t) + O\left( \rho(t) - r \right)^2 \quad (8.20)
$$

and in this way obtain the solution

$$
r^* - r = \frac{1}{1 - \dot{\rho}^2(t)} \left( \rho(t) - r \right) + O\left( \rho(t) - r \right)^2 ,
\quad (8.21)
$$

$$
t^* - t = \frac{\dot{\rho}(t)}{1 - \dot{\rho}^2(t)} \left( \rho(t) - r \right) + O\left( \rho(t) - r \right)^2 ,
\quad (8.22)
$$

$$
r \to \rho(t) .
$$
This brings Eq. (8.19) to its final form

\[ w(\mu, t, r) = \left( \frac{\varepsilon}{2\mu^2} + O\left(\frac{1}{\mu^3}\right) \right) \exp\left(-\mu \frac{|r - \rho(t)|}{\sqrt{1 - \rho^2(t)}}\right), \quad r \to \rho(t), \mu \to \infty. \]  

(8.23)

It is seen from the latter expression that, with any law of motion \( \rho(t) \), the mechanism of formation of the singularity on the shell’s surface is one and the same. At \( r = \rho(t) \), the integrand in Eq. (6.11) loses the exponential cut off and becomes \( O(1/\mu^2), \mu \to \infty \). The behaviour of \( e(t, r) \) as \( r \to \rho(t) \) can now be obtained by calculating the spectral-mass integral (6.11) with the function (8.23):

\[ \frac{\partial}{\partial \eta} e(t, r) = -\frac{e\kappa^2}{24\pi} \frac{\varepsilon}{\eta^2} \int_0^\infty dx \Gamma(x^2) \exp(-x), \quad \eta = \frac{|r - \rho(t)|}{\sqrt{1 - \rho^2(t)}} \to 0. \]  

(8.24)

The same result is obtained with Eq. (6.10). Since \( \Gamma(\infty) = 1 \), one has

\[ e(t, r)\big|_{r=\rho(t)^\pm} = \mp e \frac{\kappa^2}{24\pi} \log |r - \rho(t)|. \]  

(8.25)

This behaviour is shown in Fig. 2. The electric field, as given in Eq. (3.7), differs from \( e(t, r) \) only by the factor \( 1/r^2 \) which is finite and continuous across the shell.
9 The force exerted by the charged shell on itself

The singularity of the electric field on the shell’s surface does not affect the motion of the shell since it cancels in the sum

$$e_+(t) + e_-(t) = e(t, \rho(t) + 0) + e(t, \rho(t) - 0) \quad (9.1)$$

which according to Eq. (3.12) determines the force exerted by the shell on itself. Making the sum (9.1) in the spectral integral (6.11) makes this cancellation unambiguous. Indeed, at $q < 0$ the points $r_+, t_+$ and $r_-, t_-$ move along smooth trajectories as the observation point crosses the shell (see Fig. 3). Therefore, the function $F(q, t, r)$ with $q < 0$ remains continuous and defines

$$\mathcal{F}(q, t) \equiv F(q, t, \rho(t)) \quad (9.2)$$

The integral

$$\int_{-\infty}^{0} dq J_0(\mu\sqrt{-2q})F(q, t, r) \quad (9.3)$$

is also continuous. The function $w(\mu, t, r)$ is discontinuous but finite and defines

$$2W(\mu, t) \equiv w(\mu, t, \rho(t) + 0) + w(\mu, t, \rho(t) - 0) \quad (9.4)$$

Then from Eqs. (6.11), (6.12) one obtains

$$e_+(t) + e_-(t) = e + e \frac{\kappa^2}{12\pi} \int_{4\pi}^{\infty} d\mu^2 \Gamma(\mu^2) \mathcal{W}(\mu, t) \quad (9.5)$$

$$\mathcal{W}(\mu, t) = \frac{1}{2\mu^2} + \int_{-\infty}^{0} dq J_0(\mu\sqrt{-2q})\mathcal{F}(q, t) \quad (9.6)$$

and the function $\mathcal{F}(q, t)$ is given by expression (6.13) with the insertion of $r = \rho(t)$.

The behaviour of the function $\mathcal{F}(q, t)$ at $q \to -\infty$ is determined by Eq. (7.2). The behaviour of this function at $q \to 0$ is obtained in Eq. (7.8), and the behaviour of the integral (9.6) with this function is obtained in Eq. (7.18). For $\mathcal{W}(\mu, t)$ this yields

$$\mathcal{W}(\mu, t) = O\left(\frac{1}{\mu^3}\right), \quad \mu \to \infty \quad (9.7)$$

As a result, the spectral-mass integral (9.5) converges, and the force exerted on the shell is finite. The effect of making the sum (9.1) in the spectral integral is a cancellation of the $1/\mu^2, \mu \to \infty$ terms (7.20) in the integrand.
Since the calculation above implies a subtraction of infinities, it should be analysed what regularization does it correspond to physically. The answer is contained in expression (8.23). Calculate $e(t, r)$ for two close points $r_1, t_1$ and $r_2, t_2$ outside and inside the shell respectively, and consider the sum

$$e(t_1, r_1) + e(t_2, r_2).$$

This is given by the spectral integral (6.11) with

$$w(\mu, t_1, r_1) + w(\mu, t_2, r_2) = \frac{1}{2\mu^2} \left[ \exp(-\mu \eta(t_1, r_1)) - \exp(-\mu \eta(t_2, r_2)) \right] + O\left(\frac{1}{\mu^3}\right),$$

$$\eta(t, r) = \frac{|r - \rho(t)|}{\sqrt{1 - \rho^2(t)}}.$$  

The result (9.5) is recovered at the limit where the exponents in Eq. (9.9) tend to zero. Since the scale for $\mu$, set up by the spectral integral, is $m$, the regularization implied in $e_+(t)$ and $e_-(t)$ before their sum is made consists in a fixation of the parameter

$$m \eta(t, r) = \text{const.} \neq 0.$$ 

As soon as the sum (9.8) is made, the points $r_1, t_1$ and $r_2, t_2$ can be brought to the shell in any succession and along any paths. Since the $1/\mu^2$ term of expression (9.9) cancels in all cases, the limit for the sum is finite and independent of the way the regularization is removed.

The difference $|r - \rho(t)|$ that figures in expression (9.10) is the proper distance from the point $r, t$ to the shell along the line $t = \text{const}$. But not this distance is made fixed in Eq. (9.11). The function $\eta(t, r)$ is the proper distance from the point $r, t$ to the shell along the line orthogonal to the world line of the shell. Indeed, this function was introduced in Eq. (8.9), and originally it had the form

$$\eta(t, r) = \sqrt{(r - r^*)^2 - (t - t^*)^2}$$

where $r^*, t^*$ is the point on the world line of the shell connected with the point $r, t$ by the line $L$ (Fig. 4). It is easy to check from Eq. (8.8) that, up to $O(r - r^*)$, the line $L$ is orthogonal to the world line of the shell at the point of their intersection.
The final inference is that the subtraction of infinities in the spectral integral is physically equivalent to giving the shell a Compton width in the direction orthogonal to its world line. The lines on the $r, t$ plane specified by Eq. (9.11):

\[
\frac{|r - \rho(t)|}{\sqrt{1 - \dot{\rho}^2(t)}} = \frac{\text{const.}}{m}
\]  

mark the band of quantum uncertainty around the world line of the shell. This band is shown in Fig. 5, and it narrows as the speed of expansion increases.
The ultrarelativistic limit

It will now be shown that the force exerted by the shell on itself is singular at the ultrarelativistic limit. This is the limit at which the world line of the shell approaches the world line of an outgoing light ray (the line $N$ in Fig. 1). To be more precise, we consider a family of functions $\rho(t)$, for which

$$1 - \rho(t) = \delta f(\delta, t), \quad \delta \to 0.$$  \hspace{1cm} (10.1)

Here $\delta$ is a parameter (function of the initial data), and it is assumed that, at all $t > t_{\text{start}}$, $f(\delta, t)$ has a finite limit as $\delta \to 0$, whereas, at $t = t_{\text{start}}$,

$$f(\delta, t_{\text{start}}) = \frac{1}{\delta}.$$ \hspace{1cm} (10.2)

The function $f$ can be normalized as $f(\delta, \infty) = 1$, and then

$$\delta = 1 - \rho(\infty).$$ \hspace{1cm} (10.3)

Eq. (10.1) generalizes the form that the classical law of motion has as $(M/E) \to 0$. Indeed, with $E$ and $r_{\text{min}}$ taken for independent data, Eq. (3.20) can be written as

$$\frac{1}{\sqrt{1 - \rho^2}} = \frac{E}{M} \left(1 - \frac{r_{\text{min}}}{\rho}\right) + 1.$$ \hspace{1cm} (10.4)

However, Eq. (10.1) does not predetermine the dependence of the velocity on energy. The limiting form of $\rho(t)$ at $\delta = 0$ is

$$\rho_{\text{lim}}(t) = \rho(t)_{\delta = 0} = \begin{cases} r_{\text{min}}, & t < t_{\text{start}}, \\ r_{\text{min}} + (t - t_{\text{start}}), & t > t_{\text{start}}. \end{cases}$$ \hspace{1cm} (10.5)

This world line is shown in Fig. 6.

Consider the function (9.2) for the shell obeying the law of motion (10.1). The line $N$ in Figs. 1 and 6 is the boundary of the region $P$ considered in Section 5. Therefore, when the point $r, t$ in the argument of the function $F(q, t, r)$ is on the line $N$, the respective points $r_+, t_+$ and $r_-, t_-$ are, at all $q$, at the static sector of the world line of the shell. For the function (9.2) with $\rho(t)$ in Eq. (10.5) this yields the result

$$\mathcal{F}(q, t)_{\delta = 0} = F_{\text{stat}}(q, \rho_{\text{lim}}(t)).$$ \hspace{1cm} (10.6)
where the function $F_{\text{stat}}(q,r)$ is given in Eq. (6.17). The integral that figures in Eq. (9.6) is then already calculated in Eq. (6.18). One obtains

$$W(\mu, t)\bigg|_{\delta=0} = -\frac{1}{2 \mu^2} + \frac{(1 + \mu \rho_{\lim}(t))}{2 \mu r_{\min}} \left\{ \exp[-\mu (\rho_{\lim}(t) - r_{\min})] - \exp[-\mu (\rho_{\lim}(t) + r_{\min})] \right\}. \tag{10.7}$$

For $t \leq t_{\text{start}}$ this brings one back to Eq. (5.15), but for $t > t_{\text{start}}$ one has

$$W(\mu, t)\bigg|_{\delta=0} = -\frac{1}{2 \mu^2} \left( 1 + \mathcal{O} \right), \quad \mathcal{O} \to 0 ; \mu \to \infty \tag{10.8}$$

and the integral in Eq. (9.5) diverges at large $\mu$ :

$$[e_+(t) + e_-(t)]\bigg|_{\delta=0} = \infty , \quad t > t_{\text{start}}. \tag{10.9}$$

The force exerted on the shell is infinite at the ultrarelativistic limit.

The null limit (10.5) for $\rho(t)$ is never reached exactly even with the classical motion, and the next task is obtaining the asymptotic behaviour of the force as $\rho(t)$ approaches the null limit. For that consider any point with a given $t > t_{\text{start}}$ on a timelike world line of the shell. When this point is sufficiently close to the line $N$, the respective point $r_-, t_-$ is, at all $q$, at the static sector of the evolution of the shell (see Fig.3). However, for the point $r_+, t_+$ this is not the case. Rather the range of variation of $q$ should be divided into two: the one for which $t_+ < t_{\text{start}}$ and the one for which $t_+ > t_{\text{start}}$. It follows from Eq. (6.1) that the former range is

$$-\infty < q < -\frac{1}{2} s^2(t), \tag{10.10}$$

and the latter is

$$-\frac{1}{2} s^2(t) < q < 0 \tag{10.11}$$

where

$$s(t) = \sqrt{(t-t_{\text{start}})^2 - (\rho(t) - r_{\min})^2}. \tag{10.12}$$

Only in the range (10.10) does one have

$$\mathcal{F}(q, t) = F_{\text{stat}}(q, \rho(t)). \tag{10.13}$$

Expression (9.6) first integrated by parts as in Eq. (8.1):

$$W(\mu, t) = \frac{1}{2 \mu^2} - \frac{2}{\mu} \int_0^\infty d\sqrt{-2q} J_1(\mu \sqrt{-2q}) q \frac{\partial}{\partial q} \mathcal{F}(q, t) \tag{10.14}$$

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may now be written in the form
\[ W(\mu, t) = \frac{1}{2\mu^2} - \frac{2}{\mu} \int_0^\infty d\sqrt{-2q} J_1(\mu\sqrt{-2q}) q \frac{\partial}{\partial q} F_{\text{stat}}(q, \rho(t)) \]
\[ + \frac{2}{\mu} \int_0^1 dx J_1(x\mu s(t)) \left[ q \frac{\partial}{\partial q} F_{\text{stat}}(q, \rho(t)) \right]_{2q=x^2s^2(t)} \]
\[ - \frac{2}{\mu} \int_0^1 dx J_1(x\mu s(t)) \left[ q \frac{\partial}{\partial q} F(q, t) \right]_{2q=x^2s^2(t)} \]  \hspace{1cm} (10.15)

It will be noted that \( s(t) \) is the two-dimensional geodetic distance between the point of start and a point on the shell. Therefore, when the latter point is on the null line \( N \), \( s(t) \) vanishes. Indeed, the insertion of the limiting form (10.5) for \( \rho(t) \) in Eq. (10.12) yields
\[ s(t) \bigg|_{\delta=0} = 0 \ , \quad t > t_{\text{start}} \right. \]  \hspace{1cm} (10.16)

With \( s(t) = 0 \), the last two integrals in Eq. (10.15) vanish, and one recovers the result (10.7). Thus \( s(t) \) serves in Eq. (10.15) as a parameter of proximity of the law \( \rho(t) \) to its ultrarelativistic limit.

For obtaining the asymptotic behaviour of \( W(\mu, t) \) as \( s(t) \to 0 \), rewrite the last two integrals in Eq. (10.15) as
\[ \frac{2}{\mu} s(t) \int_0^1 dx J_1(x\mu s(t)) \left[ q \frac{\partial}{\partial q} F_{\text{stat}}(q, \rho(t)) \right]_{2q=x^2s^2(t)} \]
\[ - \frac{2}{\mu} s(t) \int_0^1 dx J_1(x\mu s(t)) \left[ q \frac{\partial}{\partial q} F(q, t) \right]_{2q=x^2s^2(t)} \]  \hspace{1cm} (10.17)

and recall that \( W(\mu, t) \) is needed at large \( \mu \). The approximation of interest is, therefore,
\[ s(t) \to 0 \ , \quad \mu s(t) = \text{finite} \]  \hspace{1cm} (10.18)

At this limit, the behaviours of the integrals (10.17) are obtained by expanding
\[ \left[ q \frac{\partial}{\partial q} F_{\text{stat}}(q, \rho(t)) \right]_{q \to 0} = \frac{1}{2} + O(q) \ , \quad t > t_{\text{start}} \]  \hspace{1cm} (10.19)
\[ \left[ q \frac{\partial}{\partial q} F(q, t) \right]_{q \to 0} = \frac{1}{4} + O(\sqrt{-2q}) \]  \hspace{1cm} (10.20)
Here use is made of Eqs. (7.4) and (7.8). Using also the explicit form (10.7) of the first two terms in Eq. (10.15), one obtains finally

\[ \mathcal{W}(\mu, t) = -\frac{J_0(\mu s(t))}{2\mu^2} (1 + \mathcal{O}), \quad \mathcal{O} \to 0, \mu \to \infty, s(t) \to 0. \quad (10.21) \]

Eq. (10.21) is the sought for asymptotic formula for the ultrarelativistic motion. Setting in it \( s(t) = 0 \), one recovers the limiting behaviour (10.8) which caused the divergence of the integral in Eq. (9.5). With the function (10.21) this integral converges:

\[ [e_+(t) + e_-(t)] \bigg|_{s(t) \to 0} = e - e \frac{\kappa^2}{24\pi} \int_0^\infty \frac{d\mu^2}{\mu^2} \Gamma(\mu^2) J_0(\mu s(t)) \left(1 + \mathcal{O}(1)\right), \quad (10.22) \]

and its behaviour as \( s(t) \to 0 \) can be found by calculating

\[ \frac{\partial}{\partial s} \int_0^\infty \frac{d\mu^2}{\mu^2} \Gamma(\mu^2) J_0(\mu s) = -2 \int_0^\infty dx \Gamma\left(\frac{x^2}{s^2}\right) J_1(x). \quad (10.23) \]

Since \( \Gamma(\infty) = 1 \), one obtains

\[ e_+(t) + e_-(t) = e + e \frac{\kappa^2}{12\pi} \log(ms(t)) + \kappa^2O(1) \quad (10.24) \]

where \( O(1) \) denotes the terms that remain finite at the ultrarelativistic limit.

The \( s(t) \) in Eq. (10.12) can be represented in the form

\[ s(t) = (t - t_{\text{start}})\sqrt{1 - \dot{\rho}^2(t)}, \quad t > t_{\text{start}} \quad (10.25) \]

where \( \tilde{t} \) is some time instant between \( t_{\text{start}} \) and \( t \). By Eq. (10.1),

\[ \log\left(1 - \dot{\rho}^2(\tilde{t})\right) = \log\left(1 - \dot{\rho}^2(t)\right) + O(1), \quad (10.26) \]

and, therefore, expression (10.24) may finally be written as

\[ e_+(t) + e_-(t) = e + e \frac{\kappa^2}{24\pi} \log\left(1 - \dot{\rho}^2(t)\right) + \kappa^2O(1). \quad (10.27) \]

By derivation, the remainder \( O(1) \) in Eq. (10.27) is bounded uniformly in energy but not necessarily in time. Because of Eq. (10.2), one may worry about the vicinity of \( t = t_{\text{start}} \). However, also at \( t = t_{\text{start}} \), expression (10.27) yields the correct result, Eq. (5.18), provided that condition (3.15) is fulfilled. Since \( \kappa^2 \) is small, all bounded terms of order \( \kappa^2 \) may be regarded as negligible corrections. The term \( \kappa^2O(1) \) in Eq. (10.27) can then be discarded for all times and energies.
11 Vacuum back-reaction on the motion of the shell

The expression (10.27) with the term $\kappa^2 O(1)$ discarded is to be inserted in Eq. (3.12). Then the equation of motion of the shell closes and takes the form

$$M \frac{d}{dt} \left( \frac{\dot{\rho}}{\sqrt{1 - \rho^2}} \right) = \frac{e^2}{2\rho^2} \left[ 1 + \frac{\kappa^2}{24\pi} \log(1 - \rho^2) \right].$$

(11.1)

The last term on the right-hand side of this equation is the force of the vacuum reaction. As will be clear in the next section, this is the force of the back-reaction of a radiation produced by the charged shell in the vacuum.

The force of the vacuum back-reaction depends on the velocity. Nevertheless, the equation of motion (11.1) admits the energy integral:

$$M \int_{\rho}^{1/\sqrt{1 - \rho^2}} \frac{dx}{1 - \frac{\kappa^2}{12\pi} \log x} + \frac{1}{2} \frac{e^2}{\rho} = \mathcal{E}$$

(11.2)

which at $\kappa^2 = 0$ goes over into the classical law (3.20). That the constant $\mathcal{E}$ is indeed the energy of the initial state is seen from the fact that, at $\dot{\rho} = 0$, one recovers Eq. (3.16).

There is no problem with the singularity of the integral in Eq. (11.2). It is never reached. As in Eq. (3.20), for a given energy, the velocity $\dot{\rho}$ reaches its maximum value at $\rho = \infty$ but the value is now different:

$$\int_{\rho}^{1/\sqrt{1 - \rho^2}} \frac{dx}{1 - \frac{\kappa^2}{12\pi} \log x} = \frac{\mathcal{E}}{M}.$$  

(11.3)

As in Eq. (3.20), $\dot{\rho}(\infty)$ grows with $\mathcal{E}/M$ but not up to 1:

$$\dot{\rho}(\infty) = 1 - \frac{1}{2} \exp \left( -\frac{24\pi}{\kappa^2} \right), \quad \frac{\mathcal{E}}{M} \to \infty$$

(11.4)

and this is the principal consequence of the vacuum back-reaction.

Eq. (11.2) is surprising. The coupling to the vacuum charges does not change the electric potential$^5$ as one could expect. It changes the kinematics of motion as relativity theory does. Furthermore, within a given type of coupling, this change is universal. It

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$^5$Under condition (3.15).
does not depend on the parameters of the source, only on the coupling constant $\kappa^2$. There emerges a new kinematic bound on the velocity of a charged body. As shown below, this bound is crucial for the maintenance of the conservation laws.
12 Emission of charge

The singularity of \( e(t, r) \) on the shell’s surface, as calculated in Section 8, has an important feature. Namely, the coefficient of the divergent log in Eq. (8.25), or, equivalently, the coefficient of \( 1/\mu^2 \) in Eq. (7.20) is constant in time. This suggests that the singularity comes from the static contribution which is present in \( e_\pm(t) \) but cancels in the difference

\[
e_\pm(t_1) - e_\pm(t_2) . \tag{12.1}
\]

The flux of charge across the shell should, therefore, be finite.

Indeed, from Eqs. (6.11), (6.12), and (9.2) one obtains

\[
e_\pm(t_1) - e_\pm(t_2) = e \frac{\kappa^2}{24\pi} \int_{4m^2}^{\infty} d\mu^2 \Gamma(\mu^2) \left[ w(\mu, t_1, \rho(t_1) \pm 0) - w(\mu, t_2, \rho(t_2) \pm 0) \right] , \tag{12.2}
\]

\[
w(\mu, t_1, \rho(t_1) \pm 0) - w(\mu, t_2, \rho(t_2) \pm 0) = \int_{-\infty}^{0} dq J_0(\mu\sqrt{-2q}) \left[ \mathcal{F}(q, t_1) - \mathcal{F}(q, t_2) \right] . \tag{12.3}
\]

It follows that, first, the quantity (12.3) is continuous across the shell:

\[
w(\mu, t_1, \rho(t_1) + 0) - w(\mu, t_2, \rho(t_2) + 0) = w(\mu, t_1, \rho(t_1) - 0) - w(\mu, t_2, \rho(t_2) - 0) , \tag{12.4}
\]

and, therefore,

\[
e_+(t_1) - e_+(t_2) = e_-(t_1) - e_-(t_2) . \tag{12.5}
\]

Second, by Eq. (7.18) the quantity (12.3) is \( O(1/\mu^2) \), \( \mu \to \infty \), and, therefore, the difference (12.2) is finite.

For obtaining the flux of charge across the shell, no new calculation is needed. Denote

\[
\Delta e = e_+(-\infty) - e_+(\infty)
\]

\[
= e_-(\infty) - e_-(\infty) . \tag{12.6}
\]

This is the charge emitted by the shell for the whole of its history. Owing to Eq. (12.5), one may write

\[
\Delta e = \frac{1}{2} [e_+(-\infty) + e_-(\infty)]
\]

\[
- \frac{1}{2} [e_+(\infty) + e_-(\infty)] \tag{12.7}
\]
and thereby relate the radiation of charge to the force of its back-reaction. The latter has already been considered in Sections 9-11. For the ultrarelativistic shell one has from Eq. (10.27)

\[ e_+ (\infty) + e_- (\infty) = e + e \frac{\kappa^2}{24 \pi} \log \left( 1 - \dot{\rho}^2 (\infty) \right) + \kappa^2 O(1) , \tag{12.8} \]

and, from Eq. (5.18),

\[ e_+ (-\infty) + e_- (-\infty) = e + \kappa^2 O \left( \frac{1}{m r_{\text{min}}} \right) . \tag{12.9} \]

Hence

\[ \Delta e = - e \frac{\kappa^2}{48 \pi} \log \left( 1 - \dot{\rho}^2 (\infty) \right) + \kappa^2 O(1) . \tag{12.10} \]

Also the instantaneous radiation flux can readily be estimated. Let \( t_1 < t_2 \) be two time instants belonging to the epoch of the rapid expansion of the shell. The amount of charge emitted by the shell for the time between \( t_1 \) and \( t_2 \) is the quantity (12.1):

\[ e_\pm (t_1) - e_\pm (t_2) = e \frac{\kappa^2}{48 \pi} \log \frac{1 - \dot{\rho}^2 (t_1)}{1 - \dot{\rho}^2 (t_2)} + \kappa^2 O(1) . \tag{12.11} \]

From Eq. (10.1) one infers that this is a negligible amount:

\[ e_\pm (t_1) - e_\pm (t_2) = \kappa^2 O(1) , \quad t_{\text{start}} < t_1 < t_2 . \tag{12.12} \]

However, if in Eq. (12.11) one takes \( t_{\text{start}} = t_1 \), i.e., if one calculates the amount of charge emitted from the beginning of expansion by the instant \( t \), the result will be different:

\[ e_\pm (t_{\text{start}}) - e_\pm (t) = - e \frac{\kappa^2}{48 \pi} \log \left( 1 - \dot{\rho}^2 (t) \right) + \kappa^2 O(1) \]
\[ = - e \frac{\kappa^2}{48 \pi} \log \left( 1 - \dot{\rho}^2 (\infty) \right) + \kappa^2 O(1) , \quad t_{\text{start}} < t . \tag{12.13} \]

This is easy to understand. The cause of the vacuum particle creation is the acceleration of the source. The shell radiates at a short stage of its evolution near \( t = t_{\text{start}} \) where its acceleration is maximum [3]. Almost all the emitted charge \( \Delta e \) is released at this stage. Therefore, up to a small correction, the quantity (12.13) is constant.

Thus the rate of emission of charge by the ultrarelativistic shell is

\[ \frac{\Delta e}{e} = - \frac{\kappa^2}{48 \pi} \log \left( 1 - \dot{\rho}^2 (\infty) \right) , \quad \dot{\rho} (\infty) \rightarrow 1 . \tag{12.14} \]
The rate of emission of energy was calculated in Ref. [3]. Generalized properly, this calculation yields the same result as in Eq. (12.14):

\[ \frac{\Delta E}{E} = -\frac{\kappa^2}{48\pi} \log \left( 1 - \rho^2(\infty) \right), \quad \rho(\infty) \to 1. \] 

(12.15)

One sees that the radiation rate grows unboundedly as the motion of the shell approaches the ultrarelativistic limit. Inserting in Eqs. (12.14) and (12.15) the \( \rho(\infty) \) calculated from the classical law of motion (3.20), one obtains the result [3]:

\[ \frac{\Delta e}{e} = \frac{\Delta E}{E} = \frac{\kappa^2}{24\pi} \log \frac{E}{M}, \quad \frac{E}{M} \to \infty \] 

(12.16)

which manifestly contradicts the conservation laws. However, the result (12.16) does not take into account the back-reaction of radiation. The vacuum friction does not allow the velocity of the source to approach the speed of light closer than the limit (11.4). The insertion of the expression (11.4) in Eqs. (12.14) and (12.15) restores the conservation laws:

\[ \frac{\Delta e}{e} = \frac{\Delta E}{E} = \frac{1}{2}, \quad \frac{E}{M} \to \infty. \] 

(12.17)

Up to 50% of energy and charge can be extracted from the source by raising its initial energy.
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References


Figure captions

Fig.1. The world line of the shell on the $r, t$ plane. The broken lines bound the future light cone of the point of start. The broken line $N$ is the world line of the outgoing radial light ray.

Fig.2. The function $e(t, r)$ for a given $t$.

Fig.3. The world line of the shell crosses the past hyperboloid of the observation point. For definiteness, the observation point is shown inside the shell.

Fig.4. $L$ is the line specified by Eq. (8.6), and $r^*, t^*$ is the point at which it crosses the world line of the shell. The observation point $r, t$ is shown inside the shell, and the broken lines mark its light cone.

Fig.5. The band of quantum uncertainty around the world line of the shell narrows as the speed of expansion increases.

Fig.6. The world line of the shell at the ultrarelativistic limit.
Fig. 1

start

$r=0$

$r=r_{\text{min}}$

$N$
Fig. 2
Fig. 3
Fig. 6