The dynamical symmetry breaking in a two-field model is studied by numerically solving the coupled effective field equations. These are dissipative equations of motion that can exhibit strong chaotic dynamics. By choosing very general model parameters leading to symmetry breaking along one of the field directions, the symmetry broken vacua makes the role of transitory strange attractors and the field trajectories in phase space are strongly chaotic. Chaos is quantified by means of the determination of the fractal dimension, which gives an invariant measure for chaotic behavior. Discussions concerning chaos and dissipation in the model and possible applications to related problems are given.

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The study and understanding of the dynamics of fields are a timely subject and of broad interest, with applications in diverse areas like in particle physics, cosmology and in condensed matter (for a recent review, see [1] and references therein). Additional interest on the subject comes from the fact that many of the theoretical ideas and models can be tested in ongoing experiments, as those been performed in condensed matter systems, and in the future ones, in the entering of operation of the RHIC and LHC heavy ion colliders, which will be able to probe possible new phenomena at the QCD scale and space based experiments, which will be putting on test different cosmological models. It is then becoming urgent the detailed investigation of the underline field dynamics that may be common to all these very different areas of physical research.

In this Letter we are mainly concerned with the connection between the development of strong nonlinearities in the time evolving system of equations of motion of a given field theory model and the possible chaotic behavior associated to them. Lets recall that in symmetry breaking phase transitions we are usually interested in the study of the evolution of a given order parameter, for example the magnetization in spin systems in statistical physics or a vacuum expectation value of some scalar field (the Higgs) in particle physics, which gives a measure of the degree of organization of the system at the macroscopic level. However, at the microscopic level disorder is related to the nonlinearities and fluctuations responsible to chaotic behavior in the system and these chaotic motion phenomena can reflect in a nontrivial time dependence of the macroscopic quantities and, therefore, influencing all the dynamics of the system. This is clear once several properties of the system at longer times are closely related to the microscopic physics, like relaxation to equilibrium, phase ordering, thermalization and so on. Thus we expect that chaos will be not only an important ingredient in determining the final states of a given system but also in how it gets there.

Previous studies on chaos in field theory have mostly emphasized chaotic behavior in gauge theory models (for a review and additional references and applications, see [2]). In special, in homogeneous Yang-Mills-Higgs models we can reduce the system of classical equations of motion to ones analogous to those of nonlinear coupled oscillators, which is well known to exhibit chaotic motion.

While in all the previous works on chaotic dynamics of fields dealt with (conservative) Hamiltonian systems, here we will be mainly concerned with the effective field evolution equations, which are known to be intrinsically dissipative [3–9] and, therefore, the dynamical system we will be studying is non-Hamiltonian.

Thus, we will be concerned with the influence of field dissipation, due to field decaying modes, on the degree of chaoticity of the field dynamics. Typically we expect that dissipation damps the fluctuations on the system and consequently tends to suppress possible chaotic motions and makes field trajectories in phase space to tend faster to the system asymptotic states. On the other hand there are well known examples of dynamical systems, as the Lorenz system [10], which are dissipative ones and at the same time they display a very rich evolution on phase space in which, under appropriate system parameters, field trajectories may be lead towards strange attractors. The verification of the same properties in a model motivated by particle physics would be a novel result with possible consequences to, for example, particle physics phenomenology and cosmology.

We study chaos in our dynamical system of equations by means of the measure of the fractal dimension (or dimension information) [for a review and definitions, see e.g., [10]], which gives a topological measure of chaos for different space-time settings and it is a quantity invariant under coordinate transformations, providing then an unambiguous signal for chaos [11]. The method we apply in this work for quantifying chaos will then be particularly useful in our planned future applications of our model and it extensions to a cosmological context, in which case other methods may be ambiguous, like, for ex-
example, the determination of Lyapunov exponents, which does not give a coordinate invariant measure for chaos, as discussed in [11]. Also, other methods for studying chaotic systems, like for example by Poincaré sections, are not suitable in the case we are interested here, in which chaos is a transitory phenomenon as we will see.

The model we will study consists of two scalar fields in interaction with Lagrangian density given by

\[
\mathcal{L}[\Phi, \Psi] = \frac{1}{2} (\partial_{\mu} \Phi)^2 - \frac{m_{\Phi}^2}{2} \Phi^2 - \frac{\lambda_{\Phi}}{4!} \Phi^4 + \frac{1}{2} (\partial_{\mu} \Psi)^2 - \frac{m_{\Psi}^2}{2} \Psi^2 - \frac{\lambda_{\Psi}}{4!} \Psi^4 - \frac{g^2}{2} \Phi^2 \Psi^2 . \tag{1}
\]

All coupling constants are positive and \(m_{\Phi}^2 > 0\), but we choose \(m_{\Psi}^2 < 0\), such that we allow for spontaneous symmetry breaking in the \(\Psi\)-field direction. Additionally, note that from the above Lagrangian, for values of \(\Phi\) larger than a \(\Phi_{cr}\), where \(\Phi_{cr}^2 = |m_{\Psi}|^2 / g^2\), there is no symmetry breaking in the \(\Psi\)-field direction. Thus, for example, if we have an initial state prepared at \(\Phi > \Phi_{cr}\), the \(\Psi\) field will move towards zero and remain around that state till eventually \(\Phi\) crosses below the critical value inducing \(\Phi \neq 0\).

We have defined and the specific numerical implementation we use here have been described in details in Ref. [12].

The fractal dimension is the box-counting method, whose definition and the specific numerical implementation we use here have been described in details in Ref. [12].

Quantum corrections are taken into account in the effective equations of motions (EOMs) by use of Weinberg tadpole method. Let \(\phi_c\) and \(\psi_c\) be the expectation values for \(\Phi\) and \(\Psi\), respectively. Splitting the fields in (1) in the expectation values and fluctuations, \(\Phi \to \phi_c + \phi\) and \(\Psi \to \psi_c + \psi\), where \(\langle \Phi \rangle = \phi_c\) and \(\langle \Psi \rangle = \psi_c\), the EOMs for \(\phi_c\) and \(\psi_c\) are obtained by imposing that \(\langle \phi \rangle = 0\) and \(\langle \psi \rangle = 0\), which lead to the condition that the sum of all tadpole terms for each field vanishes. Let us restrict our analysis of the EOMs to homogeneous fields \((\phi_c \equiv \phi_c(t), \psi_c \equiv \psi_c(t))\). Thus, at 1-loop order we can write the following EOMs for \(\phi_c\) and \(\psi_c\), respectively, as 1

\[
\ddot{\phi}_c + m_{\phi}^2 \phi_c + \frac{\lambda_{\phi}}{6} \phi_c^3 + g^2 \phi_c \psi_c^2 + \frac{\lambda_{\phi}}{2} \phi_c \langle \phi^2 \rangle + g^2 \phi_c \langle \psi^2 \rangle + 2g^2 \phi_c \langle \phi \psi \rangle = 0 \tag{2}
\]

and

\[
\ddot{\psi}_c + m_{\psi}^2 \psi_c + \frac{\lambda_{\psi}}{6} \psi_c^3 + g^2 \psi_c \phi_c^2 + \frac{\lambda_{\psi}}{2} \psi_c \langle \psi^2 \rangle + g^2 \psi_c \langle \phi^2 \rangle + 2g^2 \psi_c \langle \phi \psi \rangle = 0 , \tag{3}
\]

where \(\langle \phi^2 \rangle\) and \(\langle \psi^2 \rangle\) are given in terms of the coincidence limit of the (causal) two-point Green’s functions \(G_{\phi}^{++}(x, x')\) and \(G_{\psi}^{++}(x, x')\), which are obtained from the (1, 1)-component of the real time matrix of full propagators which satisfy the appropriate Schwinger-Dyson equations (see, e.g., Refs. [6] and [9] for details).

By perturbatively expanding 2 the field averages in Eqs. (2) and (3), we can write for the EOMs 3

\[
\ddot{\phi}_c(t) + \ddot{\phi}_c(t) + \frac{\lambda_{\phi}}{6} \phi_c(t) + g^2 \phi_c(t) \psi_c(t) + \frac{\lambda_{\phi}}{2} \phi_c(t) \phi_c(t) \psi_c(t) + g^2 \phi_c(t) \psi_c(t) = 0 \tag{4}
\]

and

\[
\ddot{\psi}_c(t) + \ddot{\psi}_c(t) + \frac{\lambda_{\psi}}{6} \psi_c(t) + g^2 \psi_c(t) \phi_c(t) + \frac{\lambda_{\psi}}{2} \psi_c(t) \psi_c(t) + g^2 \psi_c(t) \phi_c(t) = 0 . \tag{5}
\]

1Note that the mixed field averages in (2) and (3) can only be treated within perturbation theory.

2Throughout this work we assume couplings \(g, \lambda_{\phi, \psi} \ll 1\) such that perturbation theory can be consistently formulated and subleading terms can be neglected.

3See, e.g., Ref. [6] for the explicit expressions for the two-point functions appearing in the EOMs.
where $\bar{m}_\phi^2 = m_\phi^2 + \lambda_\phi/2(\phi^2)_0 + g^2(\psi^2)_0$ and $\bar{m}_\psi^2 = m_\psi^2 + \lambda_\psi/2(\psi^2)_0 + g^2(\phi^2)_0$, with $(\ldots)_0$ meaning fields independent averages. The nonlocal terms of the type appearing in the above equations have been shown in Refs. [3-6] to lead to dissipative dynamics in the EOMs.

The two coupled nonlocal EOMs above are too complicated to directly numerically work with them. Similar time nonlocalities as the ones appearing in (4) and (5) have been dealt with in [3,6] by use of an adiabatic (or sudden) approximation for the fields. As shown in [6] this is a consistent approximation in the case the fields are in an overdamped regime. The strong field dissipation responsible for the overdamped regime can be attained by coupling both $\Phi$ and $\Psi$ fields to a set of other field degrees of freedom and then enlarging the number of field decay channels available. This idea was used in [7] for the construction of an alternative inflationary model. However, for the specific objective we have in mind here, which is to study the chaoticity in the system of equations (4) and (5), in the overdamped regime chaos gets completely suppressed. Chaotic motion can only develop in the underdamped or very weakly damped regime, in which case enough energy can be exchanged fast enough from one field to the other, making both $\phi_c$ and $\psi_c$ to fluctuate with large enough amplitudes, leading to a highly nonlinear behavior that then precludes the chaotic motion of the system. A simple example is shown in Fig. 1 for a weakly damped system, showing highly chaotic dynamics prior to symmetry breaking. In this regime another type of approximation for the nonlocal terms may be more appropriate, such as the harmonic approximation for the nonlocal terms may be more suitable for numerical analysis (and that still retain the

\[ \ddot{\phi}_c + \bar{m}_\phi^2 \dot{\phi}_c + \frac{\lambda_\phi}{6} \phi_c^3 + \bar{g}^2 \phi_c \psi_c^2 + \eta_{\phi\phi} \dot{\phi}_c \dot{\psi}_c + \eta_{\phi\psi} \phi_c \dot{\psi}_c = 0 \]  
and
\[ \ddot{\psi}_c + \bar{m}_\psi^2 \dot{\psi}_c + \frac{\lambda_\psi}{6} \psi_c^3 + \bar{g}^2 \psi_c \phi_c^2 + \eta_{\psi\phi} \dot{\phi}_c \dot{\psi}_c + \eta_{\psi\psi} \dot{\phi}_c \dot{\psi}_c = 0, \]

where $\tilde{\lambda}_{\phi,\psi}, \tilde{g}$ and $\tilde{m}_{\phi,\psi}$ denote the renormalized effective couplings and masses for the $\Phi$ and $\Psi$ fields\footnote{The renormalization of the coupling constants can be made apparent by an appropriate integration by parts in the time integrals in Eqs. (4) and (5) (see Ref. [4]).}. $\eta_{\phi}$, $\eta_{\psi}$ and $\eta_{\phi\psi}$ denote dissipation coefficients. These coefficients can be explicitly evaluated in either the adiabatic or harmonic approximations for Eqs. (4) and (5). However, we refrain ourselves from an explicit evaluation of these terms, which may depend on the various parameters of the model, temperature and on the number of other field degrees making the environment that $\Phi$ and $\Psi$ may be coupled to (in fact, the magnitude of the dissipation terms may be controlled by these additional field couplings, as shown in [6,7]). For the sake of simplicity we just take $\eta_\phi$, $\eta_\psi$ and $\eta_{\phi\psi}$ as additional free constant parameters.

Let define $a^2 = \bar{m}_\phi^2/(6|\bar{m}_\phi^2|)$, $G^2 = \bar{g}^2/(a^2 \tilde{\lambda}_\phi)$, $\lambda_x = \tilde{\lambda}_\phi/(a^4 \tilde{\lambda}_\psi)$ and the dimensionless variables $x = \sqrt{\lambda_x a^2 \bar{m}_\phi \phi_c}$, $z = a^2/(\bar{m}_\phi \langle \Psi \rangle_v) \phi_c$, $y = 1/(\langle \Psi \rangle_v \psi_c$, $w = 1/(\sqrt{6|\bar{m}_\psi^2| \langle \Psi \rangle_v} \psi_c$ and rescaling time and dissipation coefficients as $t' = \sqrt{6|\bar{m}_\psi^2|} t$, $\eta_\phi = a^4 \sqrt{\lambda_x/\langle \Psi \rangle_v} \eta_x$, $\eta_\psi = \sqrt{\lambda_x/\langle \Psi \rangle_v} \eta_y$ and $\eta_{\phi\psi} = a^2 \sqrt{\lambda_x/(\langle \Psi \rangle_v \eta_{xy}}$, respectively, we can write Eqs. (6) and (7) in terms of the following dimensionless first-order differential system of equations:

\begin{align*}
\dot{x} &= z \\
\dot{z} &= -a^2 \left( x + \frac{\lambda_x}{6} x^3 + G^2 x y^2 + \eta_x x^2 z + \eta_{xy} x y w \right) \\
\dot{y} &= w \\
\dot{w} &= \frac{y}{6} - G^2 x^2 y - \eta_y y^2 w - \eta_{xy} x y z .
\end{align*}

(8)

For convenience we also choose parameters such that $a^2 = 1$, $G^2 = 1$ and $\lambda_x = 1$ (which means we consider $\bar{m}_\phi^2 = 6|\bar{m}_\phi^2|$ and $\bar{g}^2 = \tilde{\lambda}_\phi = \tilde{\lambda}_\psi$). We also take as base values for the (dimensionless) dissipation coefficients the values $\{ \eta \} = (\eta_x, \eta_y, \eta_{xy}) = (1/120, 1/240, 1/200)$ for which the dynamics displayed by (8) happen in the weakly damped regime. We then numerically solve the dynamical system with initial conditions taken such that at $t = 0$ the potential in the Lagrangian density (1) is symmetrical in both field directions. The system is then evolved in time till the symmetry breaking in the $\Psi$-field direction occurs. We then look for chaotic regimes as $\eta_\phi$, $\eta_\psi$ and $\eta_{\phi\psi}$ are changed.

Our choice for initial conditions to numerically solving the (dissipative) dynamical system (8) is as follows. At the initial time we consider $(\phi, \phi_c, \psi, \psi_c)|_{t=0} = (4\Phi_{cr}, 0, 0, 0)$. Around this initial condition it is considered a box in phase space (for the dimensionless variables) of size $10^{-5}$, inside which a large number of random points are taken (a total of 200,000 random points were used in each run). All initial conditions are then numerically evolved by using an eighth-order Runge-Kutta integration method and the fractal dimension is obtained by statistically studying the outcome of each initial condition at each run of the large set of points. Special care is taken to keep the statistical error in the results always below $\sim 1\%$. The results obtained are shown in Table I for different values of dissipation coefficients. The uncertainty exponent $\epsilon$ (see Ref. [12]) gives a measure of
how chaotic is the system. Qualitatively speaking we can say that the closest is $\varepsilon$ of zero, the more chaotic is the system. On the contrary, the closest is $\varepsilon$ of unit, the less chaotic is the system. We see clearly the effect of dissipation on the nonlinearities of the system. It can change fast from a chaotic to an integrable regime with a relatively small increase of the dissipation. Larger dissipations tend also to destroy fast the chaotic attractors. In Fig. 2 we show an example of the structure of the chaotic attractors in the $\psi_c, \psi_e$ plane ($y, w$ plane).

It seems also that we can indirectly associate the chaoticity in the system with the equilibration rates of the fields, in close analogy with the one found between the Lyapunov exponent and the thermalization rate in perturbative thermal gauge theory [13]. We note from the results obtained here and from the numerical simulations we performed, that the smaller is the fractal dimension (or the larger is $\varepsilon$) the fastest the fields equilibrate to their asymptotic states by loosing their energies to radiation, which will then eventually thermalize.

Finally we would like to point out that the kind of model we have studied here and its generalization to larger number of fields is natural to be found in extensions of the standard model with large scalar sectors. Physical implementations of the model can be found, for example, in particle physics or condensed matter models displaying multiple stages of phase transitions, in which case the dynamics we have studied here would be likely to manifest between any of that stages and, therefore, with consequences to the phenomenology of that models. These problems are currently been examined by us and more results and details will be reported elsewhere.

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