Abstract. We examine Fermi-type acceleration at relativistic shocks, and distinguish between the initial boost of the first shock crossing cycle, where the energy gain per particle can be very large, and the Fermi process proper with repeated shock crossings, in which the typical energy gain is of order unity. We calculate by means of numerical simulations the spectrum and angular distribution of particles accelerated by this Fermi process, in particular in the case where particle dynamics can be approximated as small-angle scattering. We show that synchrotron emission from electrons or positrons accelerated by this process can account remarkably well for the observed power-law spectra of GRB afterglows and Crab-like supernova remnants. In the context of a decelerating relativistic fireball, we calculate the maximum particle energy attainable by acceleration at the external blast wave, and discuss the minimum energy for this acceleration process and its consequences for the observed spectrum.

INTRODUCTION

The spectrum of emission from GRB afterglows is well-accounted for by synchrotron emission from electrons accelerated at a decelerating relativistic blast wave. The mechanism responsible for the gamma-ray-burst emission itself is less well-established, but it has been interpreted as synchrotron emission as well, from electrons accelerated either at internal, mildly relativistic shocks, or at the ultra-relativistic external shock as it runs into a clumpy medium.

In current models of afterglow emission, however (e.g. [1]), the particle acceleration physics is simply described by two parameters which are left to be adjusted to the observations: the shock is assumed to accelerate the electrons to a power-law spectrum of index $p$, with a lower cutoff $E_{\text{min}}$ which is simply related to the efficiency of energy conversion into these accelerated electrons.
In what follows, we first examine more closely the spectral index \( p \) that can be theoretically expected for Fermi-type acceleration at relativistic shocks and compare it with observed values, and then consider the maximum and minimum energies over which this spectrum can extend, \( E_{\text{max}} \) and \( E_{\text{min}} \), and their consequences for observations.

**FERMI ACCELERATION AT RELATIVISTIC SHOCKS**

Shock-crossing kinematics and energy gain

In what follows we restrict our attention to ultra-relativistic shocks, i.e. those with Lorentz factor \( \Gamma_{\text{sh}} \gg 1 \) with respect to the upstream medium. For a weakly magnetised shock, the shock jump conditions then imply a relative Lorentz factor \( \Gamma_{\text{rel}} \approx \Gamma_{\text{sh}}/\sqrt{2} \) between the downstream and upstream media, and a shock velocity of \( c/3 \) relative to the downstream medium.

Assuming a standard Fermi-type process at the shock, in which charged particles are deflected elastically by magnetic fluctuations in both the upstream and downstream media, the energy change of a particle in a single cycle of crossing and re-crossing the shock is given by

\[
\frac{E_f}{E_i} = \Gamma_{\text{rel}}^2 (1 - \beta_{\text{rel}} \mu_{-d})(1 + \beta_{\text{rel}} \mu_{-u}).
\]

Here \( E_i \) and \( E_f \) are the particle’s initial and final energies, and \( \mu_{-d} \) and \( \mu_{-u} \) the cosine of its direction angle \( \theta \) (between its velocity and the shock normal) upon crossing the shock into the downstream and upstream media respectively. Throughout, primed and unprimed variables refer to quantities measured in the downstream and upstream rest frames, respectively.

Since kinematics require \( 1 \geq \mu_{-u}' > 1/3 \), the energy gain factor (1) will depend most sensitively on the distribution of \( \mu_{-d} \). For a pre-existing isotropic population of relativistic particles upstream, energy gains \( E_f/E_i \) of order \( \Gamma_{\text{rel}}^2 \) can be achieved in the first shock crossing cycle. However, for particles having crossed the shock from downstream, realistic deflection processes upstream yield \( \theta_{-d} \lesssim 2/\Gamma_{\text{sh}} \), so that for all subsequent shock crossing cycles, on average the particle energy is only roughly doubled by each cycle [2].

**Numerical simulations and the spectral index \( p \)**

The power-law index of the accelerated particle spectrum depends on the average energy gain per shock crossing and on the return probability, the chance that a particle crossing downstream will eventually recross the shock upstream. Both these factors are strong functions of the angular distribution
of particles crossing the shock, which as suggested by the above considerations is highly anisotropic. Thus the quasi-isotropic approximations current in non-relativistic shock acceleration do not apply, and we turn to numerical simulations.

For simplicity, we focus our attention here on the case where both the upstream and downstream particle dynamics are dominated by scattering; in other words, we assume that magnetic fluctuations, possibly amplified by turbulence downstream, dominate the effect of the regular magnetic field for transport in the shock normal direction. Since the nature of the particle transport is by assumption independent of particle energy, we decoupled the dynamical problem from the energy gains, and first computed a numerical approximation to the function $f_d(\mu'_{-u}; \mu'_{-d})$, the distribution of downstream egress angles $\mu'_{-u}$ for a given ingress angle, by Monte-Carlo simulation of the downstream scattering process for a grid of $\mu'_{-d}$ values. Along with a similarly obtained representation of the upstream dynamics, $f_u(\mu_{-d}; \mu_{-u})$, and the energy gain formula (1), these constitute the necessary ingredients for a Monte-Carlo calculation of the accelerated particle distribution.

The results of such a calculation are shown in Figure 1(a). Particles are injected at $E'_0$ with $\mu'_{-d} = -1$, but the influence of this highly anisotropic initial condition disappears after a little more than a decade in energy, where the self-consistent angular distribution is established with a smooth power-law dependence, $F(E', \mu') \propto F(\mu')E'^{-p}$. We obtained $p = 2.23$ for this case, in perfect agreement with the results of the semi-analytical eigenfunction method [3]. The angular distribution obtained with that method is compared with the simulation results in Figure 1(b), which shows both the asymptotic flux distribution $F(\mu')$ measured in the simulations and the corresponding density

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) Left panel: Steady-state flux distribution at the shock, $F(E', \mu')$, as a function of downstream particle energy $E'$ and direction angle cosine $\mu'$. (b) Right panel: Asymptotic angular distribution of the particles at shock crossing, expressed both in terms of flux $F(\mu')$ (dotted line) and density $n(\mu')$ (solid line), each normalised to unity. The dashed line shows the distribution $n(\mu')$ obtained by the eigenfunction method for the same case.}
\end{figure}
distribution, \( n(\mu') \propto F(\mu')/(\mu' - 1/3) \). The agreement is excellent except near the loss cone, \( \mu' \approx 1/3 \), where the denominator amplifies Poisson noise in the measured Monte-Carlo flux.

While the spectral index found above is valid only for pure scattering, simulations incorporating more complex transport dynamics, especially upstream where the effect of the regular magnetic field may well not be negligible, yield similar or only slightly steeper spectra in the same ultra-relativistic limit [4,5].

**Comparison with observations**

Early observations of two GRB afterglows suggested \( p = 2.3 \pm 0.1 \) [6], and detailed analysis of the GRB 970805 afterglow spectrum yielded \( p = 2.2 \) [7]. While multi-wavelength spectral analyses have not been carried out in such detail for other afterglows, a value of \( p \approx 2.2 \) seems compatible with most [8]. Moreover, in another class of astrophysical objects where an ultra-relativistic shock is thought to accelerate particles, namely Crab-like supernova remnants, the inferred spectral indices are similar: the best-fit model for the Crab Nebula spectrum corresponds to \( p \) in the range 2.2–2.3 [9].

This spectral index might thus be considered a signature of ultra-relativistic shock acceleration, as the spectral index for acceleration at mildly relativistic shocks is expected to be different [3]. In this connection, it is intriguing to note that for GRB prompt emission, the average value of 2.12 for the high-energy spectral index [10], when interpreted as cooled synchrotron emission, corresponds to \( p = 2.24 \), in excellent agreement with the theoretical value found above. This could perhaps be viewed as spectral evidence that the prompt GRB emission also originates at an ultra-relativistic shock, such as the external shock, rather than a mildly relativistic one, such as internal shocks.

**MAXIMUM AND MINIMUM ENERGIES**

**Age limit and ultra-high-energy cosmic rays**

We first consider the maximum particle energy attainable by Fermi acceleration at a relativistic blast wave in the absence of energy loss processes; this is set by the constraint that the acceleration time \( t_{\text{acc}} \) be shorter than the age of the system. Since the fractional energy gain per shock crossing cycle is typically of order unity, the acceleration time is roughly the cycle time, which is the sum of the upstream and downstream residence times, \( t_u \) and \( t_d \). When the downstream magnetic field is simply the shock-compressed upstream field, one can show that \( t_d \sim t_u \); if the downstream field is amplified from this value, e.g. by turbulence, \( t_d \) is correspondingly shorter, so that the total cycle time is of order \( t_u \). Comparing this with the age of the fireball yields a maximum
energy which is attained at the beginning of the deceleration phase, and has value

$$E_{\text{max}}^{(\text{age})} \simeq 5 \times 10^{15} B_{-6} \left( \frac{\mathcal{E}_{52} \Gamma_3}{n_0} \right)^{1/3} \text{eV},$$

(2)

where $B_{-6}$ is the upstream magnetic field, $\mathcal{E}_{52}$ the isotropic fireball energy, $\Gamma_3$ its initial Lorentz factor and $n_0$ the upstream density, respectively in units of microgauss, $10^{52}$ erg, $10^3$ and cm$^{-3}$. This upper limit has important implications for the hypothesis that ultra-high-energy cosmic rays might be produced in GRBs [2].

**Synchrotron loss limit and spectral upper cutoff**

While the above upper limit is appropriate for cosmic-ray protons, for the electrons responsible for the observed afterglow emission synchrotron losses must also be taken into account. This yields the additional criterion that the energy lost to synchrotron radiation in time $t_u$ upstream and $t_d$ downstream must be less than the energy gained per shock crossing cycle. If the downstream field is simply the compressed upstream one, synchrotron losses upstream and downstream will be comparable; otherwise, the downstream losses will dominate, so we need only consider the latter. Using this criterion yields another upper limit $E_{\text{max}}^{(\text{syn})}$, which is more stringent than (2) when the downstream magnetic field $B'$ exceeds about $0.1 \Gamma_3^{10/9} \xi^{2/3} G$, with weak dependences on the other fireball parameters $\mathcal{E}_{52}$ and $n_0$, where $\xi$ is the field amplification factor above simple compression, i.e. $B' = \xi \sqrt{5} \Gamma_{\text{sh}} B$.

The maximum synchrotron photon energy emitted by electrons of energy $E_{\text{max}}^{(\text{syn})}$ will be roughly 150 MeV in the proper (downstream) frame, independently of the value of the magnetic field. This $B$-independence is a generic result for acceleration times scaling like the gyrotime, with longer $t_{\text{acc}}$ yielding correspondingly lower maximum photon energies. Boosting to the observer’s frame yields

$$h\nu_{c,\text{max}}^{(\text{syn})} \simeq 150 \Gamma_3 \text{ GeV};$$

(3)

the same result also holds for synchrotron-limited acceleration at internal shocks. This suggests that establishing the presence of a cutoff to the synchrotron spectrum in the range of $EGRET$ to TeV gamma-ray energies could place direct constraints on the fireball Lorentz factor.

**Minimum energy and electron pre-acceleration**

The Fermi process requires that particles feel the shock as a discontinuity, which requires that their Larmor radius be larger than the shock thickness,
which is in turn roughly the downstream thermal ion Larmor radius. This sets a minimum electron energy for Fermi acceleration $E'_{\text{min}} \simeq \Gamma_b m_i c^2$, where $m_i$ is the ion mass, corresponding to an observed synchrotron photon energy

$$h\nu_{c,\text{min}} \simeq 160 \xi B_{-6} \Gamma_3^4 \text{keV}.$$  \hspace{1cm} (4)

It is interesting to note that for our fiducial parameters this falls in the BATSE break energy range, although it is unclear how the strong dependence on $\Gamma_3$, in particular, could keep $h\nu_{c,\text{min}}$ constrained to a narrow range of values.

The energy $E'_{\text{min}}$ exceeds by a factor $m_i/m_e$ that resulting from randomisation of the bulk upstream electron energy, and electrons must thus be pre-accelerated by some other process before Fermi acceleration can operate. One candidate for this pre-acceleration mechanism is the resonant ion cyclotron wave acceleration process [11], which efficiently accelerates electrons over precisely the required energy range, and yields harder power-law spectra than those obtained above, providing a possible explanation for the BATSE low-energy spectral indices.

**SUMMARY**

We examined the particle energy gain per shock crossing cycle for Fermi-type acceleration at ultra-relativistic shocks, and found that while the initial shock crossing cycle can yield a very large energy gain, in all subsequent crossing cycles the particle energy on average roughly doubles. We used Monte-Carlo simulations to obtain the spectrum of accelerated particles, and found a power-law spectral index $p = 2.23$ for the case of small-angle scattering, in agreement with the results of the semi-analytical eigenfunction method. This value is compatible with those inferred from observations of GRB afterglows and Crab-like supernova remnants, and might thus be considered a signature of ultra-relativistic shock acceleration.

For protons, the maximum energy attainable by Fermi acceleration at a relativistic blast wave is set by the age limit, and is of order $5 \times 10^{15}$ eV for typical fireball parameters. For electrons, synchrotron losses can become the dominant limiting factor for moderately amplified downstream magnetic fields. In that case the maximum observed synchrotron photon energy is independent of the magnetic field, and is proportional only to the fireball Lorentz boost factor. Finally, electrons must be pre-accelerated to a minimum energy comparable with the thermal ions’ before they can undergo Fermi acceleration; a good candidate for the pre-acceleration mechanism is the resonant ion cyclotron wave acceleration process.
This work was supported by the Netherlands Foundation for Research in Astronomy (ASTRON) under project 781–76–014, and by the European Commission under TMR programme contract ERBFMRX–CT98–0168.

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