A new approach to the parametrization of the 
Cabibbo-Kobayashi-Maskawa matrix

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Abstract

The CKM-matrix $V$ is written as a linear combination of the unit 
matrix $I$ and a matrix $U$ which causes intergenerational-mixing. It 
is shown that such a $V$ results from a class of quark-mass matrices. 
The matrix $U$ has to be hermitian and unitary and therefore can 
depend at most on 4 real parameters. The available data on the CKM-
matrix including CP-violation can be reproduced by $V = (I+iU)/\sqrt{2}$. 
This is also true for the special case when $U$ depends on only 2 real 
parameters. There is no CP-violating phase in this parametrization. 
Also, for such a $V$ the invariant phase $\Phi \equiv \phi_{12} + \phi_{23} - \phi_{13}$, satisfies a 
criterion suggested for 'maximal' CP-violation.

It is more than twenty-five years since the first explicit parametrization 
for the six quark case was given [1] for the so called Cabibbo-Kobayashi-
Maskawa (CKM) matrix. Since then many different parametrizations have 
been suggested [2, 3]. In this note, we wish to suggest a new approach to 
parametrizing the unitary CKM matrix $V$. For this purpose, we write $V$ as 
a linear combination of the unit matrix $I$ and another matrix $U$, so that

$$V(\theta) = \cos \theta I + i \sin \theta U \quad (1)$$

It is clear that for $V$ to be unitary, $U$ has to be both hermitian and 
unitary. Here $\theta$ is a parameter which will be fixed later. In Eq. (1), for the 
first term the physical (or the quark mass-eigenstate) and the gauge bases
are the same. The second term, through $U$, represents the difference in the
two bases. It also causes inter-generational mixing and makes it possible for
$V$ to give CP-violating processes. The break-up of $V$ in two parts makes it
possible to have a simple parametrization. We now show that knowing $V(\theta)$
allows us to construct the quark-mass matrices in terms of the parameters
of $V$ and the quark-masses.

Form of the quark-mass matrices. In the gauge-basis, the part of the
standard model Lagrangian relevant for us can be written as

$$
\mathcal{L} = \bar{q}'_u M_u q'_u + \bar{q}'_d M_d q'_d + \frac{g}{\sqrt{2}} \bar{q}'_u \gamma_\mu q'_d W^\mu_\mu + H.c. 
$$

where $q'_u = (u', c', t')$ and $q'_d = (d', s', b')$. By suitable redefinition of the
right-handed quark fields one can make the quark-mass matrices $M_u$ and $M_d$
hermitian. Let the diagonal forms of the hermitian $M_u$ and $M_d$ be given by

$$
\tilde{M}_u = V_u^\dagger M_u V_u, \quad \tilde{M}_d = V_d^\dagger M_d V_d.
$$

In the physical basis, defined by $q_\alpha = V^\dagger q'_\alpha$ ($\alpha = u$ or $d$), one has

$$
\mathcal{L} = \sum_\alpha \bar{q}_{\alpha L} \tilde{M}_u q_{\alpha R} + \frac{g}{\sqrt{2}} \bar{q}_{\alpha L} \gamma_\mu V q_{dL} W^\mu_\mu + H.c.
$$

where

$$
V = V_u^\dagger V_d
$$

is the CKM-matrix.

For a $V$ given by Eq.(1), one can easily find $V_u$ and $V_d$ which satisfy Eq.
(5) In general,

$$
V_u = V(\theta_u) = \cos \theta_u I - i \sin \theta_u U
$$

$$
V_d = V(\theta_d) = \cos \theta_d I + i \sin \theta_d U
$$

will give $V(\theta)$ provided $\theta_u + \theta_d = \theta$. This is so since $V(\theta_1) V(\theta_2) = V(\theta_1 + \theta_2)$
because $U = U^\dagger$ and $U^2 = I$.

Given these $V_u$ and $V_d$, Eq.(3) then determines $M_u$ and $M_d$ in terms
of the quark masses and the experimentally accessible parameters of the CKM-
matrix. More formally, this means that in the spectral decomposition of
$M_u(M_d)$ the projectors depend only on the parameters in $V(\theta)$ and $\theta_u(\theta_d)$.
There is a freedom in the choice of the values $\theta_u$ and $\theta_d$ as only their sum
$\theta_u + \theta_d = \theta$ is determined from knowing $V(\theta)$. 

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It is clear that our form of $V(\theta)$ provides an explicit solution for a class of quark mass matrices.

*Form of $U$ in the standard model.* To determine the general form of the hermitian and unitary $3 \times 3$ matrix $U$ we start with a general hermitian matrix

$$ U = \begin{pmatrix} u_1 & \alpha^* & \beta^* \\ \alpha & u_2 & \gamma^* \\ \beta & \gamma & u_3 \end{pmatrix} \quad (8) $$

where $u_i \ (i = 1, 2, 3)$ are real and $\alpha, \beta$ and $\gamma$ are complex numbers. Requiring $U$ to be unitary as well implies that $U^2 = I$. Explicitly this gives

$$ u_1^2 + |\alpha|^2 + |\beta|^2 = 1, \quad (9) $$

$$ u_2^2 + |\alpha|^2 + |\gamma|^2 = 1, \quad (10) $$

$$ u_3^2 + |\beta|^2 + |\gamma|^2 = 1; \quad (11) $$

and

$$ |\alpha| (u_1 + u_2) + |\beta\gamma| \exp(i\phi) = 0, \quad (12) $$

$$ |\beta| (u_1 + u_3) + |\alpha\gamma| \exp(-i\phi) = 0, \quad (13) $$

$$ |\gamma| (u_2 + u_3) + |\alpha\beta| \exp(i\phi) = 0. \quad (14) $$

Here $\phi \equiv \phi_\alpha - \phi_\beta + \phi_\gamma$ while $\phi_\alpha, \phi_\beta$ and $\phi_\gamma$ are the phases of $\alpha, \beta$ and $\gamma$. Eqs. (12-14) immediately imply that $\sin \phi = 0$ or $\phi = 0$ or $\pi$. The resulting $U$ in the two cases differ by an overall sign [4]. For definiteness we consider the case $\phi = 0$. Eqs. (12-14) determine the diagonal elements in terms of $|\alpha|$, $|\beta|$ and $|\gamma|$ and substituting these in Eqs. (9-11) gives the constraint

$$ \left| \frac{\alpha\beta}{\gamma} \right| + \left| \frac{\alpha\gamma}{\beta} \right| + \left| \frac{\beta\gamma}{\alpha} \right| = 2. \quad (15) $$

Using this one has

$$ u_1 = \left| \frac{\alpha\beta}{\gamma} \right| - 1, \quad u_2 = \left| \frac{\alpha\gamma}{\beta} \right| - 1 \quad \text{and} \quad u_3 = \left| \frac{\beta\gamma}{\alpha} \right| - 1. \quad (16) $$

For a more convenient form of $U$, we put

$$ \alpha = -2bc^*, \quad \beta = -2ac, \quad \text{and} \quad \gamma = -2a^*b. \quad (17) $$

Since, $\phi_\alpha = (\phi_b - \phi_c) + \pi$ etc., the condition $\phi = 0$ translates into

$$ \phi_\alpha - \phi_b + \phi_c = \frac{\pi}{2}, \quad (18) $$
where $\phi_a$, $\phi_b$ and $\phi_c$ are the phases of the complex numbers $a$, $b$ and $c$. The constraint of Eq. (15) becomes

$$|a|^2 + |b|^2 + |c|^2 = 1.$$  \hspace{1cm} (19)

The general expression of the hermitian and unitary $U$ in terms of $a$, $b$ and $c$ is

$$U = I - 2 \begin{pmatrix} |a|^2 + |b|^2 & b^*c & a^*c^* \\ b*c^* & |a|^2 + |c|^2 & ab^* \\ ac & a^*b & |b|^2 + |c|^2 \end{pmatrix}.$$  \hspace{1cm} (20)

Given the two constraints in Eq. (18) and Eq. (19), we note that a general hermitian and unitary $3 \times 3$ matrix depends on at most four real parameters. This is the form of $U$ we will use \cite{4}.

The Jarlskog invariant \cite{5} for $U$, viz. $J(U) = Im(U_{11}U_{22}U_{12}^*U_{21}^*) = 0$. However, the $V(\theta)$ in Eq. (1) does give CP-violation, since

$$J(V(\theta)) = 8 \cos \theta \sin^3 \theta |abc|^2 = \cos \theta |V_{12}||V_{13}||V_{23}|$$  \hspace{1cm} (21)

In our case, there is no ‘CP-violating phase’ which governs the finitess of $J$. One of the off-diagonal elements of $V(\theta)$ has to be zero for $J$ to vanish. Note, that $J$ is just given in terms of $|V_{ij}|(i \neq j)$ unlike usual parametrizations \cite{3}. It is interesting to note, that even when $a,b,c$ are pure imaginary \cite{6} so that $V(\theta)$ depends on only 3 real parameters, $J(V(\theta))$ is non-zero. In this case, $U$ becomes real and symmetric and the only complex number in $V(\theta)$ is $i$ in Eq.(1)!

Since $U$ is hermitian it requires that $|V_{ij}| = |V_{ji}|$ for $V(\theta)$ in Eq. (1). The experimentally determined CKM-matrix $V_{EX}$ given by the Particle Data Group \cite{3}

$$V_{EX} = \begin{pmatrix} 0.9745 - 0.9760 & 0.2170 - 0.2240 & 0.0018 - 0.0045 \\ 0.2170 - 0.2240 & 0.9737 - 0.9753 & 0.0360 - 0.0420 \\ 0.0040 - 0.0130 & 0.0350 - 0.0420 & 0.9991 - 0.9994 \end{pmatrix}.$$  \hspace{1cm} (22)

The entries correspond to the ranges for the moduli of the matrix elements. It is clear that $|V_{12}| = |V_{21}|$ and $|V_{23}| = |V_{32}|$ are satisfied for the whole range, while the equality $|V_{13}| = |V_{31}|$ is suggested by the data. Given the fact that $|V_{13}|$ and $|V_{31}|$ are the hardest to determine experimentally, it is possible they might turn out to be equal. We adopt a common numerical value viz.
$|V_{13}| = |V_{31}| = 0.005825 \pm 0.002925$. This numerical value is obtained by first converting the range of values in $V_{EX}$ into a central value with errors, so that $|V_{13}| = 0.00315 \pm 0.00135$ and $|V_{31}| = 0.0085 \pm 0.0045$. The average of these two gives the common numerical value above. Ranges for other moduli also are converted into a central value with errors.

To confront $V(\theta)$ with experiment we need to specify $\theta$. A physically appealing choice is to give equal weight to the generation mixing term ($U$) and the generation diagonal term ($I$) in $V(\theta)$, so that $\theta = \pi/4$ and the CKM-matrix

$$V(\pi/4) = \frac{1}{\sqrt{2}}(I + iU).$$

(23)

We use this for numerical work.

**Numerical results** Experimentally, $|V_{12}|$ and $|V_{23}|$ are well determined. We take their average (or central) value in the range given in Eq. (22) as inputs; that is, $|V_{12}| = |V_{21}| = 0.2205$ and $|V_{23}| = |V_{32}| = 0.039$. Given these, one has

$$|a| = |V_{23}|/(2 \sin \theta |b|),$$

(24)

$$|c| = |V_{12}|/(2 \sin \theta |b|).$$

(25)

The constraint Eq. (19), gives a quadratic equation for $|b|^2$ with the solutions,

$$|b|^2 = \frac{1}{2} \left[ 1 \pm \sqrt{1 - (|V_{12}|^2 + |V_{23}|^2) \csc^2 \theta} \right].$$

(26)

Note, for real $|b|^2$, above input implies $\sin^2(\theta) \geq 0.05014$ or $\theta \geq 12.94^\circ$. Since, $|V_{12}| > |V_{23}| > |V_{13}|$ it is clear we need the positive sign in Eq. (26) so that $|b| > |c| > |a|$. For $\theta = \pi/4$, Eqs. (24-26) yield,

$$|a| = 0.02794, \quad |b| = 0.98705, \quad |c| = 0.15796.$$ (27)

The values of the $|V_{ij}|$ for $V(\pi/4)$ in Eq. (23) are given in Table I. The values in the table should be compared with the average values of $|V_{ij}|$ obtained from $V_{EX}$. For example, average of $V_{11}$ from Eq. (22) is $\frac{1}{2}(0.9745 + 0.9760) = 0.97525$. This is given as $0.97525 \pm 0.00075$. The ‘error’ indicates the range for $|V_{11}|$. The experimental $|V_{ij}|$ are given in column 2, while the calculated values are given in column 3. The agreement is quite satisfactory
suggesting that a CKM-matrix with $|V_{ij}| = |V_{ji}|$ may fit the data. We did not attempt a best fit in view of our assumption $|V_{13}| = |V_{31}|$.

The value of $J$ for $V_{EX}$ and $V(\pi/4)$ are also given in the Table. $J(V_{EX})$ was calculated using the formula [7]

$$J^2 = |V_{11}V_{22}V_{12}V_{21}|^2 - \frac{1}{4} \left[ 1 - |V_{11}|^2 - |V_{22}|^2 - |V_{12}|^2 - |V_{21}|^2 + |V_{11}V_{22}|^2 + |V_{12}V_{21}|^2 \right]^2$$

(28)

with the central values of $|V_{ij}|$, $i = 1, 2$ and since these four are best measured. The value $J(V(\pi/4))$ was calculated using Eq. (21) and is about $3 - 4$ times smaller. This is reasonable considering the slight differences in values of $|V_{ij}|$ $i = 1, 2$ in the two cases and also since there is a strong numerical cancellation between the two terms on the r.h.s of Eq. (28).

It is important to note that calculated values require only the knowledge of $|a|$, $|b|$ and $|c|$. Thus, the numerical results are valid even when $a$, $b$ and $c$ are pure imaginary [6] and $V(\pi/4)$ depends on only 2 real parameters [8].

**Concluding remarks** Apart, form providing a good numerical fit with 4 or possibly 2 parameters, the CKM-matrix $V(\pi/4)$ has an interesting feature connected with a criterion [9] for ‘maximal’ CP-violation.

It was noted [9] that physically the relevant phase for CP-violation in the CKM-matrix $V$ is $\Phi = \phi_{12} + \phi_{23} - \phi_{13}$, where $\phi_{ij}$ is the phase of the matrix element $V_{ij}$. The reason for this is because $\Phi$ is invariant under rephasing transformations of $V$. So, a value of $\Phi \equiv |\pi/2|$ was suggested as corresponding to ‘maximal’ CP-violation. This is so in our case because of the constraint in Eq. (18) since $\Phi = 2(\phi_{a} + \phi_{c} - \phi_{b}) - \pi/2$. So, $\cos \Phi = 0$ for $V(\pi/4)$. Note that, $\Phi = \pi/2$ is automatic when $a$, $b$ and $c$ are pure imaginary [6] and in that case $V(\pi/4)$ depends on only 2 real parameters.

It is remarkable that $V(\pi/4)$ with only 2 real parameters fits the available data. This may be because only the absolute values $|V_{ij}|$ are known at present. Future information on the full $V_{ij}$ will tell us if the relations [10] implied by the two parameter parametrization given here are viable or the more general four parameter parametrization would be needed. It would be very interesting if the symmetry relations $|V_{ij}| = |V_{ji}|(i \neq j)$ are confirmed experimentally.

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References


[4] In case of the choice $\phi = \pi$, $u_i$ have the opposite sign, so that $u_1 = 1 - |a|^2$ etc. Then putting $\alpha = +2bc$ etc. gives $\phi_a - \phi_b + \phi_c = -\frac{\pi}{2}$ and the resulting matrix is just -1 times $U$ given in Eq. (20). Note in either case, $\cos (\phi_a - \phi_b + \phi_c) = 0$.


[6] Note that for pure imaginary $a$, $b$ and $c$, one has $\phi_a = \phi_b = \phi_c = \pi/2$. So Eq. (18) is automatically satisfied. In this case, $U$ and $V(\pi/4)$ depend on only two real parameters.


[8] For a two parameter fit with a different approach see paper by P. Kielanowski cited in ref. 1.


[10] For example $|V_{ij}| = |V_{ji}| (i \neq j)$. Also phases $\phi_{ij}$ are fixed and other relations between absolute magnitudes exist. For $V(\pi/4)$, an interesting relation (implied by Eq.(19) and Eq.(18)) is $2J = \sqrt{2}|V_{12}V_{13}V_{23}| = |V_{12}V_{23}|^2 + |V_{12}V_{13}|^2 + |V_{23}V_{13}|^2$
Table I. Numerical values of the moduli of the matrix elements of $V(\theta)$ for $\theta = \pi/4$. Experimental values are average values obtained from $V_{EX}$ in Eq. (13). The ‘errors’ reflect the large of values for $|V_{ij}|$. Note, since $|V_{13}| = 0.00315 \pm 0.00135$ and $|V_{31}| = 0.0085 \pm 0.0045$, we got the average of these in the Table. $J$ is the Jarlskog invariant (see text).