Multi-Valued Logic Gates for Quantum Computation

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We consider multi-valued logic for quantum computation, where the fundamental units of memory are d-level quantum systems. It is shown that arbitrary unitary operations consisting of d-valued inputs and outputs can be decomposed into logic gates that operate on no more than two inputs and outputs at a time, generalizing the result from binary quantum logic. The advantages offered by multi-valued quantum computing are discussed, and the implementation of multi-valued logic gates in the context of the Cirac-Zoller linear ion-trap scheme is outlined.

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Binary logic gates and Boolean algebra play an important role in classical and quantum theories of computation. The unit of memory for binary quantum computation is the qubit, a quantum system existing in a linear superposition of two basis states, labeled |0⟩ and |1⟩. Universal logic gates that operate on a small number of bits or qubits are known to be sufficient for building arbitrary computational networks. In the quantum case, arbitrary unitary operations on a large number of qubits can be built from two-qubit logic gates alone [1–4], a result that has no analog in classical reversible logic where ternary or qubit operations are needed to simulate reversible Boolean functions [5].

We consider the extension of universal quantum logic to the multi-valued domain, where the unit of memory will be called a qudit [6], a d-dimensional quantum system with the basis states, |0⟩, |1⟩, . . . , |d − 1⟩. As in the binary case, a tensor product of many such qudits is needed for efficient storage of information, since the number of dimensions in the Hilbert space scales exponentially with the number of qudits in the system. Allowing d to be arbitrary enables a trade-off between the number of separate quantum systems making up the quantum computer and the number of levels in each of these systems. In the case of the linear ion-trap scheme [7], for example, only two levels in each ion are actually used for computing, although more levels are needed for processing and reading out the information [8]. Using d computational levels in each ion reduces the number of ions needed for a computation by \( \log_2 d \), since the Hilbert space of n qudits has the same dimensionality as \( n(\log_2 d) \) qudits, since \( d^n = 2^{n \log_2 d} \). As the inability to trap a large number of ions coherently in their vibrational ground state is a major limiting factor in this implementation, a reduction in the number of ions offered by a multi-valued memory is advantageous.

Although a discrete quantum system with an infinite number of energy levels allows unlimited information capacity in principle, there are practical limits to the number of levels that can be resolved and manipulated coherently. As in the binary case, a tensor product of multiple qudits is indispensable to simulating a computation efficiently, allowing unitary transforms on the whole system to be built from logic gates operating within and between qudits, creating entangled superpositions, rather than transforming subsets of a unary Hilbert space. Multi-valued information processing is necessarily more complex at the single qudit level, as we will see, involving a controlled unitary operation on all the levels of each qudit. However, a network of multi-valued logic gates is potentially simpler at this expense, invoking a trade-off between the complexity of each logic gate and the number of gates needed for a computation, analogous to the situation in classical multi-valued logic design [9]. The polynomial increase in the number of levels manipulated in each qudit contrasts with a logarithmic reduction in the number of qudits in the computation, but the time complexity of the simulation, measured in terms of the number of elementary gates performed as a function of the input size of the computation, is unaffected. In fact, if each multi-level gate can be computed in less time than it takes to implement the equivalent binary gates sequentially, a multi-valued implementation potentially enables larger computations within the decoherence time of the system.

A binary simulation of multi-valued logic becomes impractical when dealing with multi-level systems. A base-d representation is ideal for describing the information stored in a d-level quantum system, since a measurement will collapse the system to one of these d levels. This gives only one value per read-out of a qudit, rather than the \( \log_2 d \) values characteristic of a binary representation of the same Hilbert space. Moreover, the processing of multi-level information is difficult to implement in terms of binary logic gates. A single-qubit gate on a multi-level system involves all the levels of the system, not just two, and is effectively a multi-valued gate. Implementing such a gate becomes difficult when the target and control levels of the same atom coincide. By contrast, a two-qudit gate will use the first qudit as control and the second qudit as target in the transformation.

Quantum error correction in non-binary systems has been investigated, and generalizations of binary logic gates have been introduced [6,10–12]. In this paper, we describe a set of universal logic gates that are shown to be sufficient for multi-valued quantum computation. Gen-
However, we note that the property (4) is sufficient to determine the coefficients, and thus does not uniquely determine \( Y_d \). We can show that two-qudit logic gates are sufficient for multi-valued quantum logic. The properties of these multi-valued gates are examined, and an implementation is presented in the context of the linear ion-trap scheme.

The two-qudit gates that are known to be universal for quantum logic belong to a family of unitary transforms described by three parameters \([3]\), which we label as \( \lambda \), \( \nu \), and \( \phi \). This derives from the fact that, except for an overall phase factor, any two-dimensional unitary matrix can be written in the form \([4]\),

\[
Y_2(\lambda, \nu, \phi) = \begin{bmatrix}
\cos \lambda & e^{i(\phi-\nu)} \sin \lambda \\
e^{i\nu} \sin \lambda & e^{i\phi} \cos \lambda
\end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

(1)

expressed here in the basis states of a qubit, \( |0\rangle \) and \( |1\rangle \). \( Y_2 \) can transform any state in this basis to \( |1\rangle \),

\[
Y_2(\arccos |c_0|, \arg[c_0/c_1], \arg[c_1]) : c_0|0\rangle + c_1|1\rangle \mapsto |1\rangle,
\]

(2)

and \( Y_2^{-1}(\arccos |c_0'|, \arg[c_0'/c_1'], \arg[c_1']) \) takes \( |1\rangle \) to another state, \( c_0'|0\rangle + c_1'|1\rangle \), showing that the family of gates \( Y_2 \) is universal for single-qudit transforms. Using this property, it can be shown that the conditional two-qudit analogs of \( Y_2 \),

\[
\Gamma_2[Y_2] = \begin{bmatrix}
\hat{1} & 0 \\
0 & Y_2
\end{bmatrix} \begin{bmatrix} 0,0 \\ 0,1 \\ 1,0 \\ 1,1 \end{bmatrix},
\]

(3)

which applies \( Y_2 \) to the second qudit only when the first qudit is in \( |1\rangle \), are universal for quantum logic in the sense that a unitary transform on any number of qubits can be simulated by repeated applications of the gates in Eq. (3) on no more than two qubits at a time. The parameters \( \lambda \), \( \nu \), and \( \phi \) in \( Y_2 \) can be considered as control variables in the simulation, or else fixed as irrational multiples of \( \pi \) and of each other \([3]\). Generalizing property (2), we define a family of \( d \)-dimensional transforms that map an initial state of a qudit to \( |d-1\rangle \),

\[
Y_d(c_0, c_1, \ldots, c_{d-1}) : c_0|0\rangle + c_1|1\rangle + \cdots + c_{d-1}|d-1\rangle \mapsto |d-1\rangle,
\]

(4)

where the complex coefficients, \( c_0, c_1, \ldots, c_{d-1}, \) are assumed normalized. Unlike definition (1) for the family of gates \( Y_2 \), the multi-valued gates \( Y_d \) defined above do not describe all possible unitary transforms in \( d \) dimensions (which would require \( d^2 \) real parameters), but rather a subset of them that has the property (4). For a given set of values for the coefficients \( c_0, c_1, \ldots, c_{d-1} \), Eq. (4) specifies the transformation of only one of the \( d \) states of the system, namely the superposition state with these coefficients, and thus does not uniquely determine \( Y_d \). However, we note that the property (4) is sufficient to show that the set of conditional two-qudit gates,

\[
\Gamma_2[Y_d] = \begin{bmatrix}
\hat{1} & 0 \\
0 & Y_d
\end{bmatrix} \begin{bmatrix} 0,0 \\ 0,1 \\ 1,0 \\ 1,1 \end{bmatrix},
\]

(5)

which applies \( Y_d \) to the second qudit only when the first qudit is in \( |d-1\rangle \), is universal for quantum logic. The proof of this relies on the fact that property (4) allows a sequence of \( Y_d \) gates to simulate any single-qudit unitary transform. This is because an arbitrary initial state of a qudit, \( c_0|0\rangle + \cdots + c_{d-1}|d-1\rangle \), can be transformed to \( |d-1\rangle \) by \( Y_d(c_0, \ldots, c_{d-1}) \), and \( |d-1\rangle \) can in turn be transformed to an arbitrary final state, \( c_0'|0\rangle + \cdots + c_{d-1}'|d-1\rangle \), by \( Y_d^{-1}(c_0', \ldots, c_{d-1}') \). Henceforth, when we refer to \( Y_d \) in general, we mean any of the family of gates described by Eq. (4), which we know to be universal for single-qudit transforms. We use the convention in Eq. (5) and throughout the rest of the paper that multi-qudit gates (like \( \Gamma_2 \)) are written in bold font but single-qudit gates (like \( Y_d \)) are written in regular font.

In what follows, we will be concerned with the simulation of an arbitrary \( N \)-dimensional (or \( n \)-qudit) unitary transform \( U_N \) using elementary two-qudit gates of the form (5). Since each qudit has \( d \) dimensions, the total number of qudits comprising \( U_N \) is \( n = \log_d N \). Each state in the computational Hilbert space can be written as a tensor product of the \( n \) qudits,

\[
|k\rangle = |k_1\rangle |k_2\rangle \cdots |k_n\rangle,
\]

(6)

where \( k_1 k_2 \ldots k_n \) is the base-\( d \) representation of \( k \), with \( |k_i\rangle \) denoting the state of the \( i^{th} \) qudit. We use the notation \( |k_1 k_2 \ldots k_n\rangle \) for \( |k_1\rangle |k_2\rangle \ldots |k_n\rangle \). Let the eigenstates of \( U_N \) be \( |\Psi_m\rangle \), for \( m = 1, 2, \ldots, N \), with corresponding eigenvalues \( e^{i\varphi_m} \). Each such eigenstate can be expanded in the computational basis,

\[
|\Psi_m\rangle = c_0|0\rangle + \cdots + c_{N-1}|N-1\rangle = c_0|0\rangle + \cdots + c_{N-1}|d-1, \ldots, d-1\rangle.
\]

(7)

Our simulation of \( U_N \) follows an argument given by Deutsch \([13]\), in which a network of gates is created that has the same eigenstates and eigenvalues as that of \( U_N \). Suppose we are able to simulate the unitary transform,

\[
W_m : \begin{cases}
|\Psi_m\rangle & \mapsto e^{i\varphi_m} |\Psi_m\rangle \\
|\Psi_m'\rangle & \mapsto |\Psi_m'\rangle 
\end{cases} \text{ when } m' \neq m.
\]

(8)

Then, we can write \( U_N \) as

\[
W_m : \begin{cases}
|\Psi_m\rangle & \mapsto e^{i\varphi_m} |\Psi_m\rangle \\
|\Psi_m'\rangle & \mapsto |\Psi_m'\rangle 
\end{cases} \text{ when } m' \neq m.
\]

(8)
which states that $W_1W_2\ldots W_N$ shares the same eigenstates and eigenvalues with $U_N$. The problem thus reduces to simulating $W_m$ for an arbitrary $m$. We can decompose $W_m$ into two transforms that are easier to simulate using elementary gates,

$$W_m = Z_m^\dagger X_m Z_m,$$

(10)

where $Z_m$ is assumed unitary and is defined (not unlike $Y_d$) by the property that it takes the $m$th eigenstate to $|N-1\rangle$,

$$Z_m : |\Psi_m\rangle \mapsto |N-1\rangle,$$

(11)

and $X_m$ is a phase gate that rotates the phase of $|N-1\rangle$ by the eigenphase $\Psi_m$, leaving all other computational states unchanged,

$$X_m : \begin{cases} |(N-1)\rangle \mapsto e^{i\Psi_m}|N-1\rangle \\ |m\rangle \mapsto |m\rangle \quad \text{when} \ m' \neq N-1. \end{cases}$$

(12)

We need to show that $W_m = Z_m^{-1}X_m Z_m$ satisfies definition (8). When $m' = m$,

$$Z_m^{-1}X_m Z_m |\Psi_m\rangle = Z_m^{-1}e^{i\Psi_m}|N-1\rangle = e^{i\Psi_m}|\Psi_m\rangle.$$  

(13)

When $m' \neq m$, note that

$$\langle N-1|Z_m|\Psi_m| \rangle = \langle \Psi_m|Z_m^\dagger Z_m|\Psi_m| \rangle = \langle \Psi_m|\Psi_m'| \rangle = 0,$$

which shows that the state $Z_m|\Psi_m| \rangle$ has no projection along $|N-1\rangle$, which means that $X_m$ will not affect it. Hence,

$$Z_m^\dagger X_m Z_m |\Psi_m| = Z_m^\dagger Z_m |\Psi_m| = |\Psi_m|.$$  

(14)

We are now left with the task of simulating $X_m$ and $Z_m$, each of which address the entire $n$-qudit Hilbert space, from elementary two-qudit gates. We first show that $X_m$ and $Z_m$ can be built from the $n$-qudit logic gates,

$$\Gamma_n[Y_d] \equiv \text{Apply } Y_d \text{ to the } n\text{th qudit if and only if }$$

the first $n-1$ qudits are in $|d-1\rangle$.

which are the generalization of the $n = 2$ elementary gates in Eq. (5). First note that since $Y_d$ is universal for single-qudit unitary transforms, it contains the transform that advances the phase of $|d-1\rangle$ in the $n$th qudit and leaves all other states in this qudit, $|0\rangle, |1\rangle, \ldots, |d-2\rangle$, unchanged. This means that $\Gamma_n[Y_d]$ can simulate $X_m$ by advancing the phase of $|d-1, d-1, \ldots, d-1\rangle = |N-1\rangle$ without affecting any of the other computational states, $|0, 0, \ldots, 0\rangle, \ldots, |d-1, d-1, \ldots, d-2\rangle$.

We next show that $Z_m$ is contained in $\Gamma_n[Y_d]$. $Z_m$ and $Y_d$ are similar in their transformation properties in that both take a superposition to a single state. However, $Y_d$ acts within the state space of a single qudit while $Z_m$ acts on the Hilbert space of all $n$ qudits. The conditional gate, $\Gamma_n[Y_d]$, acting in the basis of the last qudit, can reduce a superposition of the $d$ states, $|N-1\rangle$, $|N-d+1\rangle, \ldots, |N-1\rangle$, to $|N-1\rangle$. To reduce the next $d-1$ states, $|N-2d+1\rangle, \ldots, |N-2d+2\rangle, \ldots, |N-1\rangle$, we can permute these states with $|N-1\rangle$, $|N-2\rangle, \ldots, |N-1\rangle$, and apply $\Gamma_n[Y_d]$ again. In this way, blocks of $d-1$ states can be permuted with $|N-1\rangle$, $|N-1\rangle, \ldots, |N-2\rangle$ and reduced to $|N-1\rangle$, until the entire Hilbert space of the $n$ qudits has been so reduced, completing the simulation of $Z_m$. We are still left with the operation of permuting any two computational states, which we now show is also contained in $\Gamma_n[Y_d]$. Within a single qudit, $Y_d$ can permute any two states. Let $Y_d(p, q)$ denote the permutation of $|p\rangle$ and $|q\rangle$ for $p, q = 0, 1, \ldots, d-1$. Consider permuting two $n$-qudit computational states,

$$|j_1, j_2, \ldots, j_n\rangle \leftrightarrow |k_1, k_2, \ldots, k_n\rangle;$$

(16)

This can be done one qudit at a time. Suppose we start with the state, $|j_1, j_2, \ldots, j_n\rangle$. We first apply $n-1$ single-qudit gates $Y_d(p, q)$ to send all but the first qudit in this state to $|d-1\rangle$. Then we apply $\Gamma_n[Y_d(j_1, k_1)]$, which permutes the first qudit conditional on all the other qudits being in $|d-1\rangle$. The other qudits can be restored to their original values by the same $Y_d(p, q)$ gates used earlier. This can be represented symbolically as

$$|j_1, j_2, \ldots, j_n\rangle \xrightarrow{Y_d(p, q)} |j_1, d-1, \ldots, d-1\rangle \xrightarrow{\Gamma_n[Y_d(j_1, k_1)]} |k_1, d-1, \ldots, d-1\rangle.$$  

(17)

We repeat this procedure for each of the $n$ qudits in $|j_1, j_2, \ldots, j_n\rangle$ until the entire state has been transformed to $|k_1, k_2, \ldots, k_n\rangle$. The permutation (16) can thus be simulated using only $\Gamma_n[Y_d]$ and single-qudit $Y_d$ gates.

We have shown that any $n$-qudit unitary operator $U_N$ can be decomposed into the $n$-qudit logic gates, $\Gamma_n[Y_d]$. Our final decomposition is to show that $\Gamma_n[Y_d]$ can be built from the two-qudit gates $\Gamma_2[Y_d]$, shown in Eq. (5). One way of doing this is illustrated in Fig. 1 for $d > 2$. In addition to the $n$ computational qudits, the simulation uses $r = \lceil (n-2)/(d-2) \rceil$ auxiliary qudits (where $\lceil x \rceil$ means the largest integer less than $x$), that are assumed initialized to $|0\rangle$. The horizontal lines denote the qudits, with solid lines denoting the computational qudits and dashed lines denoting the auxiliary qudits. The vertical lines represent the two-qudit conditional gates, originating from the control qudit (which is required to be in $|d-1\rangle$ for the gate to apply) and terminating in a box on the target qudit. The two conditional gates represented are $\Gamma_2[Y_d]$, and $\Gamma_2[Y_d(p, q)]$, where $Y_d$ represents a generic single-qudit gate with the property (4), while $Y_d(p, q)$ is the single-qudit exchange gate which permutes states $|p\rangle$ and $|q\rangle$, with $p, q = 0, 1, \ldots, d-1$.  


FIG. 1. Construction of $\Gamma_n[Y_d]$ from $\Gamma_2[Y_d]$ for $d > 2$.

Vertical lines represent $\Gamma_2$ gates: the symbol $[\frac{p}{q}]$ represents $Y_d(p, q)$, and $[\frac{p}{q}' \frac{r}{s}']$ represents any gate $Y_d$. Horizontal lines represent qudits; solid lines are computational qudits and dashed lines are auxiliary qudits initialized to $0$. The quantity $r = (n - 2)/(d - 2)$ is assumed for simplicity to be an integer.

We want the combination of all the gates in Fig. 1 to implement $\Gamma_n[Y_d]$, which applies $Y_d$ to qudit $n$ if and only if the first $n - 1$ qudits are in $|d - 1\rangle$. Reading from left to right in the figure, the first exchange gate, $\Gamma_2[Y_d(0, 1)]$, increments auxiliary qudit $n+1$ from $|0\rangle$ to $|1\rangle$ only when qudit 1 is in $|d - 1\rangle$. The second exchange gate, $\Gamma_2[Y_d(1, 2)]$, increments qudit $n+1$ from $|1\rangle$ to $|2\rangle$ only when qudit 2 is in $|d - 1\rangle$, and so on. In this way, we see that qudit $n+1$ reaches $|d - 1\rangle$ only when all of the first $d - 1$ computational qudits are in $|d - 1\rangle$. This information is then transferred to the second auxiliary qudit, $n+2$, by the gate $\Gamma_2[Y_d(0, 1)]$, which increments qudit $n+2$ from $|0\rangle$ to $|1\rangle$ provided qudit $n+1$ is in $|d - 1\rangle$. This procedure is carried out sequentially through all the computational states, until finally we have the auxiliary qudit $n+r$ reaching the state $|d - 1\rangle$ (in the case when $r = (n - 2)/(d - 2)$ is an integer) only when all of the first $n - 1$ computational qudits are in $|d - 1\rangle$. A two-qudit gate $\Gamma_2[Y_d]$, targeted on this last auxiliary qudit, acts on the last computational qudit (labeled $n$), completing the simulation of $\Gamma_n[Y_d]$.

We have shown that two-qudit gates of the form $\Gamma_2[Y_d]$ can simulate arbitrary unitary transforms on any number of qudits. The fundamental property of $Y_d$ that allows this simulation is the same as for qubits, namely that $Y_d$ can transform an arbitrary superposition of quantum states to a single state. We note the formal analogy of this property with the result achieved by Grover’s algorithm [14], where a database of quantum information is coherently evolved to a single sought state. The diffusion and sign flip transforms used in this algorithm are sufficient for a binary simulation of $Y_d$.

The multi-valued simulation of $Y_d$ can be thought of as a problem in optimal control [15]. The property (4) describes $Y_d$ as a family of gates parameterized by the $d$ complex coefficients, $c_0, c_1, \ldots, c_{d-1}$, which are assumed normalized. If there are $2d - 1$ real parameters available for controlling the states of a qudit in a physical system, an optimization of these parameters can be done with the fidelity governed by (4). One solution to this control problem in the case of a d-level atomic system was given by Noel and Stroud [16], where the control parameters were the amplitudes and time delays of a sequence of $d$ electric field pulses, and the goal was to create an arbitrary superposition of the atomic states. An extension of this control to the nonlinear case where the ground state population in the atom becomes significantly depleted has been studied by Araujo et al. [17]. An implementation of ternary quantum logic using polarization states of a correlated photon pair generated via collinear degenerate spontaneous parametric down-conversion is suggested by Burlakov et al. [18].

In the rest of this paper we consider a multi-valued extension of the binary quantum computing scheme proposed by Cirac and Zoller [7]. In this scheme, $q$ identical ions are confined in a linear trap and interact with standing wave lasers (see Fig. 2). By laser cooling, the ions are prepared in their lowest energy vibrational ground state, corresponding to the center-of-mass (CM) normal mode, with frequency $\nu_x$ along the trap axis. In this regime, each ion vibrates about its equilibrium position in the trap with an amplitude that is small compared to an optical wavelength, and the trap is characterized by a Lamb-Dicke parameter $\eta$ that is much smaller than unity, $\eta = [\hbar k^2_x/2mu_x]^{1/2} \ll 1$, where $m$ is the mass of each ion and $k_x$ is the laser wave vector along the trap axis. If each laser field is detuned from the associated atomic resonance by $\nu_x$, only the CM mode is excited in the absence of power broadening. Using $\hat{a}^\dagger$ and $\hat{a}$ as the creation and annihilation operators for this mode, and $\hat{\sigma}_{jj} = |j\rangle\langle j|$ as the internal projection operators for each ion, the Hamiltonian for a $d$-level ion in the trap in the absence of the interaction fields is

$$\hat{H}_0 = \hbar \nu_x (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \sum_{j=0}^{d-1} \hbar \omega_j \hat{\sigma}_{jj}. \tag{18}$$

The computational level scheme considered is shown in Fig. 3, with the transition frequencies $\omega_{0,1}, \omega_{1,2}, \ldots, \omega_{d-1,d-2}$ assumed distinct compared to the linewidths of the levels. Only neighboring levels in the manifold are assumed to be dipole-connected. The coupling between the internal levels and the CM motion of the ion is accomplished by $d - 1$ lasers tuned near resonance to these transitions, and having a standing-wave configuration along the trap axis. In the semi-classical limit, the electric field along the trap is given by
When \( \omega_{j,j+1} \) levels raising (lowering) operators for the transition between the ion is at the antinode of standing-wave lasers, \( \omega_{j,j+1} = \frac{\hbar}{2m\nu_z} \) along the trap axis. The field amplitudes and polarizations of the \( d-1 \) modes are given by \( E_{j,j+1} \) and \( E_{j,j+1}^* \). When \( \varphi = \pi/2 \) or 0, the standing waves make a node or antinode at the ions equilibrium position, \( \langle \hat{x} \rangle = 0 \). The field dependence along \( \hat{y} \) and \( \hat{z} \) have been suppressed due to the strong trap confinement along these directions. We can expand the cosine function in Eq. (19) about \( \langle \hat{x} \rangle = 0 \),

\[
\cos(k_{j,j+1} \hat{x} + \varphi) = \cos(\varphi) - k_{j,j+1} \hat{x} \sin(\varphi) + \mathcal{O}[(k_{j,j+1} \hat{x})^2]
\]

where we used \( \hat{x} = [\hbar/2m\nu_z]^{1/2}(\hat{a}^\dagger + \hat{a}) \), and identified the Lamb-Dicke parameters, \( \eta_{j,j+1} = [\hbar k_{j,j+1}^2/2m\nu_z]^{1/2} \). When \( \eta_{j,j+1} \ll 1 \) for all the modes,

\[
\mathbf{E} \approx \sum_{j=0}^{d-2} \left\{ \mathbf{e}_{j,j+1} \right\} \times [\cos(\varphi) - \frac{\eta_{j,j+1}}{\sqrt{d}}(\hat{a}^\dagger + \hat{a}) \sin(\varphi)] \quad (20)
\]

The internal dipole moment for each ion is given by

\[
\mathbf{d} = \sum_{j=0}^{d-2} [\mathbf{d}_{j,j+1} \sigma_j^{(\dagger)} + \mathbf{d}_{j,j+1}^* \sigma_j^{(\dagger)}]
\]

where \( \mathbf{d}_{j,j+1} = \langle j | \mathbf{d} | j+1 \rangle \), and \( \sigma_j^{(\dagger)} \) are the raising (lowering) operators for the transition between levels \( |j\rangle \) and \( |j+1\rangle \). In the interaction picture, the operators evolve according to Eq. (18), \( \hat{d}(t) = \hat{d}(0) e^{-i\nu_z t} \) and \( \hat{d}^*(t) = \hat{d}^*(0) e^{-i\nu_{j+1} t} \). In the dipole approximation, the interaction Hamiltonian is of the form,

\[
\hat{H}_{\text{int}} \approx -\mathbf{d} \cdot \mathbf{E}(\hat{x}, t).
\]

When the ion is at the node of standing-wave lasers, \( \varphi = 0 \), that are all tuned to resonance, \( \alpha_{j,j+1} = \omega_{j,j+1} \) for all \( j \), the time-independent Hamiltonian in the interaction picture under the rotating-wave approximation is

\[
\hat{H}_{\text{int}, V} = -\frac{\hbar}{4} \sum_{j=0}^{d-2} [\Omega_{j,j+1} \sigma_j^{(\dagger)} + \Omega_{j,j+1}^* \sigma_j^{(\dagger)}]
\]

where \( \Omega_{j,j+1} = (\mathbf{d}_{j,j+1} \cdot \mathbf{e}_{j,j+1})/\hbar \) is the Rabi frequency corresponding to the transition, \( |j\rangle \rightarrow |j+1\rangle \). When the ion is at the antinode of standing-wave lasers, \( \varphi = \pi/2 \), that are all detuned from resonance by the trap frequency, \( \alpha_{j,j+1} = \omega_{j,j+1} \pm \nu_z \) for all \( j \), we get instead

\[
\hat{H}_{\text{int}, U_-} = \hbar \sum_{j=0}^{d-2} \eta_{j,j+1} \frac{\sqrt{q}}{2} [\Omega_{j,j+1} \sigma_j^{(\dagger)} \hat{a} + \Omega_{j,j+1}^* \sigma_j^{(\dagger)} \hat{a}^\dagger]
\]

\[
\hat{H}_{\text{int}, U_+} = \hbar \sum_{j=0}^{d-2} \eta_{j,j+1} \frac{\sqrt{q}}{2} [\Omega_{j,j+1} \sigma_j^{(\dagger)} \hat{a}^\dagger + \Omega_{j,j+1}^* \sigma_j^{(\dagger)} \hat{a}]
\]

which shows that \( \hat{V} \) contains the binary gate \( Y_2 \) shown in Eq. (2), except for an overall phase factor. The control parameters in this case are the complex Rabi frequency \( \Omega_{0,1} \) and the time of interaction \( t \). The ternary case is when all Rabi frequencies except for \( \Omega_{0,1} \) and \( \Omega_{1,2} \) are set to zero, leaving a three-level \( \Delta \)-system (see Fig. 2). In this case, levels \( |0\rangle \) and \( |1\rangle \) undergo two-level Rabi oscillations,

\[
\hat{V} \rightarrow \Omega^{-1} \begin{bmatrix} \Omega C & i \Omega_0 S \Omega_1^{*} S & i \Omega_0 S \\ i \Omega_0 S & \Omega C & \Omega_1^{*} S \\ i \Omega_0 S & \Omega_1^{*} S & \Omega_0 \end{bmatrix} \begin{bmatrix} |0\rangle \\ |1\rangle \\ |1\rangle \end{bmatrix},
\]

where \( C = \cos \Omega t, S = \sin \Omega t, \) and \( \Omega = |\Omega_{0,1}| \). Given a state \( |c_0 \rangle |c_1 \rangle \) when

\[
\Omega_{0,1} = -i \frac{\hbar}{\nu_z} \arg \cos \Omega t \text{ and } \Omega = |c_1|,
\]

In this case, levels \( |0\rangle \) and \( |1\rangle \) evolve according to

\[
\hat{V} \rightarrow \Omega^{-2} \begin{bmatrix} |\Omega_{0,1}|^2 + |\Omega_{1,2}|^2 C & i \Omega_0 S \Omega_1^{*} S & i \Omega_0 S \\ i \Omega_0 S & \Omega^2 C & \Omega_1^{*} S \\ i \Omega_0 S & \Omega_1^{*} S & |\Omega_{0,1}|^2 + |\Omega_{1,2}|^2 C \end{bmatrix} \begin{bmatrix} |0\rangle \\ |1\rangle \\ |1\rangle \end{bmatrix},
\]
Given a state \( \psi \), the parameters in this case are the complex Rabi frequencies in Eq. (4), except for an overall phase factor. The control parameters are the auxiliary manifold, conditional on exciting the trap. In the computational and auxiliary bases in the control ion, we move the population from all the states except \( \psi \) to the auxiliary manifold conditional on de-exciting the trap. First, applying \( \hat{Y} \), we have

\[
\hat{Y} : c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle \rightarrow e^{i \arg c_2} |2\rangle
\]

where \( \Omega = \cos \Omega t \), \( S = \sin \Omega t \), and \( \Omega^2 = |\Omega_{0,1}|^2 + |\Omega_{1,2}|^2 \). Given a state \( c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle \),

\[
\begin{bmatrix}
\Omega_{0,1} \\
\Omega_{1,2}
\end{bmatrix} = \frac{i S}{C-1} \begin{bmatrix}
|c_1| & e^{i \arg c_2} \\
|c_2|
\end{bmatrix} \begin{bmatrix}
\Omega_{2,1} \\
\Omega_{0,2}
\end{bmatrix} = \frac{i c_1}{S} e^{i \arg c_2} \begin{bmatrix}
\Omega_{2,1} \\
\Omega_{0,2}
\end{bmatrix},
\]

(28)

and \( \cos \Omega t = \frac{|c_0|^2 + |c_2|^2 - 1}{|\Omega_{0,1}|} \),

which shows that \( \hat{V} \) contains the ternary gate \( \Gamma_3 \) defined in Eq. (4), except for an overall phase factor. The control parameters in this case are the complex Rabi frequencies, \( \Omega_{0,1} \) and \( \Omega_{1,2} \), and the common time of interaction \( t \). In the general \( d \)-level case, the explicit solutions corresponding to Eqs. (26,28) will be more complicated but the identification of the transformation \( \hat{V} \) with the single-qubit gate \( \Gamma_d \) proceeds analogously. The number of control parameters are the \( d-1 \) complex Rabi frequencies, \( \Omega_{0,1}, \Omega_{1,2}, \ldots, \Omega_{d-2,d-1} \), and the common time of interaction \( t \). These suffice to insure that property (4) can be simulated given \( d \) arbitrary normalized coefficients, \( c_0, c_1, \ldots, c_{d-1} \), up to an overall phase factor.

A two-qudit gate of the form \( \Gamma_2[Y_d] \) can be accomplished using both \( \hat{U}_2 \) and \( \hat{V} \) interactions and an auxiliary manifold of \( d \) levels in each ion. The original state of the two-ion system can be written as

\[
|\psi\rangle_C |\Phi\rangle_T |0\rangle
\]

(29)

where \( |\psi\rangle_C \) is the original control ion state, \( |\Phi\rangle_T \) is the original target ion state, and \( |0\rangle \) is the trap ground state. Using \( \pi \)-pulses of the \( \hat{U}_2 \) interactions linking the computational and auxiliary bases in the control ion, we move the population from all the states except \( |d-1\rangle_C \) to the auxiliary manifold, conditional on exciting the trap. In the meantime, \( \pi \)-pulses of the \( \hat{V} \) interaction can revert the population transfer in the control ion to the original configuration without affecting the trap, leaving an entangled state of the form,

\[
|\psi_{d-1}\rangle_C |\Phi\rangle_T |0\rangle + |\psi_{\text{other}}\rangle_C |\Phi\rangle_T |1\rangle
\]

(30)

where \( |\psi_{d-1}\rangle_C = |\psi_{d-1}\rangle_C + |\psi_{\text{other}}\rangle_C \). Expression (30) says that all the control states except for \( |d-1\rangle_C \) are entangled with the trap state \( |1\rangle \). Applying \( \pi \)-pulses of the \( \hat{U}_2 \) interactions to the target ion now, we can move its population to the auxiliary manifold conditional on de-exciting the trap,

\[
|\psi_{d-1}\rangle_C |\Phi\rangle_T |0\rangle + |\psi_{\text{other}}\rangle_C |\Phi_d\rangle_T |0\rangle
\]

(31)

where \( |\phi_d\rangle_T \) is the original target state written in the auxiliary basis. The first term in expression (30) is not affected by this operation since the trap ground state cannot be de-excited. Next, applying \( \hat{V} \) in the computational basis of the target ion, we simulate the single qudit \( Y_d \) gate, transforming \( |\Phi\rangle_T \) but not affecting \( |\phi_d\rangle_T \),

\[
|\psi_{d-1}\rangle_C \{\hat{Y}_d |\Phi\rangle_T \} |0\rangle + |\psi_{\text{other}}\rangle_C |\phi_d\rangle_T |0\rangle
\]

(32)

which effectively accomplishing the two-qudit gate \( \Gamma_2[Y_d] \), with only that part of the target state entangled with \( |d-1\rangle_C \) affected by \( Y_d \). The rest of the target ion that is left in the auxiliary manifold in expression (32) can be restored to the computational basis by reversing the operations that took us from state (30) to state (31) and state (29) to state (30), giving

\[
|\psi_{d-1}\rangle_C \{\hat{Y}_d |\Phi\rangle_T \} |0\rangle + |\psi_{\text{other}}\rangle_C |\Phi\rangle_T |0\rangle
\]

(33)

Comparing states (33) and (29), we see that the target ion has been operated on by \( \hat{Y}_d \) conditional on the control ion being in \( |d-1\rangle_C \), which is equivalent to \( \Gamma_2[Y_d] \). The key to this simulation was the conditional transformation \( \hat{U}_2 \), which used the excited trap state as an intermediate marker via entanglement to distinguish \( |\psi_{d-1}\rangle_C \) from \( |\psi_{\text{other}}\rangle_C \) in the control ion. Since \( \Gamma_2[Y_d] \) were shown to be universal for quantum logic, a multi-valued simulation of quantum logic has been demonstrated in principle in the context of the linear ion trap.

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