than in [4].

We shall use this hypersurface orthogonal unit-time-like eigenvector $v^a$ to set up the Raychaudhuri equation which now reads,

$$\frac{1}{3} \dot{\theta}^2 + 2 \sigma^2 + \kappa(T_{00} - \frac{1}{2} g_{00} T) v^a v^b = - \dot{\varepsilon}^{\alpha}_a - \dot{\theta}$$

The metric with the time coordinate along this vector will have the form

$$ds^2 = g_{\mu\nu} dt^2 + g_{ab} dx^a dx^b$$

The scalars $\theta$, $\sigma$, $\varepsilon^{\alpha}_a$, $\dot{\theta}$ are built up from $v^a$ and its covariant derivatives. Thus with our choice of $v^a$, these scalars will be algebraic combination of scalars formed from the Ricci tensor and its covariant derivatives. Hence a blow up of any of these scalars would indicate a blow up of some Ricci scalars and hence signal the presence of a singularity.

We now enunciate and prove the following theorem:

The space time will be singular in the sense that some scalar built from the Ricci tensor will blow up if

(a) the strong energy condition is satisfied,

(b) the timelike eigenvector of the Ricci tensor is hypersurface orthogonal. (We are excluding the case of null Ricci tensor.)

(c) the space average of any of the scalars occurring in the Raychaudhuri equation does not vanish.

Here the condition (b) allows a foliation of the space time into space sections and the averages referred to in (c) are defined as follows. Space average of any scalar $\chi$ is

$$\langle \chi \rangle_s \equiv \frac{\int \chi \sqrt{|g|} d^3 x}{\int \sqrt{|g|} d^3 x} \text{ limit over entire space}$$

$\langle \chi \rangle_s$ is thus invariant for all transformations involving the space coordinate $x^i$ only.

We can orient the coordinates such that $v^a$ has only one nonvanishing component say along the coordinate $x^1$. As $\varepsilon^{\alpha}_a v^a = 0$, $x^1$ is a spacelike coordinate. Again since with (2),

$$\dot{\varepsilon} = \frac{1}{2} [\ln(g_{00})]_t, \dot{v} = 0$$

the three space vector $\dot{v}_i$ is a gradient vector and hence hypersurface orthogonal. Hence with the above stipulation

$$ds^2 = g_{00} dt^2 + g_{11} dx^2 + g_{ab} dx^a dx^b$$

where $a, b$ run over the indices 2 and 3. One may wonder whether the metric forms (2) and (5) are globally valid with a single coordinate system. However, we note that all known regular solutions with nonvanishing $T_{\mu\nu}$ admit a single coordinate system if there be no discontinuity in $T_{\mu\nu}$. Such discontinuities, although not inconsistent with the condition of regularity, seems unappealing in a cosmological model and in our discussion we shall assume that the forms (2) and (5) are valid over entire space time with a single coordinate system. Obviously $g_{\mu\nu}$ is a function of $x^1$ and maybe $x^i$ and $g_{11}, g_{ab}$ may be functions of all the four coordinates. As the tangent vector to $x^1$ coordinate line is a gradient, $x^1$ lines cannot form a closed loop. They may either run from $-\infty$ to $+\infty$ or in case they diverge from a point (as the radial lines in case of spherical or axial symmetry) they may run from zero to infinity. In any case if $\int_0^{\infty} \sqrt{|g_{11}|} dx^1$ converges to a finite value (i.e., $g_{11} \to 0$ as $x^1 \to +\infty$), then if there be no singularity at infinity one can see by a transformation that the $x^1$ lines are closed. (cf. the closed Friedmann universe in which $\int_0^{\infty} \frac{dz}{(1+z)^{\frac{1}{2}}}$ converges and one can transform $x$ to an angular coordinate $\chi$ with domain 0 to $2\pi$.) Again this would make the gradient vector vanish everywhere. We thus have a nontrivial $\dot{v}^a_\alpha$ only if $\int_0^{\infty} \sqrt{|g_{11}|} dx^1$ diverges.

Again, the scalar $\dot{v}^a_\alpha$ must vanish or oscillate about a mean vanishing value as $x^1 \to \pm \infty$ as otherwise the norm of $\dot{v}^a_\alpha$ would blow up - this is apparent when one recalls that in absence of singularity, the covariant divergence reduces to the ordinary divergence in a locally Lorentzian coordinate system. Now the space average of $\dot{v}^a_\alpha$ is

$$\langle \dot{v}^a_\alpha \rangle_s \equiv \frac{\int \dot{v}^a_\alpha \sqrt{|g_{11}|} \sqrt{|g|} d^3 x}{\int \sqrt{|g|} d^3 x} \text{ limit over entire space}$$

If $\sqrt{|g|}$ diverges or remains finite at infinity, the denominator integral diverges and the vanishing of $\dot{v}^a_\alpha$ (or its mean value) at infinity will make the divergence of the numerator integral weaker. Consequently in the limit $\langle \dot{v}^a_\alpha \rangle_s$ would vanish. In case $\sqrt{|g|}$ vanishes at infinity, this will be due to the vanishing of the two dimensional determinant $|g_{ab}|$ as we have seen that for nontrivial $\dot{v}^a_\alpha$, $g_{11}$ cannot vanish. Thus, in this case, as this factor is common to both the denominator and the numerator integrals, the vanishing of $\dot{v}^a_\alpha$ at infinity again ensures $\langle \dot{v}^a_\alpha \rangle_s = 0$.

Note that in eq (1), all the terms on the left are positive definite as we are assuming the strong energy condition. Hence with $\langle \dot{v}^a_\alpha \rangle_s = 0$, it follows,

$$- < \dot{\theta} >_s \geq \frac{1}{3} < \theta^2 >_s$$

It may happen that $\dot{\theta}$ and $\theta^2$ both vanish at spatial infinity such that the relation (7) is an equality with both sides vanishing. That will lead to the result that space average of all the scalars in (1) vanish and thus prove our theorem. If that is not so, then either at every point

$$- \dot{\theta} \geq \frac{1}{3} \theta^2$$

Note that in eq (1), all the terms on the left are positive definite as we are assuming the strong energy condition. Hence with $\langle \dot{v}^a_\alpha \rangle_s = 0$, it follows,
which will lead to a blow up of $\vartheta$ in the finite past or future or that in some regions of each space section

$$- \dot{\vartheta} > \frac{1}{3} \dot{\vartheta}^2$$  \hspace{1cm} (3)

Integrating over the $x^0$ lines one finds a blowing up of $\vartheta$ in the finite past or future. As already noted, $\vartheta$ is a scalar formed from the Ricci tensor components, and so it cannot blow up in a nonsingular solution. Hence we conclude that (7) must be an equality with both sides vanishing and thus our theorem is proved.

In particular we note that if the space is closed so that the total spatial volume is finite, the theorem implies that the positive definite scalars in (1) will vanish everywhere or in other words there is no nontrivial singularity free solutions in case of closed space sections.

It may be worthwhile to make a comparison of the present theorem with that of Hawking and Penrose. As we have already remarked the Hawking-Penrose theorem is of little relevance so far as realistic models of the universe are concerned as closed trapped surfaces seem inevitable. On the other hand the present theorem depends on the consideration of infinite space integrals and hence it may overlook localized singularities which do not affect the infinite integrals. Such singularities are apparently taken care of by the trapped surface condition in Hawking-Penrose theorem.

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