Colliding Plane Impulsive Gravitational Waves

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Abstract

When two non-interacting plane impulsive gravitational waves undergo a head-on collision, the vacuum interaction region between the waves after the collision contains backscattered gravitational radiation from both waves. The two systems of backscattered waves have each got a family of rays (null geodesics) associated with them. We demonstrate that if it is assumed that a parameter exists along each of these families of rays such that the modulus of the complex shear of each is equal then Einstein’s vacuum field equations, with the appropriate boundary conditions, can be integrated systematically to reveal the well-known solutions in the interaction region. In so doing the mystery behind the origin of such solutions is removed. With the use of the field equations it is suggested that the assumption leading to their integration may be interpreted physically as implying that the energy densities of the two backscattered radiation fields are equal. With the use of different boundary conditions this approach can lead to new collision solutions.

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1 Introduction

This paper is a study of the space–time describing the vacuum gravitational field left behind after the head–on collision of two plane impulsive gravitational waves. The known exact solutions are the classical solution of Khan and Penrose [1] and its generalisation by Nutku and Halil [2]. No details of the derivation were given by Khan and Penrose. The Nutku and Halil solution was obtained using a harmonic mapping technique. The latter solution was subsequently rederived by Chandrasekhar and Ferrari [3] (to quote from [3]: “This paper is addressed, principally, to a more standard derivation of the Nutku-Halil solution than the one sketched by the authors.”) who developed an Ernst–type formulation for vacuum space–times admitting two space–like Killing vectors. They demonstrated that “in some sense, the Nutku–Halil solution occupies the same place in space–times with two space–like Killing vectors as the Kerr solution does in space–times with one time–like and one space–like Killing vector”. Thus the origin of the solution is still quite mysterious. Of course the task of solving Einstein’s vacuum field equations with appropriate boundary conditions for the space–time after the waves collide is mathematically very complex. On the other hand the physical picture, at least up to the appearance of a curvature singularity, is quite simple: two non–interacting plane impulsive waves undergo a head–on collision and the interaction region afterwards contains backscattered radiation from both waves, neither of which remain plane. Our aim in this paper is to introduce a simple assumption based on this picture which provides a key to the origin of the exact solution describing the collision of two completely arbitrary plane impulsive gravitational waves.

The backscattered radiation in the interaction region of space–time between the histories of the waves after the collision determines two intersecting congruences of null geodesics. These are the ‘rays’ associated with the two systems of backscattered radiation. Both congruences have expansion and shear. The ratio of the expansions of each congruence is immediately determined from Einstein’s vacuum field equations and the boundary conditions. The shear of each congruence (the modulus of a complex variable in each case) depends in general in a simple way on the choice of parameter along the null geodesics (in the sense that a change of parameter induces a rescaling of the shear). We assume that a parameter exists along each of the two families of null geodesics such that the shear (i.e. the modulus of the ‘complex shear’) of each congruence is equal. This is the only assumption made apart from the usual assumption of analyticity of the solution of Einstein’s field equations [4]. We show that it implies, together with the field equations, an equation that could be interpreted physically as saying that the energy
density of the backscattered radiation from each wave after collision is the same. In addition we demonstrate how this assumption leads to the complete integration of the vacuum field equations.

The outline of the paper is as follows: In section 2 the collision problem is formulated as a boundary-value problem. At this stage, to make the paper as self-contained as possible, reference is made to an Appendix A giving a brief summary of the construction of a plane impulsive gravitational wave in the manner of Penrose [7]. The backscattered radiation fields, existing after the collision, are introduced in section 3. The assumption central to this study is also introduced in this section and a physical implication is explored. Although the key assumption is simple the full integration of Einstein’s vacuum field equations emerging from it is still quite complicated and this is described in section 4. The complications arise because the incoming waves are not in general linearly polarised. For readers who do not want to work through section 4 the considerably simpler case of linearly polarised incoming waves is treated in Appendix B. It is shown there that our basic assumption leads to the Khan and Penrose [1] solution. Finally in section 5 the results of section 4 are summarised and contact is made with the known exact solutions [1], [2].

With the use of different boundary conditions to those employed here the approach of this paper can lead to new collision solutions of the field equations. The present authors have published a new collision solution of the Einstein–Maxwell field equations in [5] derived using the ideas described in the present paper.

2 The Boundary–Value Problem

The line-element of the space–time describing the vacuum gravitational field of a single plane impulsive gravitational wave having the maximum two degrees of freedom of polarisation may be written in the form (see Appendix A)

\[ ds^2 = -2 \left| d\zeta + v_+ (a - ib) d\bar{\zeta} \right|^2 + 2 \, du \, dv \]

where \( a, b \) are real constants. Here and throughout this paper a bar will denote complex conjugation. The history of the wave is the null hyperplane \( v = 0 \) and \( v_+ = v \theta(v) \) where \( \theta(v) \) is the Heaviside step function. \( u \) is a second null coordinate. The space–time to the future \((v > 0)\) of the history of the wave is Minkowskian and so is the space–time to the past \((v < 0)\) of
\( v = 0 \). Writing \( \sqrt{2} \zeta = x + iy \) we see that (2.1) is in the Rosen–Szekeres form
\[
\begin{align*}
ds^2 &= -e^{-U} \left( e^V \cosh W \, dx^2 - 2 \sinh W \, dx \, dy + e^{-V} \cosh W \, dy^2 \right) + 2 e^{-M} du \, dv,
\end{align*}
\]
with
\[
\begin{align*}
e^{-U} &= 1 - (a^2 + b^2) v_+^2, \\
e^{-V} &= \left[ \frac{1 + (a^2 + b^2) v_+^2 - 2av_+}{1 + (a^2 + b^2) v_+^2 + 2av_+} \right]^{\frac{1}{2}}, \\
\sinh W &= \frac{2bv_+}{1 - (a^2 + b^2) v_+^2}, \\
M &= 0.
\end{align*}
\]
We consider the head–on collision of this wave with a wave of similar type. This latter wave is described by a space–time with line–element (2.2) but with
\[
\begin{align*}
e^{-U} &= 1 - (\alpha^2 + \beta^2) u_+^2, \\
e^{-V} &= \left[ \frac{1 + (\alpha^2 + \beta^2) u_+^2 - 2\alpha u_+}{1 + (\alpha^2 + \beta^2) u_+^2 + 2\alpha u_+} \right]^{\frac{1}{2}}, \\
\sinh W &= \frac{2\beta u_+}{1 - (\alpha^2 + \beta^2) u_+^2}, \\
M &= 0,
\end{align*}
\]
with \( \alpha, \beta \) real constants and \( u_+ = u \theta(u) \). The history of the wave front is the null hyperplane \( u = 0 \) in this case. For the collision we consider the space–time to have line–element (2.2) with \( U, V, W, M \) given by (2.3)–(2.6) in the region \( u < 0, v > 0 \) and given by (2.7)–(2.10) in the region \( v < 0, u > 0 \). The region \( u < 0, v < 0 \) has line–element (2.2) with \( U = V = W = M = 0 \) (which agrees with (2.3)–(2.10) when both \( v < 0 \) and \( u < 0 \)). The line–element in the region \( u > 0, v > 0 \) (after the collision) has the form (2.2) with \( U, V, W, M \) functions of \( (u, v) \) satisfying the O’Brien–Synge [8] junction conditions: If \( u = 0, v > 0 \) then \( U, V, W, M \) are given by (2.3)–(2.6) with \( v_+ = v \) and if \( v = 0, u > 0 \) then \( U, V, W, M \) are given by (2.7)–(2.10) with \( u_+ = u \). Einstein’s vacuum field equations have to be solved for \( U, V, W, M \) in the interaction region \( (u > 0, v > 0) \) after the collision subject to these boundary (junction) conditions. These equations are [9] (with subscripts denoting partial derivatives):
\[
U_{uv} = U_u U_v ,
\]
\[ (2.11) \]
\[ 2V_{uv} = U_u V_v + U_v V_u - 2 (V_u W_v + V_v W_u) \tanh W , \]  
\[ 2W_{uv} = -U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W , \]  
\[ 2U_u M_u = -2U_{uu} + U_u^2 + W_u^2 + V_u^2 \cosh^2 W , \]  
\[ 2U_v M_v = -2U_{vv} + U_v^2 + W_v^2 + V_v^2 \cosh^2 W , \]  
\[ 2M_{uv} = -U_{uv} + W_u W_v + V_u V_v \cosh^2 W . \]  
(2.12)  
(2.13)  
(2.14)  
(2.15)  
(2.16)

The first of these equations can immediately be solved \cite{10} in conjunction with the boundary conditions to be satisfied by \( U \) on \( u = 0 \) and on \( v = 0 \) to yield, in \( u > 0, v > 0 , \)

\[ e^{-U} = 1 - (a^2 + b^2) v^2 - (\alpha^2 + \beta^2) u^2 . \]  
(2.17)

The problem is to solve (2.12)–(2.13) for \( V, W \) subject to the boundary conditions and then to solve (2.14) and (2.15) for \( M \). Equation (2.16) is the integrability condition for (2.14) and (2.15).

### 3 The Backscattered Radiation Fields

We shall for the moment focus attention on the two field equations (2.12) and (2.13). All of our considerations from now on will apply to the interaction region of space–time \( u > 0, v > 0 \) after the collision. Introducing the complex variables

\[ A = -V_u \cosh W + iW_u , \quad B = -V_v \cosh W + iW_v , \]  
(3.1)

we can rewrite the two real equations (2.12) and (2.13) as the single complex equation

\[ 2A_v = U_u B + U_v A - 2i A V_v \sinh W , \]  
(3.2)

or equivalently as the single complex equation

\[ 2B_u = U_u B + U_v A - 2i B V_v \sinh W . \]  
(3.3)

Given the form of the line–element (2.2) it is convenient to introduce a null tetrad \( \{ m, \bar{m}, l, n \} \) in the region \( u > 0, v > 0 \) defined by

\[ m = \frac{e^{U/2}}{\sqrt{2}} \left[ e^{-V/2} \left( \cosh \frac{W}{2} - i \sinh \frac{W}{2} \right) \frac{\partial}{\partial x} + e^{V/2} \left( \sinh \frac{W}{2} - i \cosh \frac{W}{2} \right) \frac{\partial}{\partial y} \right] , \]  
(3.4)

\[ l = e^{M/2} \frac{\partial}{\partial v} , \]  
(3.5)

\[ n = e^{M/2} \frac{\partial}{\partial u} \]  
(3.6)
with \( \bar{m} \) the complex conjugate of \( m \). The integral curves of the vector fields \( l \) and \( n \) are twist–free, null geodesics. The coordinate \( v \) is not an affine parameter along the integral curves of \( l \) and these curves have complex shear \( \sigma_l \) and real expansion \( \rho_l \) (we use the standard definitions for these quantities given in §4.5 of [11] for example) given by
\[
\sigma_l = \frac{1}{2} e^{M/2} B , \quad \rho_l = \frac{1}{2} e^{M/2} U_v , \tag{3.7}
\]
with \( B \) as in (3.1). Likewise the coordinate \( u \) is not an affine parameter along the integral curves of \( n \) and these curves have complex shear \( \sigma_n \) and real expansion \( \rho_n \) given by
\[
\sigma_n = \frac{1}{2} e^{M/2} A , \quad \rho_n = \frac{1}{2} e^{M/2} U_u , \tag{3.8}
\]
with \( A \) as in (3.1). We thus see from (2.17), (3.7) and (3.8) that the ratio \( \rho_l/\rho_n \) is known in the region \( u > 0, v > 0 \). In terms of the variables introduced above the non–identically vanishing scale–invariant components [9] of the Riemann tensor in Newman–Penrose notation are \( \Psi_0, \Psi_2, \Psi_4 \) given by
\[
2 \Psi_0 = B_v + (M_v - U_v) B + i B V_v \sinh W , \tag{3.9}
\]
\[
2 \Psi_2 = M_{uv} - \frac{1}{4} \left( A \bar{B} - \bar{A} B \right) , \tag{3.10}
\]
\[
2 \Psi_4 = A_u + (M_u - U_u) A + i A V_u \sinh W . \tag{3.11}
\]
When these are non–zero we interpret \( \Psi_0 \) as describing radiation, having propagation direction \( n \) in space–time, backscattered from the wave with history \( u = 0, v > 0 \) and we interpret \( \Psi_4 \) as describing radiation, having propagation direction \( l \) in space–time, backscattered from the wave with history \( v = 0, u > 0 \). Thus the integral curves of the null vector fields \( n \) and \( l \) are the ‘rays’ associated with the backscattered radiation from the two separating waves after collision.

We now look for an interesting assumption to make regarding the rays associated with the backscattered radiation fields. Since \( \rho_l/\rho_n \) is known from (3.7), (3.8) and (2.17) we focus attention on the complex shears \( \sigma_l \) and \( \sigma_n \) in (3.7) and (3.8). Let us write
\[
A = |A| e^{i \theta} , \quad B = |B| e^{i \phi} , \quad f = \theta - \phi , \tag{3.12}
\]
with \( \theta \) and \( \phi \) real. From (3.2) and (3.3) we can obtain the equations
\[
\theta_v = - \frac{|B|}{2 |A|} U_u \sin f - V_v \sinh W , \tag{3.13}
\]
\[
\phi_u = \frac{|A|}{2 |B|} U_v \sin f - V_u \sinh W , \tag{3.14}
\]
and

\[ 2f_{uv} + i \left( A \dot{B} - B \dot{A} \right) = - \left( U_u \frac{|B|}{|A|} \sin f \right)_u - \left( U_v \frac{|A|}{|B|} \sin f \right)_v . \tag{3.15} \]

The arguments \( \theta, \phi \) of \( A, B \) respectively are tetrad dependent. If we transform the tetrad \( \{ m, \dot{m}, l, n \} \) by the rotation

\[ m \rightarrow \dot{m} = e^{i\psi} m , \tag{3.16} \]

with \( \psi \) a real-valued function, then

\[ \theta \rightarrow \dot{\theta} = \theta - 2\psi , \quad \phi \rightarrow \dot{\phi} = \phi - 2\psi , \quad f \rightarrow f . \tag{3.17} \]

On account of the field equation (3.15) we can, without loss of generality, arrange to have \( \theta + \phi = \text{constant} \). To see this we deduce from (3.13) and (3.14) that

\[ \left( \dot{\theta} + \dot{\phi} \right)_u = \frac{|A|}{|B|} U_v \sin f + f_u - 2V_u \sinh W - 4\psi_u , \tag{3.18} \]
\[ \left( \dot{\theta} + \dot{\phi} \right)_v = -\frac{|B|}{|A|} U_u \sin f - f_v - 2V_v \sinh W - 4\psi_v . \tag{3.19} \]

We are free to choose \( \psi \) to make the right hand sides of (3.18) and (3.19) vanish because the integrability condition for the resulting pair of first order partial differential equations for \( \psi \) is the field equation (3.15). Hence it is always possible to choose a tetrad so that \( \theta \) and \( \phi \) in (3.12) have the property that \( \theta + \phi = \text{constant} \). This result suggests that to discover an interesting assumption to make about the rays associated with the backscattered radiation fields we should consider the ratio (because it depends upon \( \theta - \phi \) and not \( \theta + \phi \))

\[ \frac{A}{B} = \frac{|A|}{|B|} \psi^f = \frac{\sigma_n}{\sigma_l} , \tag{3.20} \]

with the last equality following from (3.7) and (3.8). We note that \( f \) satisfies the second order equation (3.15) to which we shall return later. It is clear from (3.1) that a change of parameters \( u \rightarrow \tilde{u} = \tilde{u}(u) \) and \( v \rightarrow \tilde{v} = \tilde{v}(v) \) along the integral curves of \( n \) and \( l \) rescales \( A \) and \( B \) by a function of \( u \) and a function of \( v \) respectively. This change of parameter obviously leaves the form of the line-element (2.2) invariant. Also from the field equation (3.2) we deduce that

\[ \left( |A|^2 \right)_v - U_v |A|^2 = \frac{1}{2} U_u \left( A \dot{B} + A B \right) , \tag{3.21} \]
and from the equivalent equation (3.3) we find

\[
\left( |B|^2 \right)_u - U_u |B|^2 = \frac{1}{2} U_v \left( A \bar{B} + \bar{A} B \right), \tag{3.22}
\]

From these two equations we obtain

\[
2 \left[ \log \left( \frac{|A|^2}{|B|^2} \right) \right]_{uv} = \left( U_u \frac{A \bar{B} + \bar{A} B}{|A|^2} \right)_u - \left( U_v \frac{A \bar{B} + \bar{A} B}{|B|^2} \right)_v, \tag{3.23}
\]

which is a partner for the equation (3.15) for \( f \). This suggests that it might be interesting to explore the following assumption concerning the rays associated with the backscattered radiation: there exist parameters \( \bar{u}, \bar{v} \) along the integral curves of \( n \) and \( l \) respectively such that

\[
|A|^2 = |B|^2 \quad \text{(or equivalently) \quad |\sigma_n|^2 = |\sigma_l|^2).}
\]

This is equivalent to the assumption that there exist functions \( C(u), D(v) \) such that

\[
\frac{|A|^2}{|B|^2} = C(u) D(v). \tag{3.24}
\]

When \( u = 0 \) it follows from (2.4) and (2.5) that

\[
|B|^2 = \frac{4 (a^2 + b^2)}{(1 - (a^2 + b^2) v^2)^2}. \tag{3.25}
\]

and when \( v = 0 \) it follows from (2.8) and (2.9) that

\[
|A|^2 = \frac{4 (a^2 + \beta^2)}{(1 - (a^2 + \beta^2) u^2)^2}. \tag{3.26}
\]

Also when \( u = 0 \) we see from (2.17) that the right hand side of (3.21) vanishes and thus solving (3.21) for \( |A|^2 \) when \( u = 0 \) we obtain

\[
|A|^2 = \frac{4 (a^2 + \beta^2)}{1 - (a^2 + b^2) v^2}, \tag{3.27}
\]

with the constant numerator here (the constant of integration) chosen so that the two expressions (3.26) and (3.27) for \( |A|^2 \) agree when \( u = 0 \) and \( v = 0 \). Similarly when \( v = 0 \) the right hand side of (3.22) vanishes and we readily obtain, when \( v = 0 \),

\[
|B|^2 = \frac{4 (a^2 + b^2)}{1 - (a^2 + \beta^2) u^2}. \tag{3.28}
\]

Thus (3.24) together with the boundary conditions at \( u = 0 \) and at \( v = 0 \) results in

\[
\frac{|A|^2}{|B|^2} = \left( \frac{a^2 + \beta^2}{a^2 + b^2} \right) \left[ \frac{1 - (a^2 + b^2) v^2}{1 - (a^2 + \beta^2) u^2} \right]. \tag{3.29}
\]
Hence there exist parameters \((\bar{u}, \bar{v})\) given by
\[
\bar{u} = \sin^{-1}\left(u \sqrt{\alpha^2 + \beta^2}\right), \quad \bar{v} = \sin^{-1}\left(v \sqrt{\alpha^2 + b^2}\right),
\] (3.30)
such that
\[
\frac{V_{\bar{u}}^2 \cosh^2 W + W_{\bar{u}}^2}{V_{\bar{v}}^2 \cosh^2 W + W_{\bar{v}}^2} = 1.
\] (3.31)
We note that when \(\bar{u} = 0\) we have \(u = 0\) and when \(\bar{v} = 0\) we have \(v = 0\).
We shall express (3.31) by saying that, in the coordinates \((\bar{u}, \bar{v})\), (3.29) reads
\[
|A| \cosh W + W|A| = 1.
\] (3.32)
We note from (2.17) and (3.30) that in the barred coordinates
\[
e^{-M} = \frac{\cos \bar{u} \cos \bar{v}}{\sqrt{\alpha^2 + \beta^2 \sqrt{\alpha^2 + b^2}}} e^{-M}.
\] (3.33)
Also in these coordinates (3.23) becomes
\[
(U \cos f)_{\bar{u}} = (U \cos f)_{\bar{v}} \quad \iff \quad U_{\bar{u}} f_{\bar{u}} = U_{\bar{v}} f_{\bar{v}},
\] (3.34)
(the equivalence here following from (3.33) since now \(U_{\bar{u}} = U_{\bar{v}}\)) from which we conclude that
\[
f = f(\lambda) \quad \text{with} \quad \lambda = \frac{\cos(\bar{u} - \bar{v})}{\cos(\bar{u} + \bar{v})}.
\] (3.35)
We see that \(\lambda = 1\) when \(\bar{u} = 0\) and/or when \(\bar{v} = 0\). Also (3.21) and (3.22) become
\[
\left(|A|^2\right)_{\bar{v}} - U_{\bar{v}} |A|^2 = U_{\bar{u}} |A|^2 \cos f ,
\] (3.36)
\[
\left(|A|^2\right)_{\bar{u}} - U_{\bar{u}} |A|^2 = U_{\bar{v}} |A|^2 \cos f.
\] (3.37)
It thus follows that, again in the barred coordinates,
\[
|A|^2 = |B|^2 = e^{U} g(\lambda),
\] (3.38)
for some function \(g(\lambda)\) satisfying
\[
\lambda g' = g \cos f,
\] (3.39)
with the prime here and henceforth denoting differentiation with respect to \( \lambda \).

In order to interpret physically the implications of our assumption that there exists \((\bar{u}, \bar{v})\) such that \( |A|^2 = |B|^2 \), we proceed as follows: using (3.20) and the field equations (2.14) and (2.15) in the expressions (3.9) and (3.11) for \( \Psi_0 \) and \( \Psi_4 \) we find that we can write

\[
U_v \frac{\Psi_0}{B} = \frac{1}{2} \left\{ -iU_v f_u - U_{vv} + \frac{1}{2} |B|^2 + \frac{B}{2A} U_u U_v + U_v \left( \log \frac{|B|}{|A|} \right)_v \right\},
\]

and

\[
U_u \frac{\Psi_4}{A} = \frac{1}{2} \left\{ iU_u f_u - U_{uu} + \frac{1}{2} |A|^2 + \frac{A}{2B} U_u U_v + U_u \left( \log \frac{|A|}{|B|} \right)_u \right\}.
\]

When these are expressed in the coordinates \((\bar{u}, \bar{v})\) we can put \( |A|^2 = |B|^2 \) and as above \( U_{\bar{u}\bar{u}} = U_{\bar{v}\bar{v}} \) and \( U_{\bar{u}} f_{\bar{u}} = U_{\bar{v}} f_{\bar{v}} \) and thus

\[
U_{\bar{u}} \frac{\Psi_4}{A} = U_{\bar{v}} \frac{\Psi_0}{B}.
\]

From (3.42) it follows, using the second of (3.7) and of (3.8) that

\[
\rho_l^{-2} |\Psi_4|^2 = \rho_n^{-2} |\Psi_0|^2.
\]

This equation, which is a consequence of our basic assumption, deserves some physical interpretation. \(|\Psi_0|^2\) and \(|\Psi_4|^2\) are analogous to the energy densities of electromagnetic waves propagating in the \(n\) and \(l\) directions respectively in space–time, in the same sense that the Bel–Robinson tensor [11] is analogous to the electromagnetic energy–momentum tensor. However with the coordinates carrying the dimensions of length these quantities have the dimensions of \((\text{length})^{-4}\). The quantities \(\rho_l^{-2} |\Psi_4|^2\) and \(\rho_n^{-2} |\Psi_0|^2\) both have dimensions \((\text{length})^{-2}\) of energy density and are positive definite expressions in terms of the backscattered radiation fields [we note that the backscattered radiation has non–vanishing shear and expansion and thus does not consist of systems of plane waves]. Hence it seems reasonable to suggest the following interpretation for the equation (3.43): in the coordinate system (the barred system) in which \(|A|^2 = |B|^2\) the energy density of the backscattered radiation from each of the separating waves after the collision is the same. We note that (3.43) also holds for our solution [5] which describes the collision of an impulsive gravitational wave with an impulsive gravitational wave sharing its wave front with an electromagnetic shock wave. This latter solution contains the Khan and Penrose [1] solution and a solution of Griffiths [12] as special.
cases. Also (3.43) holds (trivially) for the Bell and Szekeres [13] solution describing the collision of two electromagnetic shock waves. These examples demonstrate that (3.43) holds for a class of collision problems involving gravitational impulse waves and/or electromagnetic shock waves.

4 Integration of the Field Equations

We begin by writing (3.15) in the barred system \((\bar{u}, \bar{v})\). Using (3.20) with \(|A| = |B|\), (3.35) and (3.38) we find that

\[
(1 - \lambda^2) f'' - 2 \lambda f' + \frac{g}{\lambda} \sin f = \left(\frac{1 + \lambda^2}{\lambda^2}\right) \sin f - \frac{(1 - \lambda^2)}{\lambda} f' \cos f ,
\]

(4.1)

with \(g\) given in terms of \(f\) by (3.39). We can simplify (4.1) to read

\[
g = -\frac{1}{2} \frac{\lambda}{\sin f} \frac{d}{d\lambda} \left[ (1 - \lambda^2) \left( f' + \frac{\sin f}{\lambda} \right) \right] .
\]

(4.2)

We get a single third order equation for \(f\) by eliminating \(g\) (taken to be non-zero) between (3.39) and (4.2). Since we are working in the barred coordinate system (3.20) gives

\[
\frac{A}{B} = \frac{-V_\phi \cosh W + iW_\phi}{-V_\phi \cosh W + iW_\phi} = e^{if} = \frac{1 - ih}{1 + ih} ,
\]

(4.3)

where, for convenience, we have introduced \(h(\lambda)\) by the final equality. After eliminating \(g\) from (3.39) and (4.2) it is useful to write the resulting equation as a differential equation for \(h(\lambda)\). Then defining

\[
G = -\frac{2}{1 + h^2} (1 - \lambda^2) \left( h' - \frac{h}{\lambda} \right) ,
\]

(4.4)

the equation for \(h(\lambda)\) can be put in the form

\[
\lambda G'' + G' - \frac{4}{\lambda} G = -\frac{GQ}{1 - \lambda^2} ,
\]

(4.5)

where

\[
Q = \frac{\lambda}{2h} (1 - h^2) G' + \frac{1}{h} (1 + h^2) G .
\]

(4.6)

We remark that if we define

\[
P = \frac{\lambda}{2h} (1 + h^2) G' + \frac{1}{h} (1 - h^2) G ,
\]

(4.7)
then
\[ P' = Q \quad \text{and} \quad P^2 - Q^2 = \lambda^2 (G')^2 - 4G^2. \] 
(4.8)

In studying the differential equation (4.5) for \( h \) we found it helpful to write (4.5) and the second of (4.8) in the form
\[ \lambda G'' + G' - \frac{4}{\lambda} G = \frac{-\lambda G P'}{1 - \lambda^2}, \]
(4.9)
\[ P^2 - \lambda^2 (P')^2 = \lambda^2 (G')^2 - 4G^2, \]
(4.10)
and to work with these equations. Before proceeding further however we need to know \( h \) and \( h' \) when \( u = 0 \) and/or when \( v = 0 \), i.e. we require \( h(1) \) and \( h'(1) \). To find \( h(1) \) start by writing (4.3), using (3.30), as
\[ 1 - i h(\lambda) = \frac{\sqrt{a^2 + b^2} \sqrt{1 - (\alpha^2 + \beta^2) u^2}}{\sqrt{\alpha^2 + \beta^2} \sqrt{1 - (a^2 + b^2) v^2}} \left( \frac{-V_u \cosh W + iW_u}{-V_v \cosh W + iW_v} \right), \]
(4.11)
and evaluate this equation when \( u = 0 \) and \( v = 0 \). From the boundary conditions on \( V \) and \( W \) given by (2.4), (2.5), (2.8) and (2.9) we have that when \( u = 0 \) and \( v = 0 \):
\[ V_u = 2\alpha, V_v = 2a, W_u = 2\beta, W_v = 2b, \]
(4.12)
and thus from (4.11),
\[ 1 - i h(1) = \frac{\sqrt{a^2 + b^2} (\alpha - i\beta)}{\sqrt{\alpha^2 + \beta^2}} e^{i(\hat{\alpha} - \hat{\beta})}, \]
(4.13)
where
\[ e^{i\hat{\alpha}} = \frac{\alpha - i\beta}{\sqrt{\alpha^2 + \beta^2}}, \quad e^{i\hat{\beta}} = \frac{a - ib}{\sqrt{a^2 + b^2}}. \]
(4.14)
It thus follows from (4.13) that
\[ h(1) = -\tan \left( \frac{\hat{\alpha} - \hat{\beta}}{2} \right) = k \text{ (say)}. \]
(4.15)
Next to find \( h'(1) \) we begin with (4.11) and by two differentiations obtain from it
\[ - \frac{4i}{1 + i h(1)} h'(1) = \left[ \frac{\partial^2}{\partial u \partial v} \left( \frac{-V_u \cosh W + iW_u}{-V_v \cosh W + iW_v} \right) \right]_{(u=0,v=0)}. \]
(4.16)
To evaluate the right hand side here we first note from (2.17) that when \( u = 0 \) and \( v = 0 \), \( U_u = 0 \) and \( U_v = 0 \). Also from the boundary conditions on
W we have \(W = 0\) when \(u = 0\) and \(v = 0\). Now evaluating the field equations (2.12) and (2.13) when \(u = 0\) and \(v = 0\) we easily see that in this case

\[
V_{uv} = 0, \quad W_{uv} = 0. \quad (4.17)
\]

From the boundary conditions satisfied by \(V\) and \(W\) we have, when \(u = 0\) and \(v = 0\):

\[
V_{vv} = V_{uu} = W_{vv} = W_{uu} = 0. \quad (4.18)
\]

Next differentiating (2.12) and (2.13) with respect to \(u\) we find that when \(u = 0\) and \(v = 0\):

\[
V_{uvu} = 2 \alpha^2 a - 6 \beta^2 a - 8 b \alpha \beta, \quad W_{uvu} = 2 b (\alpha^2 + \beta^2) + 8 a \alpha \beta. \quad (4.19)
\]

Finally differentiating (2.12) and (2.13) with respect to \(v\) we find that when \(u = 0\) and \(v = 0\):

\[
V_{vvv} = 2 \alpha a^2 - 6 \alpha b^2 - 8 a b \beta, \quad W_{vvv} = 2 \beta (a^2 + b^2) + 8 a b \alpha. \quad (4.20)
\]

Now substituting all of these results into the right hand side of (4.16) we obtain

\[
h'(1) = k, \quad (4.21)
\]

with \(k\) given by (4.15). Using (4.15) and (4.21) in (4.4) we see that

\[
G(1) = 0 = G'(1). \quad (4.22)
\]

We can now set about solving (4.9) and (4.10) for \(G\) and then obtain \(h(\lambda)\) from (4.4).

Differentiating (4.10) with respect to \(\lambda\) and using (4.9) we find that either (a) \(P' = 0\) or (b) if \(P' \neq 0\) then

\[
\lambda P'' + P' - \frac{1}{\lambda} P = \frac{\lambda G G'}{1 - \lambda^2}. \quad (4.23)
\]

We can quickly dispose of case (a). If \(P = \text{constant} \neq 0\) then (4.10) can be integrated to yield

\[
G = \frac{P}{4} \left( c_0^2 \lambda^{\pm 2} - \frac{1}{c_0^2 \lambda^{\pm 2}} \right), \quad (4.24)
\]

where \(c_0\) is a constant of integration. It is easy to see that this constant cannot be chosen to satisfy both boundary conditions (4.22). Also if \(P = 0\) then (4.10) integrates to

\[
G = c_1 \lambda^{\pm 2}, \quad (4.25)
\]

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with \( c_1 \) a constant of integration. Clearly we must have \( c_1 = 0 \) to satisfy (4.22). Thus the only acceptable solution in case (a) is \( G = 0 \). Turning now to case (b) with (4.23) holding we find that we can integrate this equation once (using (4.22)) to read [14]

\[
\lambda P' + \left( \frac{1 + \lambda^2}{1 - \lambda^2} \right) P = \frac{\lambda G^2}{2(1 - \lambda^2)} .
\]

From the first of (4.8) this can be written

\[
(P + Q) + \lambda^2 (P - Q) = \frac{\lambda}{2} G^2 ,
\]

and from (4.7) and (4.8) this reads

\[
\lambda G' = \frac{\lambda h}{2(1 + \lambda^2 h^2)} G^2 - 2 \frac{1 - \lambda^2 h^2}{1 + \lambda^2 h^2} G .
\]

A glance at (4.3) and (4.4) shows how \( h \) and thence \( G \) are constructed from the functions \( V \) and \( W \) appearing in the line–element (2.2) for \( u > 0, v > 0 \). On the boundaries of this region \( \lambda = 1 \) and within this region \( \lambda > 1 \) with \( \lambda \) becoming infinite when the right hand side of (2.17) vanishes. The \( \lambda = \text{constant} > 1 \) curves densely fill the interior of the region \( S \) (say) with boundaries \((b_1) u = 0, v > 0, (b_2) v = 0, u > 0 \) and \((b_3)\) the right hand side of (2.17) vanishing with \( u > 0, v > 0 \). Within \( S \) there is one curve \( \lambda = \text{constant} > 1 \) passing through each point. When the field equations are completely integrated for \((u,v) \in S\) the boundary \((b_3)\) turns out to be a curvature singularity. For \( G \) analytic in \( S \) we conclude from (4.22) and (4.28) that \( G \equiv 0 \) in \( S \). It thus follows from (4.4) with the boundary condition (4.15) that

\[
h(\lambda) = k \lambda ,
\]

for \((u,v) \in S\).

We are now at the following stage in the integration of the field equations: the function \( U \) in the line–element (2.2) is given by (2.17) in coordinates \((u,v)\) or by (3.33) in coordinates \((\bar{u}, \bar{v})\). Also on account of (4.3) and (4.29) the functions \( V \) and \( W \) in (2.2) satisfy the differential equation

\[
\frac{A}{B} = \frac{-V_{\bar{u}} \cosh W + i W_{\bar{u}}}{-V_{\bar{v}} \cosh W + i W_{\bar{v}}} = \frac{1 - i k \lambda}{1 + i k \lambda} ,
\]

with \( \lambda \) given by (3.35) and \( k \) by (4.15). We shall now solve this complex equation for \( V \) and \( W \) in terms of the barred coordinates. First we need to
note the boundary values of $V$ and $W$ in terms of the barred coordinates.

By (2.4) and (3.30) we have when $\bar{u} = 0$,

$$e^{-V} = \left[ \frac{\sqrt{a^2 + b^2 - a \sin \bar{v}}}{\sqrt{a^2 + b^2 + a \sin \bar{v}}} \right]^2 \frac{1 - e^{i\bar{\beta}} \sin \bar{v}}{1 + e^{i\bar{\beta}} \sin \bar{v}}, \quad (4.31)$$

with the second equality following from (4.14). By (2.5) and (3.30) we have when $\bar{u} = 0$

$$\sinh W = \frac{2b \sin \bar{v}}{\sqrt{a^2 + b^2 \cos^2 \bar{v}}} = -i \frac{(e^{-i\bar{\beta}} \sin \bar{v} - e^{i\bar{\beta}} \sin \bar{v})}{1 - |e^{-i\bar{\beta}} \sin \bar{v}|^2}. \quad (4.32)$$

The corresponding boundary values on $\bar{v} = 0$ are obtained by replacing $\bar{v}$ by $\bar{u}$ and $\bar{\beta}$ by $\bar{\phi}$ in the final expressions in (4.31) and (4.32). It is convenient to use a complex function $E$ (the Ernst function) in place of the two real functions $V$ and $W$ defined (in a way that is suggested by the final expressions in (4.31) and (4.32)) by

$$e^{-V} = \left[ \frac{(1 - E)(1 - \bar{E})}{(1 + E)(1 + \bar{E})} \right]^\frac{1}{2}, \quad \sinh W = -i \frac{(E - \bar{E})}{1 - |E|^2}, \quad (4.33)$$

or equivalently by

$$E = \frac{\sinh V \cosh W + i \sinh W}{1 + \cosh V \cosh W}. \quad (4.34)$$

Now (4.31) and (4.32) can be written neatly as:

when $\bar{u} = 0$, \quad $E = e^{-i\bar{\beta}} \sin \bar{v}, \quad (4.35)$

and correspondingly

when $\bar{v} = 0$, \quad $E = e^{-i\bar{\phi}} \sin \bar{u}. \quad (4.36)$

In terms of $E$, the complex functions $A, B$ can be written

$$A = -\frac{2 \cosh W}{1 - E^2} \bar{E}_{\bar{u}}, \quad B = -\frac{2 \cosh W}{1 - E^2} \bar{E}_{\bar{v}}. \quad (4.37)$$

Substitution into (4.30) simplifies this equation to

$$E_{\bar{u}} - E_{\bar{v}} = ik \lambda (E_{\bar{u}} + E_{\bar{v}}). \quad (4.38)$$
With λ given by (3.35) this equation establishes that

\[ E = E(w) \quad \text{with} \quad w = \sin(\bar{u} + \bar{v}) + i k \sin(\bar{u} - \bar{v}). \quad (4.39) \]

We can now determine \( E \) using the boundary conditions (4.35) and (4.36). To see this easily we first write \( k \) in (4.15) in the form

\[ k = -i \frac{e^{-i\alpha} - e^{-i\beta}}{e^{-i\alpha} + e^{-i\beta}}. \quad (4.40) \]

Using this in (4.39) we see that we can consider \( E \) to have the functional dependence:

\[ E = E \left( e^{-i\alpha} \sin \bar{u} \cos \bar{v} + e^{-i\beta} \cos \bar{u} \sin \bar{v} \right). \quad (4.41) \]

Now the boundary conditions (4.35) and (4.36) establish that

\[ E = e^{-i\alpha} \sin \bar{u} \cos \bar{v} + e^{-i\beta} \cos \bar{u} \sin \bar{v}, \quad (4.42) \]

and thus the functions \( V, W \) appearing in the line-element (2.2) are determined by (4.33) in the coordinates \((\bar{u}, \bar{v})\). They are then converted into the coordinates \((u, v)\) using the transformations (3.30) (see section 5 below).

Finally in the barred system the field equations (2.14) and (2.15) for \( \tilde{M} \) read (using (4.33) and (4.37))

\[ \tilde{M}_\bar{u} = \frac{U_{\bar{u}\bar{u}}}{U_{\bar{u}}} + \frac{1}{2} U_{\bar{u}} + \frac{2 E_{\bar{u}} \bar{E}_{\bar{u}}}{U_{\bar{u}} \left( 1 - |E|^2 \right)^2}, \quad (4.43) \]

\[ \tilde{M}_\bar{v} = \frac{U_{\bar{v}\bar{v}}}{U_{\bar{v}}} + \frac{1}{2} U_{\bar{v}} + \frac{2 E_{\bar{v}} \bar{E}_{\bar{v}}}{U_{\bar{v}} \left( 1 - |E|^2 \right)^2}, \quad (4.44) \]

with \( \tilde{M} \) related to \( M \) by (3.32). Since we must have \( M = 0 \) when \( u = 0 \) and when \( v = 0 \) we see from (3.32) that

when \( \bar{u} = 0 \), \[ e^{-\tilde{M}} = \frac{\cos \bar{v}}{\sqrt{\alpha^2 + \beta^2} \sqrt{a^2 + b^2}}, \quad (4.45) \]

and

when \( \bar{v} = 0 \), \[ e^{-\tilde{M}} = \frac{\cos \bar{u}}{\sqrt{\alpha^2 + \beta^2} \sqrt{a^2 + b^2}}. \quad (4.46) \]

In (4.43) and (4.44), \( U \) is given by (3.33) and \( \tilde{E} \) by (4.42). Using (4.42) we find

\[ E_{\bar{u}} \bar{E}_{\bar{u}} = 1 - |E|^2 = E_{\bar{v}} \bar{E}_{\bar{v}} \], \quad (4.47)
with

\[ 1 - |E|^2 = \cos^2 \left( \frac{\alpha - \beta}{2} \right) \cos^2 (\bar{u} + \bar{v}) + \sin^2 \left( \frac{\alpha - \beta}{2} \right) \cos^2 (\bar{u} - \bar{v}). \quad (4.48) \]

The only complication in solving (4.43) and (4.44) is in dealing with the final term in each. In the case of (4.43) this now involves evaluating the integral

\[ \int \frac{2 \, d\bar{u}}{U_a \left( 1 - |E|^2 \right)} = \int \frac{2 \lambda \, d\lambda}{(\lambda^2 - 1) \left\{ \cos^2 \left( \frac{\alpha - \beta}{2} \right) + \lambda^2 \sin^2 \left( \frac{\alpha - \beta}{2} \right) \right\}}, \quad (4.49) \]

where we have changed the variable of integration from \( \bar{u} \) to \( \lambda \), given in (3.35), with \( \bar{v} \) held fixed. This integral is easy to evaluate and using (4.48) again we obtain from (4.43)

\[ e^{-\bar{u}} = \frac{1 - |E|^2}{F(\bar{v}) \sqrt{\cos(\bar{u} - \bar{v}) \cos(\bar{u} + \bar{v})}}, \quad (4.50) \]

with \( F(\bar{v}) \) a function of integration. By (4.45) we find that in fact \( F \) is a constant given by

\[ F = \sqrt{\alpha^2 + \beta^2} \sqrt{a^2 + b^2}. \quad (4.51) \]

It is straightforward to see that (4.50) with (4.51) also satisfies (4.44). The integration of Einstein’s vacuum field equations is now complete.

5 Discussion

The purpose of this paper has been to propose a simple key to open up the boundary–value problem which is involved in deriving a model in General Relativity of the vacuum gravitational field left behind after the head–on collision of two plane impulsive gravitational waves. This ‘key’ has been provided, with some motivation, in section 3 (following equation (3.23)) and a physical interpretation has been suggested there for the interesting equation (3.43), which is a consequence of the key assumption and some of the vacuum field equations. Our approach has been to focus attention on properties of the backscattered gravitational radiation present after the collision. This appears to be a non–linear phenomenon whose presence therefore ought to be expected to play a central role in the development of a scattering theory for gravitational radiation.

Notwithstanding the simplicity of our key assumption, the derivation of the line–element of the vacuum space–time in the region \( S \) in section 4 [the
region $S$ is defined following equation (4.28) is complicated. It is therefore useful to summarise the result: the vacuum space–time in the region $S$ has line–element of the form (2.2) with $U$ given by (2.17), $V$ and $W$ given by (4.33) with the complex function $E$ in (4.42) expressed in coordinates $(u, v)$ as

$$E = (\alpha + i\beta) u \sqrt{1 - (\alpha^2 + b^2) v^2} + (\alpha + ib) v \sqrt{1 - (\alpha^2 + \beta^2) u^2},$$  \hspace{1cm} (5.1)

while $M$ is reconstructed in coordinates $(u, v)$ using (3.30), (3.32), (4.50) and (4.51). The result is

$$e^{-M} = \frac{\left(1 - |E|^2\right) e^{U/2}}{\sqrt{1 - (\alpha^2 + \beta^2) u^2} \sqrt{1 - (\alpha^2 + b^2) v^2}},$$  \hspace{1cm} (5.2)

with $e^U$ given in (2.17). If in (2.17), (5.1) and (5.2) we put $a^2 + b^2 = 1 = \alpha^2 + \beta^2$ we recover the original form of the Nutku and Halil [2] solution. If in addition $b = \beta = 0$ (and thus $E = \bar{E}$ and so $W = 0$) we recover the original form of the Khan and Penrose [1] solution. We note that in the region $S$ a curvature singularity [1], [2] is encountered on the boundary where $(\alpha^2 + b^2) v^2 + (\alpha^2 + \beta^2) u^2 = 1$ and the solution above is valid only up to this space–like subspace.

Finally we wish to emphasise that the approach to solving collision problems in General Relativity developed in this paper is not confined to the examples worked through above. With different boundary conditions to those described by equations (2.3)–(2.10) and the paragraph following (2.10), the technique is capable of solving new collision problems (see, for example [5]).

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References


It is well known (see [9]) that (2.11) results in $e^{-U} = f_1(u) + f_2(v)$ for some functions $f_1, f_2$ and these functions are immediately determined by the boundary values (2.3) and (2.7) of $U$.


This follows because (4.23) can be written $\frac{d}{dx} \left[ \left( \frac{i}{\lambda} - \lambda \right) X \right] = 0$ where $X = \lambda^2 P' + \left( \frac{1}{\lambda^2} \right) P - \frac{\lambda G^2}{2(1-\lambda^2)}$. The constant of integration then vanishes on account of (4.22), which in particular implies by (4.7) that $P(1) = 0$.

A Plane Impulsive Wave

The line–element of the vacuum space–time describing the gravitational field of an impulsive pp–wave [6] is given by

$$ds^2 = -2dZ d\bar{Z} + 2dV \left( dU + \delta(V) f(Z, \bar{Z}) dV \right). \quad (A.1)$$

Here $f$ is a real–valued function which is harmonic in $Z, \bar{Z}$:

$$\frac{\partial^2 f}{\partial Z \partial \bar{Z}} = 0. \quad (A.2)$$
The only non-vanishing component of the Riemann tensor in Newman–Penrose notation is
\[ \Psi_0 = \frac{\partial^2 f}{\partial Z^2} \delta (V) , \] (A.3)
which is therefore Petrov type N with \( \partial / \partial U \) as degenerate principal null direction. Integral curves of \( \partial / \partial U \), which have vanishing expansion and shear, generate the history of the wave–front \( V = 0 \). Hence \( V = 0 \) is a null hyperplane. Following the example of Penrose [7] a discontinuous coordinate transformation removes the \( \delta \)–function from the line–element (A.1) and introduces a coordinate system \( (v, u, \zeta, \bar{\zeta}) \) in which the metric tensor is continuous \( (C^0) \) across \( V = 0 \). This transformation is given by
\[ V = v, \quad U = u - \theta(v) f(\zeta, \bar{\zeta}) + |g|^2 v_+ , \quad Z = \zeta + v_+ \bar{g}(\bar{\zeta}) , \] (A.4)
where \( \theta(v) \) is the Heaviside step function, \( v_+ = v \theta(v) \) and \( g(\zeta) = -\partial f / \partial \zeta \) is an analytic function of \( \zeta \). The line–element (A.1) is transformed under (A.4) into the Rosen form
\[ ds^2 = -2 \left| d\zeta + v_+ \bar{K}(\bar{\zeta}) d\bar{\zeta} \right|^2 + 2 du dv , \] (A.5)
with \( K(\zeta) = dg / d\zeta = -\partial^2 f / \partial \zeta^2 \) and (A.3) becomes
\[ \Psi_0 = \frac{\partial^2 f}{\partial \zeta^2} \delta (v) . \] (A.6)

For a plane impulsive wave with two degrees of freedom of polarisation \( \partial^2 f / \partial \zeta^2 \) is a complex constant. We take \( f = -\text{Re} \{(a + ib) \zeta^2 \} \) with \( a, b \) real constants for this case. For a linearly polarised plane impulsive gravitational wave either \( a = 0 \) or \( b = 0 \). We note that (A.4) incorporates Penrose’s geometrical construction of the impulsive pp–wave whereby the history of the wave is formed in Minkowskian space–time by first subdividing the space–time into two halves \( v > 0 \) and \( v < 0 \) each with boundary \( v = 0 \) and then reattaching the halves on \( v = 0 \) identifying the points \( (v = 0, u, \xi) \) and \( (v = 0, u - f(\zeta, \bar{\zeta}), \xi) \). This is a mapping of the two copies of \( v = 0 \) where points on the same generators \( \zeta = \text{constant} \) of \( v = 0 \) are mapped one to the other by the translation \( u \to u - f(\zeta, \bar{\zeta}) \) and the mapping preserves the intrinsic degenerate metric on \( v = 0 \).

## B Incoming Waves Linearly Polarised

If the incoming waves are linearly polarised \( b = 0 \) in (2.3)–(2.5) and \( \beta = 0 \) in (2.7)–(2.9). In the interaction region \( (u > 0, v > 0) \) after the collision the
function $W = 0$. This is the problem solved by Khan and Penrose [1]. Our
basic assumption following (3.23) in this case reads: there exist parameters
$(\tilde{u}, \tilde{v})$ along the integral curves of $n$ and $l$ respectively such that $V_{\tilde{u}} = V_{\tilde{v}}$.
These parameters are given by (3.30) with $b = \beta = 0$. Now $V = V(\tilde{u} + \tilde{v})$ in
$u > 0, v > 0$. In the barred coordinates the boundary value of $V$ when $\tilde{u} = 0$
is obtained from (2.4) with $b = 0$ to be:

$$e^V = \frac{1 + \sin \tilde{v}}{1 - \sin \tilde{v}}.$$  \hspace{1cm} (B.1)

Thus for $u > 0, v > 0$ we have

$$e^V = \frac{1 + \sin(u + \tilde{v})}{1 - \sin(u + \tilde{v})} = \frac{(\cos \tilde{u} + \sin \tilde{v})(\cos \tilde{u} + \sin \tilde{u})}{(\cos \tilde{u} - \sin \tilde{v})(\cos \tilde{v} - \sin \tilde{u})}.$$  \hspace{1cm} (B.2)

Using (3.30) with $b = \beta = 0$ we can write (B.2) as

$$e^V = \frac{(\sqrt{1 - a^2u^2} + av)(\sqrt{1 - a^2v^2} + au)}{(\sqrt{1 - a^2u^2} - av)(\sqrt{1 - a^2v^2} - au)},$$  \hspace{1cm} (B.3)

which is the Khan and Penrose [1] expression for $V$ in the interaction region.

From (3.30) and (3.33) we have $U$ in the coordinates $(u, v)$ given by

$$e^U = 1 - \alpha^2 u^2 - \delta^2 v^2.$$  \hspace{1cm} (B.4)

The only remaining function to be determined is $M$. In the coordinates
$(\tilde{u}, \tilde{v})$ this is replaced by $\tilde{M}$ with the latter related to $M$ via (3.32) with
$b = \beta = 0$. If we let $\tilde{Q} = \tilde{M} + \frac{1}{2}U$ then in the barred coordinates the field
equations (2.14) and (2.15) reduce to the following equations for $\tilde{Q}$:

$$\cos \tilde{u} \cos \tilde{v}:$$  \hspace{1cm} (B.5)

Since $M = 0$ when $u = 0$ or $v = 0$ it is easy to see that when $\tilde{u} = 0$, $\tilde{Q} = \log (\cos^2 \tilde{v}/\alpha a)$ and when $\tilde{v} = 0$, $\tilde{Q} = \log (\cos^2 \tilde{u}/\alpha a)$. It thus follows
from (B.5) that

$$\tilde{Q} = - \log \left( \frac{\cos^2(\tilde{u} + \tilde{v})}{\alpha a} \right).$$  \hspace{1cm} (B.6)

Hence by (3.32) with $b = \beta = 0$:

$$e^{-M} = \frac{\alpha a}{\cos \tilde{u} \cos \tilde{v}} e^{-\tilde{Q} + \frac{1}{2}U} = \frac{\cos(\tilde{u} + \tilde{v}) \cos(\tilde{u} - \tilde{v})}{\cos \tilde{u} \cos \tilde{v} \cos^2(\tilde{u} - \tilde{v})}.$$  \hspace{1cm} (B.7)

In the $(u, v)$ coordinates this reads

$$e^{-M} = \frac{1 - \alpha^2 u^2 - \alpha^2 v^2}{\sqrt{1 - \alpha^2 u^2} \sqrt{1 - \alpha^2 v^2} \sqrt{1 - \alpha^2 u^2 \sqrt{1 - a^2 v^2} + a \alpha u v}^2},$$  \hspace{1cm} (B.8)

which is the Khan and Penrose [1] expression.