We have evaluated the observational constraints on the spectral index $n$, in the context of the CDM model which is the simplest viable choice. If $n$ is practically scale-independent, as predicted by most models of inflation, present data require $n \approx 1.0 \pm 0.1$ at something like the 2-$\sigma$ level. We also exhibit the much tighter constraint, obtained if one fixes the epoch of reionization, and the height of the first acoustic peak in the cmb anisotropy. The former has a preferred range 20 to 30, while the latter may soon be accurately known. We have also investigated the two-parameter scale-dependent spectral index, predicted by running-mass inflation models. Present data allow significant variation of $n$ in this case, which occurs in a physically reasonable regime of parameter space.

PACS numbers: 98.80.Cq DESY 00/029, astro-ph/0002397

I. INTRODUCTION

It is generally supposed that structure in the Universe originates from a primordial gaussian curvature perturbation, generated by slow-roll inflation. The spectrum $\delta^2_H(k)$ of the curvature perturbation is the point of contact between observation and models of inflation. Its overall normalization $\delta_H \approx 10^{-5}$ provides a strong constraint, whose nature depends on the model under consideration. Taking that for granted, we here are interested in the scale-dependence of the spectrum, defined by the, in general, scale-dependent spectral index $n$:

$$n(k) - 1 = \frac{2}{3} \frac{\delta_H}{\ln k}.$$  \hfill (1)

According to most inflation models, $n$ has negligible variation on cosmological scales so that $\delta_H^2 \propto k^{-1}$, but we shall also discuss an interesting class of models giving significant scale dependence.

The spectral index measures the shape of the inflaton potential $V(\phi)$, according to the formula\footnote{As usual, $M_P = 2.4 \times 10^{18}$ GeV is the Planck mass, $a$ is the scale factor and $H = \dot{a}/a$ is the Hubble parameter, and $k/a$ is the wavenumber.}

$$n - 1 = 2 M_P^2 (V''/V) - 3 M_P^2 (V'/V)^2 ,$$  \hfill (2)

The potential and its derivatives are evaluated at the epoch of horizon exit $k = aH$. To work out the value of $\phi$ at this epoch one uses the relation $\ln(k_{\text{end}}/k) = N(\phi)$, where $N$ is the number of e-folds to the end of slow-roll inflation, and $k_{\text{end}}$ is the scale leaving the horizon then. The number of e-folds is given by

$$\ln(k_{\text{end}}/k) = N(\phi) = M_P^{-2} \int_{\phi_{\text{end}}}^{\phi} (V'/V) d\phi .$$  \hfill (3)

In almost all models of inflation, Eq. (2) is well approximated by

$$n - 1 = 2 M_P^2 (V''/V) .$$  \hfill (4)

The observational constraints on the spectral index have been studied by many authors, but a new investigation is justified for two reasons. On the observational side, the cosmological parameters are at last being pinned down, and a good measurement of the height of the first peak in spectrum the cmb anisotropy may soon be available. No study has yet been given which takes on board both of these developments. On the theory side, it is known that the spectral index may be strongly scale-dependent if the inflaton has a gauge coupling, leading to what are called running-mass models. The quite specific, two-parameter prediction for the scale dependence of the spectral index in these models has not been compared with presently available data.
Observations of various types indicate that we live in a low density Universe, which is at least approximately flat [1]. In the interest of simplicity we therefore adopt the \( \Lambda \)CDM model, defined by the requirements that the Universe is exactly flat, and that the non-baryonic dark matter is cold with negligible interaction. There is no motivation at present to invoke a more complicated hypothesis, such as a significant fraction of hot dark matter. Also, essentially exact flatness is predicted by inflation, unless one invokes a special kind of model, or special initial conditions.

We shall constrain the parameters of the \( \Lambda \)CDM model, including the spectral index, by performing a least-squares fit to the key observational quantities.

II. THE OBSERVATIONAL CONSTRAINTS ON THE PARAMETERS OF THE \( \Lambda \)CDM MODEL

A. The parameters

The \( \Lambda \)CDM model is defined by the spectrum \( \delta_R^2(k) \) of the primordial curvature perturbation, and the four parameters that are needed to calculate the matter density perturbation and cmb anisotropy. The four parameters are the Hubble constant \( h \) (in units of 100 km s\(^{-1}\) Mpc\(^{-1}\)), the total matter density parameter \( \Omega_0 \), the baryon density parameter \( \Omega_b \), and the reionization redshift \( z_R \). As we shall describe, \( z_R \) can be estimated in terms of the other parameters (and \( \Omega_H \)) because it can be related to the density perturbation at the epoch of reionization. We therefore exclude it from the least-squares fit. In the case of the constant \( n \) models we fix it at various reasonable values (Figure 1) while in the case of the running mass models we assume that reionization occurs when a fixed fraction of the matter collapses.

The spectrum is conveniently specified by its value at the COBE scale \( k_{\text{CMB}} = 6.67H_0 \) (see below), and the spectral index \( n(k) \). We shall consider the usual case of a constant spectral index, and the case of running mass models where \( n(k) \) is given by a two-parameter expression. Excluding \( z_R \), the \( \Lambda \)CDM model is therefore specified by five parameters in the case of a constant spectral index, or by six parameters in the case of running mass inflation models.

B. The data

To compare the \( \Lambda \)CDM model with observation, we take as our starting point a study performed three years ago [2]. We consider the same seven observational quantities as in the earlier work, since they still summarize most of the relevant data. Of these quantities, three are the cosmological quantities \( h \), \( \Omega_0 \), \( \Omega_B \), which we are also taking as free parameters. The crucial difference between the present situation and the earlier one is that observation is beginning to pin down \( h \) and \( \Omega_0 \). Judging by the spread of measurements, the systematic error, while still important, is no longer completely dominant compared with the random error. At least at some crude level, it therefore makes sense to pretend that the errors are all random, and to perform a least squares fit. The adopted values and errors are given in Table 1, and summarised below. In common with earlier investigations, we take the errors to be uncorrelated.

a. Hubble constant  On the basis of observations that have nothing to do with large scale structure it seems very likely [1] that \( h \) is in the range 0.5 to 0.8. We therefore adopt, at notionally the 2-\( \sigma \) level, the value \( h = 0.65 \pm 0.15 \), corresponding to \( h = 0.65 \pm 0.075 \) at the notional 1-\( \sigma \) level.

b. The matter density  The case of the total density parameter \( \Omega_0 \) is similar to that of the Hubble parameter. On the basis of observations that have nothing to do with large scale structure, it seems very likely [1] that \( \Omega_0 \) lies between 0.2 and 0.5, and we adopt at the notional 1-\( \sigma \) level the value \( \Omega_0 = 0.35 \pm 0.075 \).

c. The baryon density  As described for instance in [3,4], the baryon density parameter \( \Omega_b \) has two likely ranges. At the 1-\( \sigma \) level, these are estimated in [3] to be \( \Omega_b h^2 = 0.019 \pm 0.002 \) and \( \Omega_b h^2 = 0.007 \pm 0.0015 \). When a good measurement of the height of the first peak in the cmb anisotropy becomes available, the difference between these values may start to matter, but with present data it is not significant. Except where stated, we adopt the high \( \Omega_b \) range, which is generally regarded as the most likely.

d. The rms density perturbation at 8h\(^{-1}\) Mpc  Primarily through the abundance of rich galaxy clusters, a useful constraint on the primordial spectrum is provided by the present rms density contrast, with top-hat smoothing inside a sphere of radius 8h\(^{-1}\) Mpc, usually denoted by \( \sigma_8 \). A recent estimate [5] based on low-redshift clusters gives at 1-\( \sigma \), for the linearly evolved quantity

\[
\sigma_8 = \bar{\sigma}_8 \Omega_0^{-0.47} \tag{5}
\]

\[
\bar{\sigma}_8 = 0.560 \pm 0.055 \tag{6}
\]
e. The shape parameter

The slope of the galaxy correlation function in the tens of megaparsec range is conveniently specified by a shape parameter $\Gamma$, defined by

$$\dot{\Gamma} = \Gamma - 0.28(n^8 - 1)$$

$$\Gamma = \Omega_B h \exp(-\Omega_B - \Omega_B/\Omega_0).$$

(The quantity $\Gamma$ determines, to an excellent approximation, the matter transfer function.) As indicated, we shall evaluate $n$ at $k^{-1} = 8h^{-1}$ Mpc, in the case that $n$ has significant scale dependence. A fit reported in [2] gives $\dot{\Gamma} = .23$ with a 15% uncertainty at 2-$\sigma$. A more recent fit with more data [6] gives $\dot{\Gamma} = .20$ to .25, depending on the assumed velocity dispersion, but with 15% statistical uncertainty at the 1-$\sigma$ level. We therefore adopt $\dot{\Gamma} = .23$, with 15% uncertainty at 1-$\sigma$.

f. The COBE normalization of the spectrum

We adopt the standard estimate of Bunn and White [7], dropping the quadratic term in $n - 1$ since it negligible in our case. Keeping only the linear term we have at 1-$\sigma$,

$$\delta_H(\nu_{\text{COBE}}) = 0.785 - 0.05 \ln \Omega_0, \quad 10^5 \delta_H = 1.94 \pm 0.075,$$

with $\nu_{\text{COBE}} = 6.6H_0$. This normalization is practically equivalent to specifying the spectrum of the cmb anisotropy at $\ell = 10$.

g. The peak height

The model under consideration predicts a peak in the cmb anisotropy at $\ell \simeq 210$ to 230. Presently available data [8] confirm the existence of a peak at about this position. We adopt as a crucial observational quantity $C_{\text{peak}}$, defined as the maximum value of

$$\tilde{C}_\ell \equiv \ell(\ell + 1)C_\ell/2\pi,$$

where $C_\ell$ is the mean square multipole of the cmb anisotropy. Presently available data give conflicting estimates [8] of $\sqrt{C_{\text{peak}}}$, with central values in the range 70 to 90 $\mu$K, and random errors 10 $\mu$K or so. We adopt $(80 \pm 10)$ $\mu$K with the uncertainty taken to be at 1-$\sigma$. Since a good determination may soon be available from the 1998 Boomerang data (see [9] for some test run results) we have also taken $\sqrt{C_{\text{peak}}}$ as a parameter, when considering the constant $n$ models.

C. Reionization

The effect of reionization on the cmb anisotropy is determined by the optical depth $\tau$. We assume sudden, complete reionization at redshift $z_R$, so that the optical depth $\tau$ is given by [10,11]

$$\tau = 0.035 \frac{\Omega_B h}{\Omega_0} \left( \sqrt{\Omega_0(1 + z_R)^3 + 1 - \Omega_0} - 1 \right).$$

One may choose to regard $z_R$ as a free parameter. However, it is usually supposed that reionization occurs at an early epoch, when some fraction $f \ll 1$ of the matter has collapsed, into objects with mass very roughly $M = 10^6M_\odot$. In that case, the Press-Schechter approximation gives the estimate

$$1 + z_R \simeq \frac{\sqrt{2} \sigma(M)}{\delta_c} \text{erfc}^{-1}(f) \quad (f \ll 1),$$

Here $\sigma(M)$ is the present, linearly evolved, rms density contrast with top-hat smoothing, and $\delta_c = 1.7$ is the overdensity required for gravitational collapse. Estimates of $f$ are in the range [12]

$$10^{-4} \lesssim f \lesssim 1.$$  

As an alternative, one might suppose that reionization occurs only when a fraction $f \sim 1$ of matter has collapsed, leading to the estimate

---

2See Table 3 of [6]; in the present context one should focus on the last three rows of the Table.
(This estimate is not very different from the one that would be obtained by using $f = 1$ in Eq. (13).) Finally, one may entertain the possibility that reionization occurs after most of the matter has collapsed, thought that is not considered likely.

With $n = 1$ and the other parameters in the ranges that we have indicated, the range $f < 1$ corresponds to roughly $z_R \lesssim 20$. For the constant $n$ models, this remains true for $n$ within the fairly narrow range allowed by our fits. Furthermore, we find that an acceptable fit to the data becomes impossible for $z_R \gtrsim 35$, in agreement with [13]. Therefore, in the case of constant $n$ models, the favoured range is

$$20 \lesssim z_R \lesssim 35 .$$

For the most interesting running mass model (Model (i)), these estimates cease to hold, because $n$ increases strongly as the scale decreases. Therefore, instead of fixing $z_R$, we fix $f$, exploring in detail two values. First, we fix $f = 1$, which gives the lowest possible value of $z_R$ unless reionization occurs after most of the matter has collapsed. Second, we fix $f = 10^{-2.2}$, which is about the middle of the range of $\log f$ that is generally regarded as reasonable.

### D. The predicted peak height

Using the CMBfast package [14] allows one to calculate the peak height for given values of the parameters. Following [15], we parameterize the predicted value of $\sqrt{C_{\text{peak}}}$ in the form

$$\sqrt{C_{\text{peak}}} = \sqrt{C_{\text{peak}}^{(0)}} \left( \frac{\delta H (k_{\text{COBE}})}{1.94 \times 10^{-5}} \right) \left( \frac{220}{10} \right)^{\nu/2} ,$$

where

$$\nu \equiv a_n (n_{\text{COBE}} - 1) + a_h \ln(h/0.65) + a_0 \ln(\Omega_0/0.35) + a_b h^2 (\Omega_b - \Omega_b^{(0)}) - 0.65 f(\tau) \tau .$$

$\sqrt{C_{\text{peak}}^{(0)}}$ is the value of $\sqrt{C_{\text{peak}}}$ evaluated with each term of $\nu$ equal to zero, and $\delta H = 1.94 \times 10^{-5}$. As indicated, we evaluate $n$ at $k_{\text{COBE}}$, in the case that $n$ has significant scale dependence. The coefficients for the high (low) choice $\Omega_b^{(0)} h^2 = 0.019$ (0.007) are $a_n = 0.88 (0.90)$, $a_h = -0.37 (-0.40)$, $a_0 = -0.16 (-0.16)$, $a_b = 5.4 (5.5)$, and $\sqrt{C_{\text{peak}}} = 77.5 \mu K$ (70.0 $\mu$K). The formula reproduces the CMBfast results within 10% for a 1-$\sigma$ variation of the cosmological parameters, $h, \Omega_0$ and $\Omega_b$, and $n = 1.0 \pm 0.05$.

With the function $f(\tau)$ set equal to 1, the term $-0.65 \tau$ is equivalent to multiplying $\sqrt{C_{\text{peak}}}$ by the usual factor $\exp(-\tau)$. This is a sufficiently accurate representation of the effect of reionization with present data, but it may require improvement when the peak height is known accurately. For the case of high $\Omega_b$, we use the following formula, which was obtained by fitting the output of CMBfast, and is accurate to a few percent over the interesting range of $\tau$;

$$f = 1 - 0.165 \tau/(0.4 + \tau) .$$

### III. CONSTANT SPECTRAL INDEX

Most models of inflation make $n$ roughly scale-independent, over the cosmologically interesting range. We therefore begin by considering the case that $n$ is exactly scale-independent. The results for $z_R = 20$ are shown in Table 1. In particular, $n = 1.01 \pm 0.05$ for this choice. Varying $z_R$ by $\pm 10$ changes $n$ by $\pm 0.02$, while the result for $z_R = 0$ is $n = 0.98 \pm 0.05$. The least-squares fit was performed with the CERN minuit package, and the quoted error bars invokes the usual parabolic approximation (i.e., it they are the diagonal elements of the error matrix). The exact error bars given by the same package agree to better than 10%.

These results are similar to the ones obtained in [16], but more precise because of improvements in our knowledge of the cosmological parameters. They are also similar to those obtained in [17], if we take the errors to be the ones
given by the error matrix. (We do not know why the exact error bars in [17] are about three times bigger, in conflict with both our work and that of [16].)

In anticipation of the good measurement of the peak height, that is likely to come with the publication of the 1998 Boomerang data, we have also fitted the data for various fixed values of the peak height. The results for some choices of $z_R$ are shown in Figure 1. For $70 < \sqrt{C_{\text{peak}}/\mu K} < 90$ and $10 < z_R < 30$, the variation of $n$ is represented to better than 20% accuracy by

$$n - 1 = 1.058 + 0.038\frac{z_R - 20}{10} + 0.053\frac{\sqrt{C_{\text{peak}}/\mu K} - 80}{10} \pm 0.026.$$  \hspace{1cm} (20)

Although the quality and quantity of data are insufficient for a proper statistical analysis, these bounds on $n$ are very striking when compared with theoretical expectations. As reviewed elsewhere [18,11], most models of inflation predict a 'red' spectrum $n < 1$. The only known models giving a 'blue' spectrum $n > 1$ are the tree-level hybrid inflation models, and even these typically give $n$ indistinguishable from 1. Therefore if observation requires a positive value for $n - 1$, most models of inflation will be immediately excluded.

What about models with $n < 1$? As summarized in [18,11], most models give $n$ not far below 1. The only simple
TABLE I. Fit of the ΛCDM model to presently available data. The spectral index $n$ is a parameter of the model, and so are the next four quantities. Every quantity except $n$ is a data point, with the value and uncertainty listed in the first two rows. The result of the least-squares fit is given in the lines three to five. All uncertainties are at the nominal 1-$\sigma$ level. The total $\chi^2$ is 2.4 for two degrees of freedom.

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>$\Omega_b h^2$</th>
<th>$\Omega_0$</th>
<th>$h$</th>
<th>$10^5 \delta_H$</th>
<th>$\Gamma$</th>
<th>$\sigma_8$</th>
<th>$\sqrt{C_{\text{peak}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>data</td>
<td>$-$</td>
<td>0.019</td>
<td>0.35</td>
<td>0.65</td>
<td>1.94</td>
<td>0.23</td>
<td>0.56</td>
<td>80 $\mu$K</td>
</tr>
<tr>
<td>error</td>
<td>$-$</td>
<td>0.002</td>
<td>0.075</td>
<td>0.075</td>
<td>0.075</td>
<td>0.035</td>
<td>0.055</td>
<td>10 $\mu$K</td>
</tr>
<tr>
<td>fit</td>
<td>1.01</td>
<td>0.019</td>
<td>0.36</td>
<td>0.63</td>
<td>1.95</td>
<td>0.19</td>
<td>0.58</td>
<td>72 $\mu$K</td>
</tr>
<tr>
<td>error</td>
<td>0.05</td>
<td>0.002</td>
<td>0.06</td>
<td>0.06</td>
<td>0.075</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>$-$</td>
<td>$4 \times 10^{-5}$</td>
<td>$1 \times 10^{-2}$</td>
<td>0.1</td>
<td>$5 \times 10^{-3}$</td>
<td>1.3</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

TABLE II. Predictions for the spectral index $n$, for some potentials of the form $V_0(1 - c\phi^p)$. The case $p \to 0$ corresponds to the potential $V_0(1 + c\ln \phi)$, and the case $p \to -\infty$ corresponds to $V_0(1 - e^{-q\phi})$. The parameter $N_{\text{COBE}} < 60$ depends on the cosmology after inflation.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n$</th>
<th>$N_{\text{COBE}} = 50$</th>
<th>$N_{\text{COBE}} = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \to 0$</td>
<td>0.98</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>$p \to -2$</td>
<td>0.97</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td>$p \to \pm \infty$</td>
<td>0.96</td>
<td>0.90</td>
<td></td>
</tr>
<tr>
<td>$p = 4$</td>
<td>0.94</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>$p = 3$</td>
<td>0.92</td>
<td>0.80</td>
<td></td>
</tr>
</tbody>
</table>

exception is the case of a potential dominated by a negative mass-squared,$^3$

$$V = V_0 - \frac{1}{2}m^2\phi^2 + \cdots$$

(21)

This is the form that one expects if $\phi$ is a string modulus (Modular Inflation), or a pseudo-Goldstone boson (Natural Inflation), or the radial part of a massive field spontaneously breaking a symmetry (Topological Inflation). It gives $1 - n = 2M_P^2m^2/V_0$, and barring a sudden steepening of the potential the vev is

$$\langle \phi \rangle \sim (1 - n)^{-1/2}M_P.$$  

(22)

One expects $\langle \phi \rangle \lesssim M_P$, which means that $n$ should not be very close to 1.

Of the remaining models, the lowest values of $n$ come from ‘new’ inflation potentials of the form

$$V = V_0 (1 - \mu\phi^p + \cdots),$$

(23)

with $p = 3$ or 4. With this form, and ignoring for the moment the mild scale-dependence of $n$, the prediction is

$$n - 1 = -\left(\frac{p-1}{p-2}\right) \frac{2}{N_{\text{COBE}}}.$$  

(24)

where $N_{\text{COBE}}$ is the number of $e$-folds of slow-roll inflation after the COBE scale leaves the horizon.

Depending on the history of the Universe,

$$N_{\text{COBE}} \simeq 60 - \ln(10^{16} \text{GeV}/V^{1/4}) - \frac{1}{3} \ln(V^{1/4}/T_{\text{reh}}) - \Delta N,$$

(25)

$^3$In this expression and in Eqs. (23) and (29), the remaining terms are supposed to be negligible, and $V_0$ is supposed to dominate, while cosmological scales leave the horizon.
where $T_{\text{reh}}$ is the reheat temperature and $\Delta N > 0$ allows for matter domination and thermal inflation between reheating and nucleosynthesis (and any continuation of inflation after slow roll ends). With the conventional cosmology, one expects $40 \lesssim N_{\text{COBE}} \lesssim 50$, but the possibilities of a very low inflation scale, significant inflation after slow-roll ends, and (especially) thermal inflation [19] mean that $N_{\text{COBE}}$ can be much smaller.

Just using $N_{\text{COBE}} < 60$, we learn that $n < 0.93$ for $p = 3$, and $n < 0.95$ for $p = 4$. One sees from Figure 1 that such low values may be excluded when the peak height is accurately measured. For instance, if reionization is assumed to occur before most of the matter collapses, corresponding to $z_R \gtrsim 20$, then at 2-$\sigma$ level $p = 3$ requires $\sqrt{C_{\text{peak}}} \lesssim 67\mu K$.

More generally, a lower bound on $n$ gives a lower bound on $N_{\text{COBE}}$,

$$N_{\text{COBE}} > \frac{p-1}{p-2} \frac{2}{1-n}.$$  

Even with present data, our 2-$\sigma$ result $n \gtrsim 0.9$ gives $N_{\text{COBE}} \gtrsim 40$ for $p = 3$, and $N_{\text{COBE}} \gtrsim 20$ for $p \gg 3$.

The potential Eq. (23), and the prediction Eq. (24), hold for mutated hybrid models with $-\infty < p < 1$. Also, one obtains the case of an exponential (logarithmic) potential, by taking the limit $p \to \infty$ ($p \to 0$). Some sample results are shown in Table 2. It is clear that any lower bound on $n$, significantly better than the present 2-$\sigma$ result $n \gtrsim 0.9$, will start to discriminate between models.

In principle, the potential Eq. (23) actually predicts mild scale dependence, because Eq. (24) should actually read

$$n(k) - 1 = - \frac{p-1}{p-2} \frac{2}{N(\phi)} \ln(k_{\text{end}}/k) = N_{\text{COBE}} - \ln(k/k_{\text{COBE}}).$$  

However, over the cosmological range of scales $\ln(k/k_{\text{COBE}})$ is at most a few, and in particular $\ln(8^{-1}h\text{Mpc}^{-1}/k_{\text{COBE}}) \simeq 4$, corresponding at most to a 5.5(2.5)% variation for $N_{\text{COBE}} = 20(40)$. This scale-dependence may be marginally detectable with the advent of the MAP satellite, but hardly before that time [18]. In contrast, one of the models to which we now turn can give a much more dramatic scale dependence.

### IV. RUNNING MASS MODELS

#### A. The prediction for the spectral index

We have also done fits with the scale-dependent spectral index, predicted in inflation models with a running inflaton mass [21–26]. In these models, based on softly broken supersymmetry, one-loop corrections to the tree-level potential are taken into account, by evaluating the inflaton mass-squared $m^2(\ln(Q))$ at the renormalization scale $Q \simeq \phi$,

$$V = V_0 + \frac{1}{2} m^2(\ln(Q))\phi^2 + \cdots.$$  

Over any small range of $\phi$, it is a good approximation to take the running mass to be a linear function of $\ln\phi$. This is equivalent to choosing the renormalization scale to be within the range, and then adding the loop correction explicitly,

$$V = V_0 + \frac{1}{2} m^2(\ln(Q))\phi^2 - \frac{1}{2} c(\ln(Q)) \frac{V_0}{M_P^2} \phi^2 \ln(\phi/Q).$$  

The dimensionless constant $c$ specifies the strength of the coupling, as described later.

It has been shown [24] that the linear approximation is very good over the range of $\phi$ corresponding to horizon exit for scales between $k_{\text{COBE}}$ and $8h^{-1}\text{Mpc}$. We shall want to estimate the reionization epoch, which involves a scale of order $k_{\text{reion}} \sim 10^{-2}\text{Mpc}$ (enclosing the relevant mass of order $10^8M_\odot$). Since only a crude estimate of the reionization

---

4This choice is to be made in the regime where $\phi$ is bigger than all relevant masses. At smaller $\phi$, one takes $m^2(\phi)$ to be scale-independent (the mass 'stops running'). We have a running mass model if inflation takes place in the former regime, which happens in some interesting cases [24,25], including that of a gauge coupling.
towards the origin, while in Models (ii) and (iv) the opposite is true. Even if Eq. (31) is not valid near \( c = c_0 \) in Models (iii) and (iv), \( \phi_c \) is the one introduced in [24]. In Models (i) and (ii), there are four clearly distinct models of inflation as shown in Figure 2. The labeling (i), (ii), (iii) and (iv) is useful, because the sign of \( c \) determines the direction of motion, while the sign \( c < 0 \) when \( c > 0 \) and \( c = 0 \) determines whether \( V'' \) and therefore \( n - 1 \) can change sign during slow-roll inflation.

Using Eq. (3) we find

\[
\sigma e^{-c N(\phi)} = c \ln(\phi_c / \phi),
\]

where \( \sigma \) is an integration constant. Eq. (4) then gives

\[
\frac{n(k) - 1}{2} = \sigma e^{-c N(\phi)} - c.
\]

Integrating the equation \( d \ln(\delta_H)/d \ln(k) = \frac{1}{2}(n - 1) \) we obtain the power spectrum

\[
\delta_H(k) = \delta_H(k_{\text{COBE}}) \exp\left\{ \frac{\sigma}{c} \left[ e^{-c N} - e^{-c N_{\text{COBE}}} \right] + c \left( N - N_{\text{COBE}} \right) \right\}.
\]

The normalization of the spectrum is also predicted by the potential Eq. (31),

\[
\delta_H(k_{\text{COBE}}) = \left( \frac{M_p}{\phi_c} \right) \left( \frac{V_0^{1/2}}{M_p^2} \exp\left( \frac{\sigma}{c e^{c N_{\text{COBE}}}} \right) \right) |\sigma|^{-1} e^{c N_{\text{COBE}}}. \tag{36}
\]

In general, the point \( \phi = \phi_c \) may be far outside the regime where the linear approximation Eq. (31) applies. However, in simple models the cosmological regime is sufficiently close to that point that the linear approximation is approximately valid there. In that case, we can trust the Eq. (31) and its derivatives for \( \phi = \phi_c \); since \( V' \) vanishes at that point, there are four clearly distinct models of inflation as shown in Figure 2. The labeling (i), (ii), (iii) and (iv) is the one introduced in [24]. In Models (i) and (ii), \( c \) is positive and the potential has a maximum near \( \phi_c \), while in Models (iii) and (iv), \( c \) is negative and there is a minimum. In Models (i) and (iii), \( \sigma \) is positive, and \( \phi \) moves towards the origin, while in Models (ii) and (iv) the opposite is true. Even if Eq. (31) is not valid near \( \phi = \phi_c \), this classification by the signs of \( \sigma \) and \( c \) is useful, because the sign of \( \sigma \) determines the direction of motion, while the sign of \( c/\sigma \) decides whether \( V'' \) and therefore \( n - 1 \) can change sign during slow-roll inflation.

The spectral index Eq. (34) depends on the two parameters \( c \) and \( \sigma \). As we shall see, the parameter \( c \) cannot be estimated theoretically, without a detailed knowledge of the interaction causing the loop correction. However, Eq. (33) allows one to obtain significant constraints on \( c \) and \( \sigma \), by requiring that there be no extreme fine-tuning of the parameters [24], and that Eq. (31) does actually give slow-roll inflation.

To see how this goes, note first that slow-roll requires \( |n - 1| \ll 1 \). Since this must hold over a significant range of \( N \), we learn that \( |c| \ll 1 \). Coming to \( \sigma \), assume first that the mass continues to run to the end of slow-roll inflation. Making the crude approximation that the linear approximation is still valid then, one can set \( N = 0 \) to learn that \( \sigma = c \ln(\phi_c / \phi_{\text{end}}) \). It then follows that \( |\sigma| \gg |c| \), unless the end of inflation is very fine-tuned, to occur close to the maximum or minimum. Also, since slow-roll requires \( |n - 1| \ll 1 \), and this must hold over a significant range of \( N \), we learn from Eq. (34) that \( |\sigma| \ll 1 \). If \( c \) is positive and not too small, it is reasonable to suppose that Eq. (31) remains valid until the end of slow-roll inflation, which ends when \( n - 1 \) actually becomes of order 1. In this, the simplest case, \( |\sigma| \sim 1 \).

In the case of Model (i), the mass may cease to run before the end of slow-roll inflation (but after COBE scales leave the horizon, or the running mass model would not apply). In this somewhat fine-tuned situation, one might
FIG. 2. Sketches of the potential for the different models in the case an extremum exists: the right panel shows the inflaton behaviour for Models (i) and (ii), while the left panel shows Models (iii) and (iv).

We have arrived at the following conclusions. In general, the expected range of $|\sigma|$ is

$$|c| \lesssim |\sigma| \lesssim 1. \quad (37)$$

At the expense of moderate fine-tuning, an upward (downward) extension of one or two orders of magnitude is possible in the case of Model (i) (Model (iv)). Also, the simplest versions of Models (i) and (ii) give respectively $\sigma = \pm 1$.

Next we consider the likely magnitude of $c$, assuming for simplicity that a single coupling dominates the loop correction. For definiteness, let us define $c$ so that the linear approximation is exact (at the one-loop level) at the point $\phi = \phi_{\text{COBE}}$. The value of $c$ is conveniently obtained from the well-known RGE for $d \log M^2 / d(\log Q)$, evaluated at $Q = \phi_{\text{COBE}}$. If a gauge coupling dominates one finds [22]

$$\frac{V_0 c}{M_P^2} = \frac{2C \alpha}{\pi} \alpha \tilde{m}^2. \quad (38)$$

where $C$ is a positive group-theoretic number of order 1, $\alpha$ is the gauge coupling, and $\tilde{m}$ is the gaugino mass. We see that if the loop correction comes from a single gauge coupling, $c$ is positive, corresponding to Model (i) or Model (ii). If a Yukawa coupling dominates, one finds [25] (for negligible supersymmetry breaking trilinear coupling)

$$\frac{V_0 c}{M_P^2} = -\frac{D}{16\pi^2} |\lambda|^2 m_{\text{loop}}^2, \quad (39)$$

where $D$ is a positive constant counting the number of scalar particles interacting with the inflaton, $m_{\text{loop}}^2$ is their common susy breaking mass-squared, and $\lambda$ is their common Yukawa coupling. In this case, $c$ can be of either sign. In both cases, the masses and couplings are to be evaluated at (say) the scale $Q = \phi_{\text{COBE}}$.

To estimate the masses, recall first that, to obtain the running mass model, supersymmetry in the inflaton sector should be broken softly. The traditional hypothesis is that soft supersymmetry breaking is gravity-mediated, and in the context of inflation this means that the scale $M_S$ of supersymmetry breaking will be roughly $V_0^{1/4}$. (As usual we are defining $M_S \equiv \sqrt{F}$, where $F$ is the auxiliary field responsible for spontaneous supersymmetry breaking in the hidden sector.) With gravity-mediated susy breaking, typical values of the masses are $\tilde{m}^2 \sim |m_{\text{loop}}^2| \sim V_0 / M_P^2$, which makes $c$ of order of the coupling strength. For a gauge coupling, or an unsuppressed Yukawa coupling, we expect

$$|c| \sim 10^{-1} \text{ to } 10^{-2}. \quad (40)$$

In special versions of gravity-mediated susy breaking, the masses could be much smaller, and with gauge-mediated susy breaking they could be much bigger. The latter case is forbidden, (unless the coupling is suppressed) because it would not satisfy the slow-roll requirement $|c| \lesssim 1$. In the former case, the mass would hardly run, and the spectral index would be practically scale-independent.
Finally, we consider briefly the expected values of the parameters $\phi_*$ and $V_0$, which are needed to calculate $\delta H(k_{\text{COBE}})$ (Eq. (36)). The simplest thing is to again assume gravity-mediated suy breaking, with the ultra-violet cutoff at the traditional scale around $M_P$, and the same supersymmetry breaking scale during inflation as in the true vacuum so that $V_0^{1/4} \sim 10^{10}$ GeV. In this scenario, one expects $|m^2(Q)| \sim V_0/M_P^2$ at $Q \sim M_P$. As Stewart pointed out in the first paper on the subject, with this very traditional set of assumptions, Eq. (36) can give the correct COBE normalization, with $|c|$ in the physically favoured range $10^{-1}$ to $10^{-2}$.

It is remarkable that the most traditional set of assumptions can give a model with the correct COBE normalization, and, as we shall see, with a viable spectral index. If one relaxes these assumptions, there is much more freedom in choosing $V_0$ and $\phi_*$. Such freedom may be very welcome, in coping with the difficulty of implementing inflation in the context of large extra dimensions [27].

### B. Observation constraints on the running mass models

Extremizing with respect to all other parameters, we have calculated $\chi^2$ in the $\sigma$ vs. $c$ plane and obtained contour levels for $\chi^2$ equal to the minimum value plus 2.41 and 5.99 respectively, corresponding nominally to the 70% and 95% confidence level in two variables. Since the $\chi^2$ function presents actually two nearly degenerate minima in the allowed region, one in the positive and one in the negative quadrants (Models (i) and (iii)), separated by a very low barrier, we will assume that the usual quadratic estimate of the probability content is not very far from the true value. The results are shown in Figures 3 and 4, for the choice $N_{\text{COBE}} = 50$, and the case that of reionization occurs when $f = 1$. For $c = 0$ or $\sigma = 0$ the constant $n$ result is recovered with $n - 1 = -2c$ or 2$\sigma$; our plots give in this case a slightly larger allowed interval with respect to the two sigma value in the previous section, due to the mismatch between the statistical one variable and two variables 95% CL contours.

![FIG. 3. Allowed region in the positive $\sigma$ plane at 70% and 95% CL for $N_{\text{COBE}} = 50$; the solid line is the 95% contour, while the dashed line the 70%. The theoretically favoured region is above the dotted line $\sigma = |c|$. For positive $c$ the contours in the second panel close at $\sigma \sim 200$, $c \approx 0.15$ and $\sigma \sim 1250$, $c \approx 0.19$, for the 70% and 95% CL respectively.](image)

In the case of Models (ii) and (iv), the allowed region corresponds to $|c|$ and $|\sigma|$ both small, giving a practically scale-independent spectral index, with a red and blue spectrum respectively. In contrast, the allowed region for Models (i) and (iii) allows strong scale-dependence. To demonstrate this, we show in Figures 5 and 6 the allowed region using the variables

\begin{align}
    n_{\text{COBE}} - 1 &= 2\sigma e^{-cN_{\text{COBE}}} - 2c \\
    n_{D} - 1 &= 2\sigma e^{-c(N_{\text{COBE}}-\Delta N)} - 2c,
\end{align}

5At the crudest level, one can verify this using the linear approximation Eq. (31) all the way up to the $\phi \sim M_P$, corresponding to $\log(M_P/\phi_*) \sim 1/c \sim 10$ to 100. Proper calculations [22–24] using the RGE’s lead to the same conclusion.
where \( \Delta N = \ln(8^{-1} h \text{Mpc}^{-1}/k_{\text{COBE}}) \approx 4 \). Note that the two quantities are linearly related by

\[
 n_8 - 1 = (n_{\text{COBE}} - 1)e^{c\Delta N} + 2c(e^{c\Delta N} - 1).
\]

For \( c \to 0 \) one recovers a constant spectral index.

In the case of Model (i), a large departure from a constant spectral index is allowed for large \( \sigma \); for the theoretically favored value \( \sigma \approx 1 \) the variation can be as large as 0.05, while the maximal change allowed by the data is 0.2. Note that a lower value of the fraction of collapsed matter \( f \) just reduces the allowed region at large \( n \).

In the case of Model (iii), a much larger departure from a constant spectral index is allowed \( ^6 \), but in the theoretically favored regime \( |\sigma| \geq c \) one again finds a variation at most 0.05.

V. CONCLUSION

We have evaluated the observational constraints on the spectral index \( n \), using a range of data, including the cmb peak height which we take to be \( \sqrt{C_{\text{peak}}} = 80 \pm 10 \) (nominal 1-\( \sigma \)). In the case that \( n \) has negligible scale dependence, the the 2-\( \sigma \) lower bound is at \( n \approx 0.9 \), which as we have discussed already gives non-trivial information in the case of some inflation models. Since a good determination of the peak height may soon be available, we have also done fits with fixed values of \( \sqrt{C_{\text{peak}}} \), with the results summarized in Figure 1. If the lower bound on \( n \) turns out to be bigger than 1, most models of inflation will be ruled out, and a lower bound bigger than .93 will start to rule out some ‘new’ inflation models.

We have also investigated the running mass models. In these models, the scale-dependence spectral index \( n(k) \) is given by \( n - 1 = \sigma \exp(cN) - c \), where \( \sigma = \ln(k/k_{\text{end}}) \). The parameters in this expression can be of either sign, leading to four different models of inflation. Barring fine-tuning, one expects \( \sigma \) to be in the range \( |\sigma| \lesssim 1 \). The parameter \( c \) depends on the nature of the soft supersymmetry breaking, but in the simplest case of gravity-mediation it becomes a dimensionless coupling strength, presumably of order \( 10^{-1} \) to \( 10^{-2} \) in magnitude.

Without worrying about the origin of the parameters \( c \) and \( \sigma \), we have investigated the observational constraints on them. In two of the four possible models (the two with \( c \) and \( \sigma \) the same sign) we have found that indeed \( n \) can vary by about 0.05. Moreover, if \( c \) is positive as it would be for a gauge coupling, \( n - 1 \) can change sign between the COBE and \( 8h^{-1} \text{Mpc} \) scales. It will be very interesting to see how the present situation changes with the advent of better data, starting with the Boomerang cmb data.

\( ^6 \)Beware that for very large deviations from \( n \approx 1 \) our \( \chi^2 \) estimate contains a non negligible systematic error due to the increasing uncertainty in our peak fitting formula (17) for large \( |n - 1| \). Since the theoretically favored regions correspond to small deviations from \( n = 1 \), we consider our results trustworthy in that regime.
FIG. 5. Allowed region in the $n_{\text{COBE}} - 1$ vs $n_8 - 1$ plane at 95% CL (solid line) and 70% CL (dashed line) for positive $\sigma$ and $c$ (Model (i)). The two panels correspond to different hypotheses about the reionization epoch. In the right panel, it is assumed that reionization occurs when a fraction $f = 10^{-2.2}$ of the matter has collapsed into bound structures, while in the left panel the fraction is taken to be $f \sim 1$. The contours do not depend on the value of $N_{\text{COBE}}$. The lines $\sigma = 1, 10, 100$ are also drawn for $N_{\text{COBE}} = 50$; the line $\sigma = c$ is indistinguishable from the diagonal line. In the simplest version of the model, the theoretically favoured regime is between the line $\sigma = 1$ and the line $\sigma = c$, but with some fine tuning it can extend down to $\sigma \sim 100$.

FIG. 6. Allowed region in the $n_{\text{COBE}} - 1$ vs $n_8 - 1$ plane for negative $\sigma$ and $c$ (Model (iii)). Again the two panels correspond to different reionization epoch hypothesis, as in Fig.5. The allowed region is below the dotted line $n_8 = n_{\text{COBE}}$, and above the solid (dashed) line at 95% (70%) confidence level. These lines do not depend on the value of $N_{\text{COBE}}$. The line $\sigma = c$ is also drawn for $N_{\text{COBE}} = 50$. The theoretically favoured regime $|\sigma| \geq |c|$ is the sector between this line and the $n_8 = n_{\text{COBE}}$ line. The region of positive $n_8 - 1$ and/or $n_{\text{COBE}} - 1$ is not shown, since it corresponds to $c \gg \sigma$. 
We thank Pedro Ferreira and Andrew Liddle for useful discussions.