This paper deals with two main subjects encountered in the design of linear accelerators: the beam dynamics and the electrodynamics. Although at first intended as a simple summary of relevant formulae and enumeration of important facts, it was soon realised that a better use of the paper could be obtained by including several derivations and explanations. In this way, the paper became necessarily heavier, but at the same time it evolved into both a collection of formulae and a tutorial on some critical issues. Ion linacs occupy most of the text, but electron linacs are also treated.

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Geneva, Switzerland
28 January 2000
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1. FOREWORD AND INTRODUCTION

Foreword

In the course of the last thirty years or so, linac theory and design procedure have considerably evolved. The progress is contained in many reports, conference papers, private notes and books, and it is usually time consuming and difficult to find a formula or procedure one is looking for. Also, in some cases, one would like to know how a formula has been derived and what approximations have been made. To help people studying or designing linear accelerators, in particular ion linacs, Helmut Haseroth proposed to the authors to try to group a collection of formulae and design procedures into a single paper, adding, where necessary, some explanations. This is the origin of this paper.

In order not to make the paper too long, many items considered as easily accessible in a number of publications have been left out. If necessary, they could be included later. Will this paper be useful? To increase its chances, the authors think that its content should be completed or changed, as the state of the art evolves, by contributions of other authors. In a later stage, one could put the paper on the Internet, together with links to its various references, so the reader could get a more complete information on specific problems. This is a suggestion put forward by Alessandra Lombardi.

The present version of the paper is to be considered as a first edition. All comments and suggestions are therefore welcome.

Introduction

This paper deals essentially with electromagnetic fields in accelerator cavities and beam dynamics; the former is emphasised as it is considered as a less well known subject than the latter.

The paper starts (Chapter 2) with a list of symbols and units. The symbols are the usual ones used for linear accelerators, with the drawback that sometime the same symbol represents different things. However, it is not too difficult to distinguish what a symbol refers to. Chapter 3 is a reminder of relativistic relations and Chapter 4 presents the basic theory of electromagnetic waves in bounded media, as accelerator cavities are (the boundary is a perfect conductor). Chapter 5 describes briefly some linear accelerator structures, both for ions and electrons. The beam dynamics is treated in Chapter 6, where one has left out the elementary equations and considerations of the alternating gradient focusing. The beam dynamics in ion linacs is treated in more detail than in electron linacs. A separate part, Chapter 7, has been devoted to space charge effects, dividing them into linear and non-linear categories. The suggested literature is enumerated in the Bibliography; only a few tutorial books, Linear Accelerator Conference Proceedings and Proceedings of the CERN Accelerator School are referred to, because they are relatively easy to find in libraries. Internal reports, more difficult to be accessed, are left out in this first version of the paper. Derivations of some formulae related to the subjects dealt with are included in the Appendices.
2. LIST OF SYMBOLS AND UNITS

The meaning of symbols used in this paper (dual meanings are separated by ;):

\( \alpha \) attenuation constant
\( \beta \) propagation constant; relativistic factor
\( \beta_{ph} \) phase velocity of an accelerating wave
\( \beta_s \) synchronous particle velocity
\( \delta_x, \delta_z \) space charge factors
\( \varepsilon, \varepsilon_{\text{rms}} \) emittance, rms emittance
\( \lambda \) RF wavelength
\( \lambda_{\beta}, \lambda_s \) betatron, synchrotron, wavelength
\( \mu \) space charge parameter
\( \sigma_{0T}, \sigma_{OL} \) smooth phase advance of betatron, synchrotron, oscillations per transverse focusing period without space charge
\( \sigma_T, \sigma_L \) idem with space charge
\( \tau \) field attenuation along a TW accelerator cavity
\( \phi \) phase of a particle with respect to the RF field
\( \phi_s \) synchronous phase
\( \omega, \omega_c \) angular frequency, cutoff angular frequency
\( A \) vector potential; mass number
\( B \) magnetic induction; focusing parameter
\( E \) total energy; electric field
\( E_z \) accelerating field
\( E_0 \) rest energy; average accelerating field
\( g \) gap length
\( k, k_z \) phase advance per unit length in the longitudinal direction
\( k_n \) value of \( k \) referring to structure mode \( n \)
\( k_{\beta}, k_s \) smooth betatron, synchrotron, phase advance per unit length
\( \bar{K}_x^2, \bar{K}_z^2 \) smooth focusing factors in the \( x, z \) direction
\( k_{sc} \) space charge factor
\( l \) cell length; structure period
\( L \) cavity length
\( m \) proton mass; atomic mass unit; electron mass
\( n \) space harmonic number of an accelerating mode
\( N \) number of \( \beta_s \) per focusing period
\( P \) power fed into an accelerating cavity
\( q, e \) elementary charge
\( Q \) quality factor of an accelerating cavity
\( t_f \) filling time of a cavity
\( T \) transit time factor
\( v_p, v_g, v_{ph} \) particle velocity, group velocity, phase velocity
\( w \) stored energy per unit length of a structure
\( Z \) charge number
\( Z_s \) shunt impedance
\( Z_{\text{seff}} \) effective shunt impedance

A certain number of symbols with obvious meaning, such as \( \varepsilon_0, \mu_0 \), and relativistic factors \( \beta, \gamma \) have been left out. The practical system of units, \( V \cdot A \cdot s \) or \( m \cdot kg \cdot s \), is applied.
3. RELATIVISTIC RELATIONS

Some relativistic expressions (the symbols have the usual meaning):

\[ E_0 = mc^2 \; ; \; E = E_0 \gamma = mc^2 \gamma \; ; \; p = mc \beta \gamma \; ; \; cp = mc^2 \beta \gamma = E_0 \beta \gamma \; ; \; E^2 = E_0^2 + p^2 c^2 \]

\[ \beta \gamma = \frac{cp}{E_0} \; ; \; \gamma = (1 - \beta^2)^{-1/2} \; ; \; \beta^2 \gamma^2 = \gamma^2 - 1 \; ; \; W = E - E_0 \; ; \; \frac{mc \beta \gamma}{q} = B \rho . \]

**Table 1.** Analytic relations between \( \beta, \gamma, W, cp \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( W )</th>
<th>( cp )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>( \sqrt{\gamma^2 - 1} )</td>
<td>( \sqrt{1 + \frac{W}{E_0} \left( \frac{1}{E_0} \right)^2} - 1 )</td>
<td>( \frac{cp}{mc^2} )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{1-\beta^2}} )</td>
<td>( \gamma )</td>
<td>( 1 + \frac{W}{E_0} )</td>
<td>( \frac{1}{\sqrt{1 + \left( \frac{cp}{mc^2} \right)^2 - 1}} )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{1-\beta^2}} - 1 )</td>
<td>( E_0 (\gamma - 1) )</td>
<td>( W )</td>
<td>( mc^2 \left[ \frac{1}{\sqrt{1 + \left( \frac{cp}{mc^2} \right)^2 - 1}} \right] )</td>
</tr>
<tr>
<td>( \frac{mc^2}{\sqrt{1-\beta^2}} )</td>
<td>( E_0 (\gamma^2 - 1)^{1/2} )</td>
<td>( \left[ W (2E_0 + W) \right]^{1/2} )</td>
<td>( cp )</td>
</tr>
</tbody>
</table>

Some relations concerning first derivatives of relativistic factors:

\[ \frac{d \beta}{d \gamma} = \frac{1}{\beta \gamma^3} \; ; \; \frac{d(1 / \beta)}{d \gamma} = - \frac{1}{\beta \gamma^2} \; ; \; \frac{d(\beta \gamma)}{d \beta} = \gamma^3 \; ; \; \frac{d(\beta \gamma)}{d \gamma} = \beta \; ; \]

Logarithmic first derivatives:

\[ \frac{d \beta}{\beta} = \frac{1}{\beta^2 \gamma} \frac{d \gamma}{\gamma} = \frac{1}{\gamma(\gamma + 1)} \frac{d W}{W} = \frac{1}{\gamma^2} \frac{d p}{p} \; ; \; \frac{d \gamma}{\gamma} = (\gamma^2 - 1) \frac{d \beta}{\beta} = \left( 1 - \frac{1}{\gamma} \right) \frac{d W}{W} = \beta^2 \frac{d p}{p} . \]

**Force equation; transverse and longitudinal mass**

The fundamental dynamics equation is the force equation:

\[ \tilde{F} = m \frac{d}{dt} (\gamma \vec{v}) = m \left( \gamma \frac{d \vec{v}}{dt} + \vec{v} \frac{d \gamma}{dt} \right) = m \left( \gamma \frac{d \vec{v}}{dt} + \vec{v} \beta \gamma^3 \frac{d \beta}{dt} \right) ; \]

it is assumed that the velocity is relativistic in the longitudinal, \( z \)-direction, and much smaller in the transverse direction. If the force is in the transverse direction, \( \gamma \equiv const \), and one has

\[ F_i = m \gamma \frac{dv_i}{dt} \Rightarrow \frac{dv_i}{dt} = \frac{F_i}{\gamma m} ; \]

\( \gamma m \) is the **transverse mass.** If the force is in the longitudinal, \( z \)-direction:

\[ F_i = m \left( \gamma \frac{dv_i}{dt} + \frac{v_i^2}{c^2} \gamma^3 \frac{dv_i}{dt} \right) = m \gamma \frac{dv_i}{dt} (1 + \beta^2 \gamma^2) = m \gamma^3 \frac{dv_i}{dt} \Rightarrow \frac{dv_i}{dt} = \frac{F_i}{\gamma^3 m} ; \]

\( \gamma^3 m \) is the **longitudinal mass.**
Table 2a. Numerical relations between $W, \gamma, \beta, B\rho$ for protons  
(Units: $W: [\text{MeV}]; \ B\rho: [\text{Tm}]$)

<table>
<thead>
<tr>
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<th>$\gamma$</th>
<th>$\beta$</th>
<th>$B\rho$</th>
<th>$W$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
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<th>$\gamma$</th>
<th>$\beta$</th>
<th>$B\rho$</th>
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Table 2b. Numerical relations between $W, \gamma, \beta, B\rho$ for electrons  
(Units: $W: [\text{MeV}]; \ B\rho: [\text{Tm}]$)

<table>
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<th>$\beta$</th>
<th>$B\rho$</th>
<th>$W$</th>
<th>$\gamma$</th>
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</table>

Some constants:

- permeability of vacuum $\mu_0 = 4\pi \times 10^{-7} = 1.257 \times 10^{-6}$ [Vs/Am]
- permittivity of vacuum $\varepsilon_0 = 1/\mu_0 c^2 = 8.854 \times 10^{-12}$ [As/Vm]
- elementary charge $q, e = 1.602 \times 10^{-19}$ [As]
- proton mass $m = 938.27$ [MeV/c^2]
- atomic mass unit $m = 931.49$ [MeV/c^2]
- electron mass $m = 0.511$ [MeV/c^2]
4. ELECTRODYNAMICS

4.1 RF field types and modes in cavities

In free space the electromagnetic fields are of the transverse electro magnetic, TEM, type: the electric and magnetic field vectors are \( \perp \) to each other and to the direction of propagation.

In bounded media (assume the boundary to be a perfect conductor) TEM waves are not possible; one of the field components must be in the direction of propagation to satisfy the boundary conditions. If it is an electric component, one has transverse magnetic, TM, waves; if it is a magnetic component, one has transverse electric, TE, waves. Wave propagation in cavities can be explained in terms of wave reflections from wall to wall.

The waves in a waveguide can be of different modes, which are indicated by two subscripts, as \( \text{TM}_{mn} \) or \( \text{TE}_{mn} \): in a rectangular waveguide, \( m \) and \( n \) indicate the number of half waves in the \( x \) and \( y \) direction, respectively (the \( z \) coordinate is in the direction of propagation). In cylindrical coordinates, \( m \) represents the number of azimuthal periods and \( n \) the number of zeros of the longitudinal field in the radial direction, axis excluded.

The modes in a resonator have three subscripts, \( m,n,p \). The meaning of \( m \) and \( n \) is as before; \( p \) indicates the number of half periods longitudinally.

The electromagnetic field in a cavity satisfies the wave equation with the boundary conditions \( E_t = 0 \) (tangential electric field) and \( B_n = 0 \) (normal magnetic field). The wave equation in rectangular coordinates is:

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \hat{E} = 0 ,
\]

and the same for \( \tilde{B} \). The wave equation applies to each field component and the solution is usually expressed as the product of functions of one variable; for TM waves one has e. g.

\[
E_z(x, y, z, t) = X(x)Y(y)Z(z)T(t) .
\]

For sinusoidally varying fields

\[
T(t) \propto e^{i\omega t} \quad \text{and} \quad Z(z) \propto e^{-jk_z z} ,
\]

\( \omega \) being the angular frequency and \( k_z \) the phase advance per unit length; the field component \( E_z \) is now

\[
E_z = AX(x)Y(y)e^{i(\omega t - k_z z)} ,
\]

with \( A \) being a constant. The expression \( (\omega t - k_z z) \) is typical for travelling waves; moving with the wave one can put \( (\omega t - k_z z) = 0 \) \( \rightarrow \frac{z}{t} = \frac{\omega}{k_z} = v_{ph} \); \( v_{ph} \) is the phase velocity of the wave, which is relevant for acceleration: there must be a synchronism between the particle velocity \( v_p \) and \( v_{ph} \). In empty cavities \( v_{ph} > c \) (the waves in a cavity move in fact with the velocity \( c \), but in order to satisfy the boundary conditions, they can only travel by being reflected from wall to wall; the velocity of the motion of the wave phenomenon along the longitudinal axis, \( v_{ph} \), results then to be \( > c \)): the electromagnetic energy propagates with a smaller velocity, the group velocity, given with

\[
v_g = \frac{d\omega}{dk_z} .
\]
Putting the assumed form of the solution of $E_z$ into the wave equation one obtains:

$$\frac{X'''}{X} + \frac{Y'''}{Y} + \frac{Z'''}{Z} + \frac{T''}{T} = 0,$$

(4.5)

where $X''$ indicates the second derivative with respect to $x$, $Y''$ the second derivative with respect to $y$ etc. All the terms of the above equation, depending on different variables, have to be constant to satisfy the equation. The functions $X(x)$ and $Y(y)$ are trigonometric functions (as e.g. $\sin (k_x x)$ or $\cos (k_y y)$, etc); with this, the above equation can finally be written in the standard form

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2.$$

(4.6)

The constants $k_x$, $k_y$ and $k_z$ can assume only discrete values which satisfy the boundary conditions.

The wave equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{c^2} \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial \theta^2} \bar{E} = 0.$$

(4.7)

The solution (valid for each field component) is for a TM wave:

$$E_z = R(r) \Theta(\theta) Z(z) T(t);$$

(4.8)

$Z(z)$ and $T(t)$ are as before; separating the variables, the following set of equations is obtained:

$$\frac{1}{\Theta} \frac{d^2 \Theta}{d \theta^2} + m^2 = 0; \quad \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( \frac{\omega^2}{c^2} - k_z^2 - \frac{m^2}{r^2} \right) R = 0.$$

(4.9)

The function $\Theta(\theta)$ is again a trigonometric function with $m$ azimuthal periods; the function $R(r)$, satisfying the Bessel equation, is given by the Bessel function of the first kind, of order $m$ and argument $(\omega^2/c^2 - k_z^2)_{1/2} r$:

$$R(r) = AJ_m \left( \sqrt{\frac{\omega^2}{c^2} - k_z^2} \cdot r \right) = AJ_m (K_r r), \quad \text{A being a constant.}$$

(4.10)

At the boundary (a cylinder of radius $a$), the condition for TM waves is $E_z = 0$, i.e. $J_m(K_r a) = 0$, and the first solution (lowest frequency) is for the TM$_{01}$ wave, with $K_r a = 2.405$ or $K_r = 2.405/a$; to satisfy the boundary conditions, $K_r$ is fixed. The relation

$$\frac{\omega^2}{c^2} = K_r^2 + k_z^2$$

(4.11)

is called the dispersion relation and it links, for a given wave type and mode, the frequency of oscillation $\omega$ to the phase advance per unit length $k_z$; the plot of the dispersion relation is shown in Fig.1. The curve is a hyperbola with asymptotes $\omega = \pm k_z c$; all the points of the hyperbola lying above the asymptotes have $v_{ph} = \omega/k_z > c$. For TE waves ($B_z$ instead of $E_z$), the condition on the boundary is $B_n = 0$, which is equivalent to

$$\frac{\partial B_z}{\partial n} = 0;$$

the partial derivative of the tangential field component is to be taken $\perp$ to the boundary. This means that for TE waves the derivative of the Bessel function at the boundary must be zero:

$$J_m'(K_r r) = 0.$$
The smallest value of $K_r$ satisfying the above condition is $K_r = 1.84/a$ (TE$_{11}$ wave), which is lower than that of TM waves. Figure 2 shows the cutoff frequencies of TE$_{mn}$ and TM$_{mn}$ modes, relative to the TE$_{11}$ mode.

![Fig.1 Dispersion or Brillouin diagram; $\omega_c$ is the cutoff frequency corresponding to $k_z = 0$.](Image)

**Fig.1 Dispersion or Brillouin diagram; $\omega_c$ is the cutoff frequency corresponding to $k_z = 0$.**

**Fig.2 Relative cutoff frequencies for various wave types and modes.**

### 4.2 Slow waves

Empty cavities are not suitable for particle acceleration because $v_{ph} > c$ and the required synchronism between particle and wave velocities is impossible; the waves must be slowed down and this is achieved by *loading* the cavity with *periodic obstacles*. Figure 3 shows a *disc loaded cavity*. The obstacles delimit *cells*, and each cell is a resonator, *coupled* to its neighbours through the central aperture (electric coupling). The coupling can also be magnetic, via peripheral slots made in the wall between cells. A cavity with $N$ cells has $N$ resonant frequencies.

![Fig.3 Disc loaded cavity](Image)

**Fig.3 Disc loaded cavity**

The loaded cavity is a *periodic structure* of period $l$. The solution of the wave equation is now based essentially on two points:

--- *Floquet theorem*: for a given frequency and mode of oscillation, the solution may differ from a period to the next only by a factor like $e^{-jkz}$ (with losses, $e^{-\gamma z}$, $\gamma = \alpha + j\beta$);

--- the complicated boundary conditions can be satisfied only by a whole spectrum of modes called *space harmonics*.

The nomenclature of the wave type and mode in a loaded cavity follows that of an empty cavity.
One is interested primarily in the configuration of the electromagnetic field around the axis, where the interaction with the beam takes place. For rotationally symmetric TM waves in a cylindrical disc loaded cavity one has:

\[ E_z(r, z, t) = F(r, z) e^{i(k_r r - \omega t)} \]  
(4.12)

\[ F(r, z + l) = F(r, z) \]  
(4.13)

\[ F(r, z) = \sum_n a_n(r) e^{-j(2\pi n/m)z} ; \]  
(4.14)

introducing these expressions in the wave equation, each space harmonic must satisfy

\[ \left[ \frac{d^2 a_n(r)}{dr^2} + \frac{1}{r} \frac{da_n(r)}{dr} + K_m a_n(r) \right] = 0 , \]  
(4.15)

with

\[ K_m = \left( \frac{\omega}{c} \right)^2 \left( k_z + \frac{2\pi n}{l} \right)^2 \]  
and

\[ v_{ph} = \frac{\omega}{k_z + 2\pi n / l} = \frac{\omega}{k_n} . \]  
(4.16)

For each \( n \) one has a travelling wave with its own, slowed down velocity \( v_{ph} \). In principle one can find \( v_{ph} = v_p \). For \( v_{ph} < c \), \( K_m^2 < 0 \), so \( K_m = jk_r \) and the solution of the Bessel equation is now expressed by Bessel functions of imaginary argument, which are called modified Bessel functions:

\[ a_n(r) = A_n I_0(k r r) . \]

It is not easy to find the functional relation between \( \omega \) and \( k_n \) for the dispersion diagram of periodic structures. This diagram must reflect the periodicity of the structure and it will be drastically different from that of an empty cavity. Figure 4 shows the diagram of loaded cavities:

![Dispersion diagram of a periodic structure (loaded cavity).](image)

Fig.4 Dispersion diagram of a periodic structure (loaded cavity).

Interesting features are observed:

a) for a given wave type and mode, there is a limited passband between \( \omega_0 \) and \( \omega_\pi \); at these points, \( v_g = 0 \); the stronger the coupling between cells, the larger the passband;

b) for a given frequency, there is an infinite series of space harmonics, with the same \( v_g \), but with different \( v_{ph} \);

c) when the electromagnetic energy propagates only in one direction (solid curves on the diagram) \( \rightarrow \) travelling wave accelerator, TW; when the energy is reflected back and forth at both ends of the cavity (solid and dotted curves) \( \rightarrow \) standing wave accelerator, SW;

d) if \( v_{ph} \) and \( v_g \) in the same direction \( \rightarrow \) forward wave; if in opposite directions \( \rightarrow \) backward wave;
e) travelling wave accelerators operate near the middle of the passband, where \( v_g \) is maximum and the mode spacing biggest, see point A on Fig.4;
f) standing wave accelerators operate on the lower or upper end of the passband, point B or C on Fig.4, because only there the direct and reflected wave have the same \( v_{ph} \), and they both accelerate particles. At these points the group velocity is zero, and to allow the RF energy to flow (due to transients and dissipation losses), the field pattern gets slightly distorted.

4.3 Fundamental cavity parameters

* Transit time factor \( T \) (on the axis of the cavity):

\[
T = \frac{\int_0^L E_z(z) e^{-\frac{z}{L}} \, dz}{\int_0^L E_z(z) \, dz}.
\]

(4.17)

\( T \) is a measure of how efficiently a cavity can accelerate particles of velocity \( v_p \). The factor \( T \) is independent of the amplitude of the accelerating field \( E_z \); \( L \) is the length of the cavity.

* Effective shunt impedance per unit length \( Z_{seff} \):

\[
Z_{seff} = Z_s T^2 = \left( \frac{E_0 T}{-dP/dz} \right)^2 \text{[M}\Omega\text{m}^{-1}] ,
\]

(4.18)

\( E_0 \) being the average accelerating field on axis, \( P \) the power flowing in the structure and \( Z_s \) the shunt impedance. \( Z_{seff} \) indicates how efficient the acceleration is for a given dissipated power \(-dP/dz\).

* Quality factor \( Q \):

\[
Q = \omega \frac{w}{-dP/dz}.
\]

(4.19)

with \( w \) being the stored energy in the structure per unit length. The \( Q \) factor is a measure of the stored energy for a given power loss.

* Ratio of shunt impedance to quality factor

\[
\frac{Z_s}{Q} = \frac{E_0^2}{\omega w} \text{[M}\Omega\text{m}^{-1}]
\]

(4.20)

depends only on the geometry of the structure and does not depend on the accelerating field nor on the losses on the structure walls.

* Group velocity \( v_g \):

\[
v_g = \frac{P}{w} \text{[ms}^{-1}] ;
\]

(4.21)

as \( w \propto E^2 \), a small \( v_g \) is usually preferred for having a bigger \( w \) and hence a bigger accelerating field \( E \).

* Filling time \( t_F \):
\[ t_F = \int_0^L \frac{dz}{v^*_g(z)} \text{ [s]} ; \]  

\[ t_F \] indicates the time needed for the electromagnetic energy to fill the cavity of length \( L \); the formula is used for TW accelerators. In SW accelerators, the electromagnetic energy builds up progressively in time according to

\[ t_F \propto \frac{Q}{\omega} \]  

* Duty factor \( \eta \):

It indicates the fraction of time during which the accelerator is in active mode. It is given by the product of RF pulse length and repetition rate.

### 4.4 Travelling wave accelerators

TW accelerators operate with \( n = 0 \) space harmonic, approximately in the middle of the passband, with a phase advance per period \( l \) of

\[ k_z l \equiv \pi/2 \text{ to } 2\pi/3 \ . \]

One distinguishes between constant impedance and constant gradient TW accelerators. Along the cavity, power is dissipated and the electric field is attenuated:

\[ \frac{dE_z(z)}{dz} = -\alpha(z)E_z(z) , \quad \alpha(z) \text{ is the attenuation constant}; \]  

\[ \frac{dP(z)}{dz} = -2\alpha(z)P(z) = -\frac{\omega P(z)}{Qv^*_g(z)} \rightarrow \alpha(z) = \frac{\omega}{2Qv^*_g(z)} . \]  

In a constant impedance accelerator the structure dimensions are kept constant through the accelerator; the properties are:

\[ \alpha(z) = \text{const} = \alpha \]

\[ E_z(z) = E_z(0) e^{-\alpha z} \]

The energy gain of a particle of charge \( q \) sitting on the crest of the wave is:

\[ qV = q \int_0^L E_z(0)Te^{-\alpha z} dz = qE_z(0)T L \frac{1 - e^{-\alpha L}}{\alpha L} , \]  

with \( L \) being the accelerator length. The energy gain can be expressed with the input power \( P(0) \), total attenuation \( \tau = \alpha L \) and the impedance \( Z_{\text{eff}} = \frac{E_z^2(z)T^2}{-dP(z)/dz} \):

\[ qV = q(2\tau)^{1/2} \frac{1 - e^{-\tau}}{\tau} \left[ P(0)Z_{\text{eff}} L \right]^{1/2} . \]  

The maximum energy gain is obtained when \( \tau = 1/2(e^{\tau} - 1) \approx 1.26 \); with this:

\[ qV_{\text{max}} \approx 0.903 q \left[ P(0)Z_{\text{eff}} L \right]^{1/2} = 0.57 qE_z(0)T L . \]  

In this optimised case, one has at the end of accelerator:
\[ E_z(L) = 0.28 \, E_z(0) \quad \text{and} \quad P(L) = 0.08 \, P(0). \quad (4.29) \]

In a constant gradient accelerator the structure dimensions vary slightly (but with a small effect on \( Z_s \) and \( Q \)) in order to have a more regular heating of the structure and smaller peak fields for a given energy gain. The properties are

\[ E_z(z) = \text{const} \quad \text{and} \]

\[ Z_{\text{eff}} = \frac{E_z^2 r^2}{-\left. \frac{dP(z)}{dz} \right|_{dz}} = \text{const}; \quad \rightarrow \quad -\frac{dP(z)}{dz} = \text{const} = \frac{d}{dz} \left( P(0) + \frac{P(L) - P(0)}{L} \right) = -P(0) \frac{1 - e^{-2\tau}}{L}, \quad \text{with} \quad \tau = \int_0^L \alpha(z)dz. \quad (4.30) \]

The group velocity is

\[ v_g(z) = -\frac{\omega P(z)}{Q \cdot \left. \frac{dP(z)}{dz} \right|_{dz}} \propto P(z), \quad \text{because of} \quad \frac{dP(z)}{dz} = \text{const}. \]

### 4.5 Standing wave accelerators

Standing wave accelerators use both the direct and the reflected wave to accelerate particles; this is possible only by operating at the lowest or highest frequency of the passband, where

\[ k_n l = N\pi, \quad N = 0, \pm 1, \quad \text{see Fig.}4. \]

The structure mode in standing wave accelerators is either 0 or \( \pi \).

The cavity mode is, as already mentioned, specified by three subscripts, \( \text{TM}_{mnp} \) or \( \text{TE}_{mnp} \).

The beam mode is defined by the RF phase change during the time the beam travels over a structure period

\[ k_n l = 2h\pi, \quad h = 0,1,2. \]

For example, for a zero structure mode, the beam mode is usually \( 2\pi \) (\( h=1 \)).

### 4.6 Electromagnetic field computations in cavities

The design of accelerator structures for a given electromagnetic field pattern and frequency is done by computer programs, the best known of them being the 2-dimensional (2-D) \textit{SUPERFISH} and \textit{URMEL}, and the 3-D \textit{MAFIA}. The former are more precise, because 2-D fields are better specified with a given number of mesh points. In all these programs an adequate drive point is chosen and the partial differential wave equation is solved numerically on the mesh points. For rotationally symmetric TM fields, 2-D, one uses cylindrical coordinates \((z,r)\), while 2-D TE fields are treated with rectangular coordinates \((x,y)\). The boundary conditions are either of the \textit{Dirichlet} type (electric field lines are parallel to the boundary) or of the \textit{Neumann} type (electric field lines are perpendicular to the boundary). The properties of periodic structures as in accelerator cavities are analysed with programs like \textit{PSPICE}, based on coupled circuit theory.

When designing a resonator with a 2-D program, one makes use of symmetries and it usually suffices to specify 1/4 or even 1/8 of the cell geometry. Figure 5 shows 1/4 of a drift tube linac (DTL) cell; the coordinate system is cylindrical. The mesh is diminished in regions where a better precision of the TM fields is required, see Fig.5a. Figure 5b shows electric field lines; the field is stronger where the lines are closer. Figure 6 shows 1/8 of the cross section of a 4-vane radio frequency quadrupole (RFQ); the field is of the TE type and rectangular coordinates are used. The beam axis (lower left corner) is perpendicular to the plane of the plot. The lines between the vane and the symmetry plane of the quadrant are electric field lines; as can be seen, the field is stronger at the vane tips. Figure 7
shows an example treated by the program MAFIA: a detail of a \textit{side coupled linac (SCL)} is analysed, containing an accelerator cell on the beam axis and two halves of coupling cells off axis; magnetic boundaries (Dirichlet) are specified on each side of the SCL detail, because the SCL operates with the $\pi/2$ beam mode.

\textit{Fig.5} DTL cell computed by SUPERFISH: mesh (5a) and electric field lines (5b) are shown.

\textit{Fig.6} 4-vane RFQ computed by SUPERFISH.

\textit{Fig.7} Analysis of a detail of the SCL by MAFIA.
5. LINEAR ACCELERATOR STRUCTURES

Linear accelerators are periodic structures (periodic with respect to time, quasi-periodic with respect to distance), where the individual resonating cells are coupled to each other either electrically or magnetically. The cells do not need to be all the same, and one often uses in the SW regime biperiodic structures, containing two distinct chains of resonating cells. Such structures are used to stabilise the pattern of the electromagnetic field.

5.1 Disc-loaded structure

The structure shown in Fig.3 is a disc-loaded structure, where the cells are electrically coupled through the beam aperture; big apertures increase the coupling and thus the bandwidth (desirable), lowering at the same time the shunt impedance $Z_s$ (undesirable). Sometimes the discs have noses to increase the transit time factor $T$. In this case, the electric coupling through the beam aperture becomes too small, and discs are made with coupling slots (magnetic coupling) to avoid this problem, see Fig.8. The lowest passband in such a structure is then normally of the backward wave type.

![Fig.8 Magnetically coupled disc-loaded structure (coupling slots).](image)

Disc-loaded structures operate in the TW and SW regime, in the RF frequency range of about 0.5 to 3 GHz. In the travelling wave case, the structure mode (sometimes also called operating mode) is approximately in the middle of the passband (phase advance $\pi/2$ to $2\pi/3$), where the group velocity and the mode spacing are the biggest. In the standing wave regime, the structure mode is $\pi$. The field configurations for different structure modes are shown in Fig.9.

![Fig.9 Instantaneous electric field configurations for different structure modes.](image)
5.2 Alvarez or drift tube linac (DTL)

The Alvarez linac is a SW accelerator, using usually RF frequencies of about 100 to 400 MHz, and operating in the 0 structure mode. The electric and magnetic fields in adjacent cells are in phase and the dividing walls are therefore not necessary. This increases $Z_s$, which can be several tens of MΩ/m at lower beam energies (for $\beta$ values from 0.03 to 0.4). Figure 10 shows a drift tube linac schematically; in the drift tubes magnetic quadrupoles are housed for beam focusing. The wave type and mode is TM$_{010}$. The energy gain per unit length is roughly constant and the gain/cell increases along the accelerator. This is a characteristic of TM waves. The electric and magnetic fields are linked via the integral equation

$$\oint E ds = -\frac{\partial}{\partial t} \int B df,$$

where $ds$ is the path element, $df$ the surface element and the integral at right englobes the magnetic flux in a cell, see Fig.11. The cell length is $\beta \lambda$ and the beam mode $2\pi$ ($h=1$), i.e. the particles travel from one gap to the next in an RF period. It is possible to use the linac with $1/2$ of the synchronous velocity, and particles then travel from one gap to the next in two RF periods. Here the second harmonic of the beam mode ($h=2$) is used, and such a structure is a 2$\beta \lambda$ one.

![Fig.10 Alvarez or drift tube linac.](image)

![Fig.11 Line and surface integrals in the drift tube linac operating with a TM$_{010}$ wave.](image)

5.2.1 Quasi Alvarez

The Alvarez structure operating in the 2$\beta \lambda$ mode is used for acceleration of heavier particles, with a charge to mass ratio, $Z/A$ of $\leq 1/2$. The accelerating efficiency of such a structure is usually low, due to the low transit time factor $T$. Nevertheless, the 2$\beta \lambda$ structure is often preferred to the more cumbersome low frequency linacs ($\approx 100$ MHz) with a $\beta \lambda$ structure. In some cases, the situation can be improved by using a quasi Alvarez linac, where one has combined 2$\beta \lambda$ and $\beta \lambda$ structures in the same cavity. The alternation between these structures can be optimised from case to case. It may also be possible to place quadrupoles only in the drift tubes of the 2$\beta \lambda$ cells, allowing the drift tubes of the $\beta \lambda$ cells to be smaller, increasing thus the shunt impedance of the cavity. In Fig.12 are sketched various structure combinations in a quasi Alvarez cavity; the transverse focusing period has a length of $N\beta \lambda$, where $N$ depends on the chosen combination.
5.2.2 Side-coupled drift tube linac (SCDTL)

The use of drift tube linacs in the GHz frequency range requires very small drift tubes, where quadrupoles cannot be placed. Quadrupoles are then placed between cavities, and in order to have enough focusing, the cavities must be relatively short, containing only a few drift tubes. To propagate the RF power, the cavities are coupled via coupling cells, which in addition stabilise the structure, making it less sensitive to various production and adjustment errors (see Chapter 5.4). Such a structure operating at 3 GHz and with very low beam intensities (about 10 nA average), has been studied for use as a medical proton accelerator and it has been called side coupled drift tube linac (SCDTL). The SCDTL has a very good shunt impedance in the energy range of 5 to 70 MeV; it is presented schematically in Fig.13; the quadrupoles are not inserted.

5.3 Interdigital or IH linac

The IH linac is a SW structure, operating with the TE_{110} wave type and mode. The electric field is transverse in most of the structure, but in the gaps between drift tubes the field is essentially longitudinal with only a small transverse component, caused by stems placed on opposite sides of neighbouring drift tubes, see the schematic Fig.14. The transverse component can be eliminated by an adequate design of drift tube front faces. The structure mode is π and the energy gain/cell is constant (not the energy gain/unit length), as can be understood by taking a line integral of the electric field and a surface integral of the magnetic field, as indicated in Fig.15. This is characteristic of TE waves. The IH structure is very efficient at low beam energies (β ≈ 0.02 to 0.08) and low RF frequencies (40 to 100 MHz), and is used primarily for ions with A/Z > 4. The focusing elements are placed outside between the accelerator tanks.
5.4 Stabilisation of structures; biperiodic structure

The SW accelerators usually work in the 0 or π structure mode (beam mode 2π or π) in order to accelerate particles efficiently. However, the 0 and π modes have $v_g = 0$, and as the electromagnetic energy must propagate (structure losses, beam loading), neighbouring modes with $v_g \neq 0$ mix in and the original field pattern gets easily distorted. Neighbouring modes mix also in the presence of machining and tuning errors, with amplitudes proportional to $\frac{1}{\omega_m / \omega_n - 1}$, where $\omega_m$ is the angular frequency of the perturbing mode, and $\omega_n$ of the nominal, operating mode. The frequency of the neighbouring modes is very close to the frequency of the operating mode in the 0 or π mode regime, so these regimes are very sensitive and therefore unstable. “Stable” operating points are in the middle of the passband, e. g. at π/2 structure mode, where $v_g$ is big; the perturbing modes, placed symmetrically below and above $\omega_n$, mix in with opposite amplitudes, cancelling each other to first order. However, the π/2 mode has each second cell ‘empty’ and therefore a low accelerating efficiency per unit length; to improve this, the “zero field cells” have been made either shorter or even placed sideways, as shown on Fig.16. A structure with two types of cells, accelerating and coupling cells, is called a biperiodic structure; such a structure is both efficient and stable. In the lossless approximation, coupling cells are unexcited; when energy has to flow (possible because $v_g \neq 0$), they get slightly excited, but the field pattern in the accelerating cells remains unperturbed.

5.5 Side-coupled linac (SCL)

The SCL is a SW accelerator, with a biperiodic structure operating in the π/2 mode. For protons it usually operates at frequencies below 1 GHz, and is used for beam energies from 100 MeV on. The accelerating cells have noses around the beam aperture to increase the transit time factor $T$, see
Fig.17. With noses, the electric coupling on axis between accelerating cells becomes negligibly small; another coupling, magnetic, is then ensured via the coupling cell which is placed sidewise. The coupling between the accelerating and the coupling cell is called nearest neighbour coupling. There also exists a smaller next nearest neighbour coupling which, in biperiodic structures, is either between adjacent accelerating cells or between coupling cells. This coupling should be kept small for better stability of the structure. For that reason the coupling cells are usually placed on alternate sides of the accelerating cells.

5.5.1 Cavity-coupled drift tube linac (CCDTL)

The coupled cavity drift tube linac (CCDTL) has been derived from the SCL by including drift tubes in the structure, see Fig.18. By combining an SCL with a DTL, one has obtained a linac with a reasonable shunt impedance in the range of $0.2 < \beta < 0.5$, i.e. at energies which are between an optimum use of a DTL and an SCL accelerator.
Fig. 18 CCDTL structure with one (18a) and two (18b) drift tubes per cell. The stems of the drift tubes are perpendicular to the plane of the drawing.

5.6 Post stabilised drift tube linac; confluence of passbands

The normal drift tube linac has the drawbacks of a SW structure operating in the 0 mode. To make the field pattern more stable, an ingenious method has been devised: one introduces into the tank, in an appropriate way, another chain of oscillators, making the DTL in a certain sense a biperiodic structure. This biperiodicity operates, however, on a different principle: the DTL passband and the passband of the introduced oscillators have to approach each other as shown in Fig. 19. When, by correct tuning of the introduced chain, the passbands start “touching” each other at the 0 operating mode, and if the dispersion curves have opposite curvature there, they will influence each other and give as result a dispersion curve which has a $\pi/2$ mode characteristic at the position of the operating mode. This “touching” of passbands is called confluence, and accelerators can be stabilised in this way. Before touching, there is a stop band between the passbands; if the passbands cross each other, the stability disappears, the situation is even worse than without the additional chain. Therefore, one usually prefers to leave a small stop band, impairing only slightly the confluence.

Fig. 19 Two passbands brought to confluence at the 0 operating mode.

Practically, the introduced resonators can be posts, pointing from the tank wall towards the middle of the drift tube. In order to have the confluence at the 0 mode (opposite curvature of the dispersion curves), the posts are placed on alternate sides of the drift tubes, in a plane at 90° to the plane of the stems, see Fig. 20. To allow for a tilt in the accelerating field (if wanted), the posts are terminated with small eccentric tabs, which can be rotated, thus introducing an asymmetry in the coupling to neighbouring accelerating cells. Another possibility for the posts is to place them opposite to the stems (but not at each drift tube), but the stabilisation is less efficient.
5.7 Radio frequency quadrupole (RFQ) linac

The RFQ is a SW accelerator operating with the TE$_{210}$ wave type and mode; the electromagnetic field does not only bunch and accelerate the particles, but also focuses them, so no external focusing is needed. There are essentially two types of RFQ, the *four vane RFQ* and the *four rod RFQ*.

**Four-vane RFQ**

Figure 21 shows the TE$_{210}$ wave type and mode in the four vane RFQ and in an empty cavity (the longitudinal accelerating electric field which exists between the *modulated* vane tips is not shown). The TE$_{210}$ field configuration can easily be distorted by a neighbouring dipole mode TE$_{110}$, see Fig.22. The dipole and quadrupolar frequencies, $f_d$ and $f_q$, are related through:

$$f_d = \frac{f_q}{\sqrt{1 + k}}$$

$$f_q = \frac{1}{2\pi\sqrt{LC}}$$

where $k$ is the coupling constant defined as $k = C_o/C$, see the equivalent circuit (simplified) of the RFQ in Fig.23. As usually $C_o \ll C$, $k$ is very small (of the order of 0.01) and the frequencies $f_q$ and $f_d$ are therefore relatively close together.

![Fig.21 TE$_{210}$ mode in a four vane cavity and in an empty cavity.](image)

![Fig.22 TE$_{110}$ mode in a four vane cavity and in an empty cavity.](image)
Four-rod RFQ

The four-rod RFQ is shown schematically in Fig.24. It resembles a double transmission line, and in fact the operating frequency is essentially determined by the capacitance between the “lines” and by the inductance of the “supports”; the vacuum tank, which is placed around, has little influence on the resonant frequency, contrary to the four-vane RFQ. The four-rod RFQ is therefore preferred at lower frequencies (≤ 100 MHz). An example of a MAFIA plot for a four-rod RFQ with symmetric supports is shown in Fig. 25.
6. BEAM DYNAMICS

The beam is represented in six-dimensional phase space by two sets of 3-dimensional coordinates, \( p \) and \( q \). The coordinates are conjugate if they satisfy the conditions

\[
\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}
\]

where \( H = H(p,q,t) \) is a function called the hamiltonian. The coordinate \( p \) usually indicates the momentum of a particle, the coordinate \( q \) its position.

The conjugate coordinates of the beam particles occupy a volume in the phase space which is constant; this is Liouville’s theorem. The 6-D phase space is analysed through its two-dimensional phase plane projections. The projections containing a \( q \) and the corresponding \( p \) component are like ellipses (rigorously only for linear equations of motion) and the surface of the ellipse \( S \) divided by \( \pi \) is called normalised emittance. The normalised emittance ellipse is also a constant of motion provided the forces are linear and there is no coupling between the coordinates. Often the phase plane variables are not conjugate (as, e.g. the position \( x \) and the divergence \( x' \)) and the corresponding emittance ellipse is not constant; to normalise it, one has to multiply the \( x,x' \) emittance by \( \beta \gamma \). In beam dynamics one usually establishes equations of motion for the longitudinal and transverse phase planes; these equations are different for ions and electrons, see Sections 6.1 to 6.4.

6.1 Longitudinal beam dynamics in ion linacs

To simplify, the formulae are presented for protons; for ions, unless otherwise indicated, the relevant charge and mass numbers, \( Z \) and \( A \), have to be introduced. Note that in the case of ions, \( m \) represents the atomic mass unit, slightly different from the proton mass. The kinetic energy is quoted per nucleon, which is the total kinetic energy of the ion divided by \( A \); in this case, the formulae for ions differ from those for protons by a factor \( Z/A \).

Energy gain in an RF gap

Ion linacs are usually SW accelerators. The method of energy gain computation in such accelerators is shown by taking a drift tube linac (DTL) as example. The DTL operates in the \( TM_{010} \) cavity mode, and has field components \( E_z \), \( E_r \) and \( B_0 \); the field \( E_z \) is assumed symmetric around the median plane of the gap, see Fig.26. This symmetry is necessary for the present computation method, where one period (cell) of the structure is considered at a time.

![Fig.26 Cell of a drift tube linac, schematic.](image)

According to our assumption, the field \( E_z \) can be represented by a Fourier series (as done before) or in a more general way by a Fourier integral (leaving out the time dependence)
\[ E_z(r,z) = \int_{-\infty}^{\infty} A(k) I_0(k,r) \cos k z \cdot dk, \quad k_r = \left[ k^2 - \left( \frac{\omega}{c} \right)^2 \right]^{\frac{1}{2}}; \quad (6.2) \]

note that we wrote for simplicity \( k^2 \) instead of \( k_z^2 \); by inversion of the Fourier integral

\[ A(k) I_0(k,r) = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} E_z(r,z) \cos k z \cdot dz, \quad (6.3) \]

and using the approximation that in the gap \( E_z(a,z) = \text{const} = E \), \( a \) is the drift tube bore radius and \( g \) the gap length) one finds \( A(k) \):

\[ A(k) = \frac{E g}{2\pi} \sin \frac{kg}{2} \frac{1}{I_0(k,a)}; \quad (6.4) \]

and the field \( E_z \)

\[ E_z(r,z) = \frac{E g}{2\pi} \int_{-\infty}^{\infty} \sin \frac{kg}{2} \frac{1}{I_0(k,r)} I_0(k,a) \cos k z \cdot dk. \quad (6.5) \]

To compute the energy gain in the \( i \)-th cell of length \( l_i \), assume that \( r = \text{constant} \) and include the time factor \( \cos(\omega t + \phi) \), where \( \phi \) is the phase of the particle at mid gap, counted from the crest of the accelerating field:

\[ W_i - W_{i-1} = q \int_{-\frac{1}{2}}^{\frac{1}{2}} E_z(r,z,t) \cdot dz = q \int_{-\frac{1}{2}}^{\frac{1}{2}} E_z(r,z) \cos(\omega t + \phi) \cdot dz; \quad (6.6) \]

putting \( t = z/v_p \) (\( v_p \) is the particle velocity, considered constant in the first approximation)

\[ W_i - W_{i-1} = q \int_{-\frac{1}{2}}^{\frac{1}{2}} E_z(r,z) \left[ \cos \frac{\alpha z}{v_p} \cos \phi - \sin \frac{\alpha z}{v_p} \sin \phi \right] \cdot dz = q \cdot E g \frac{\sin kg}{2} \frac{1}{I_0(k,r)} I_0(k,a) \cos \phi, \]

with \( k = \omega/v_p \). The result has been obtained by recognising that the first term under the integral gives the inverse Fourier integral (multiplied by \( \cos \phi \)) and the second term gives zero (an even function \( E_z \) multiplied by the odd function \( \sin(\omega z / v_p) \)). Replacing \( E \cdot g \) by \( E_0 \cdot l_i \) \( (E_0 \) is the average longitudinal field on axis in a cell) and introducing the transit time factor on axis \( T(k) \)

\[ T(k) = \frac{1}{E_0 l_i} \int_{-\frac{1}{2}}^{\frac{1}{2}} E_z(0,z) \cos k z \cdot dz = \frac{\sin kg}{2} \frac{1}{I_0(k,a)}, \quad (6.7) \]

gives finally for the energy gain or energy change in cell \( i \):

\[ \partial W_i = W_i - W_{i-1} = q \cdot E_0 l_i \cdot T(k) I_0(k,r) \cdot \cos \phi. \quad (6.8) \]

The above formula is valid for a particle which crosses the mid gap at a radius \( r \) and with a phase \( \phi \) and velocity \( v_p \); for a particle on axis \( I_0(0) = 1 \). Note that the field in a gap is not exactly as assumed in the derivation. Nevertheless, the procedure is a very good approximation, provided that in the formula for the transit time factor \( T(k) \) one introduces for \( g \) the expression \( g = g_r + 0.85 \rho_c \), where \( g_r \) is the real gap length and \( \rho_c \) is the radius of curvature of the drift tube at the bore end. Often one uses also values of \( T(k) \) computed for the real field with programs like Superfish.
In general, the action of the electromagnetic field on a particle in a cell is treated as a “kick”
given at the mid-gap; this applies to all particle coordinates in phase space. The distance from
a mid-gap to the next is then considered as a simple drift. Hence, the “real motion” of a
particle in an accelerator is replaced by an “equivalent motion”, consisting of drift spaces and
mid-gap “kicks”.

**Phase change in an RF gap**

According to what was said above, the change of phase of a particle in a cell is computed by letting
the particle drift from mid gap $i-1$ to mid gap $i$, with a velocity corresponding to $W_{i-1}$, (output energy
of cell $i-1$); at mid gap $i$ one applies a correction or kick of

$$\delta \varphi = \frac{qE_0 l_i}{mc^2 \beta \gamma} k \frac{\partial}{\partial k} \left[T(k)I_0(k,r)\right]\sin \varphi,$$

and then the particle drifts again to mid gap $i+1$ with a velocity corresponding to $W_i$. The derivation
of the formula for phase correction $\delta \varphi$ is presented in Appendix 1.

Computing with the above mid-gap corrections $\delta W$ and $\delta \varphi$, the variables $W$ and $\varphi$ are canonically
conjugate and satisfy Liouville’s theorem to first order, as shown in Appendix 1.

There is, however, one snag: for $\beta$, $\gamma$, $k$, $k_r$ and $\varphi$ in the formulae, one must put in mid-gap values,
which are not known. To compute them, additional field integral factors have to be used, as e. g.

$$S(k) = \frac{2}{E_0 l_i} \int_0^{l/2} E_z(0,z) \sin kz \cdot dz;$$

combining $T(k)$ and $S(k)$ one can annul the field in the second half of the gap and thus compute the
mid-ap values for the energy and phase. Details are given in Appendix 1, where also a method is
described to take into account the effect of the slope of a trajectory at mid-gap. The mid-gap values
are computed only for the synchronous particle, wherefrom the values for other particles are derived
by series expansion.

A newer beam dynamics method exists which does not require the knowledge of the mid-gap energy
and phase nor asks for the mid-gap field symmetry. In addition, this method can treat several gaps at a
time. It is discussed in Appendix 2.

**Synchrotron equations in an ion linac**

The difference in energy gain in an accelerator cell, between a test particle on axis and the
synchronous one is

$$\Delta W = qE_0 T(k) l \cdot (\cos \varphi - \cos \varphi_s);$$

$\varphi$ and $\varphi_s$ are the respective RF phases at the moment the particles cross the mid-gap. Instead of
treating each cell separately, it is often interesting to consider the linac as a whole. To do so, a
continuous acceleration is assumed (it corresponds to an average accelerating force), and used to
take, for the variation of $\Delta W$ with $s$ (coordinate along the axis or orbit), the expression

$$\frac{d}{ds} \Delta W = qE_0 T \cdot [\cos(\varphi_s + \Delta \varphi) - \cos \varphi_s],$$

where $\Delta \varphi = \varphi - \varphi_s$, i. e. the difference in phase between the test particle and the synchronous one.
The variation of $\Delta \varphi$ with $s$ is
\[
\frac{\partial}{\partial s} \Delta \varphi = \omega \left( \frac{\partial}{\partial s} - \frac{1}{\beta_s} \frac{dt}{ds} \right) = \frac{\omega}{c} \left( \frac{1}{\beta} - \frac{1}{\beta_s} \right) \cong \frac{\omega}{\beta_s c} \frac{\Delta \beta}{\beta_s} = -\frac{\omega}{mc^3 \beta_s^3 \gamma^3_s} \Delta W. \tag{6.13}
\]

The coupled first order differential equations (6.12) and (6.13) are called *synchrotron equations*. The variables with a subscript \(s\) refer to the synchronous particle. For a constant frequency \(\omega\), \(\Delta W\) and \(\Delta \varphi\) are canonically conjugate variables.

Linearizing (6.12), i.e. putting \(\cos(\varphi_s + \Delta \varphi) - \cos \varphi = - \sin \varphi \Delta \varphi\), valid for \(\Delta \varphi \ll 1\), a set of linear equations is obtained

\[
\frac{d}{ds} \Delta W = b^2(s) \cdot \Delta \varphi, \quad \frac{d}{ds} \Delta \varphi = -a^2(s) \cdot \Delta W; \tag{6.14}
\]

\(a(s)\) and \(b(s)\) are defined by the linearized equation (6.12) and (6.13); they are slowly varying with \(s\). Eliminating \(ds\) and integrating, one gets the equation of an ellipse:

\[
\frac{(\Delta \varphi)^2}{a^2} + \frac{(\Delta W)^2}{b^2} = \text{const}, \tag{6.15}
\]

which represents the particle trajectory in the \(\Delta \varphi, \Delta W\) phase plane for small \(\Delta \varphi\), see Fig. 27c. The ellipse including all the particles in the \(\Delta \varphi, \Delta W\) phase plane is called *longitudinal emittance* \(\varepsilon_l(\Delta \varphi, \Delta W)\) and it is normalised. The axis ratio of this ellipse is

\[
\frac{\Delta \hat{W}}{\Delta \hat{\varphi}} = \frac{b}{a} = \sqrt{-\frac{mc^3 \beta_s^3 \gamma_s^3 qE_0 T \sin \varphi_s}{\omega}} \propto (\beta_s^3) \frac{3}{2}. \tag{6.16}
\]

From (6.14) a second-order differential equation is obtained

\[
\frac{d}{ds} \left( \beta_s^3 \gamma_s^3 \frac{d}{ds} \Delta \varphi \right) - \frac{\omega qE_0 T \sin \varphi_s}{mc^3} \Delta \varphi = 0. \tag{6.17}
\]

By multiplying by \(\beta_s^3 \gamma_s^3\) and substituting \(\beta_s^3 \gamma_s^3 \frac{d}{ds} = \frac{d}{du}\) one gets the equation

\[
\frac{d^2}{du^2} \Delta \varphi + K(u) \Delta \varphi = 0, \tag{6.18}
\]

where \(K(u)\) is slowly varying with \(u\). The above equation is familiar and can be solved by the usual BKW method, which assumes solutions of the form

\[
w(u) e^{j\varphi(u)}. \tag{6.19}
\]

The final solution in the \(s\) coordinate is:

\[
\Delta \varphi(s) = C \cdot \Delta \hat{\varphi}(s) \cdot \sin \left( \int_0^s k_s(s') ds' + \varphi_0 \right), \tag{6.20}
\]

where \(C\) and \(\varphi_0\) are integration constants, and \(k_s\) is the synchrotron phase advance per unit length. For small phase oscillations, \(k_s\) is given by

\[
k_s(s) = \frac{2\pi}{\Lambda_s(s)} = \left[ -\frac{\omega qE_0 T \sin \varphi_s}{mc^3 \beta_s^3 \gamma_s^3} \right]^{\frac{1}{2}} \propto (\beta_s \gamma_s^3) \frac{3}{2}, \tag{6.21}
\]

where the synchrotron wavelength \(\Lambda_s\) has been introduced. The amplitude of small phase oscillations is

\[
\Delta \hat{\varphi}(s) \propto \left[ -\frac{mc^3}{\omega qE_0 T \sin \varphi_s \beta_s^3 \gamma_s^3} \right]^{\frac{1}{2}} \propto (\beta_s \gamma_s^3) \frac{3}{2}. \tag{6.22}
\]
One sees that the phase excursion $\Delta\phi(s)$ of particles diminishes with acceleration. This effect is called *adiabatic phase damping*. The synchrotron phase advance $k_s$ diminishes also, while, on the contrary, the energy spread increases as $\Delta\hat{W} \propto (\beta_s \gamma_s)^{3/4}$.

From (6.12) and (6.13), without linearization, but assuming $\beta_s, \gamma_s \equiv \text{const}$, one can obtain a second order differential equation

$$\frac{d^2}{ds^2} \Delta \phi + \frac{\omega q E_0 T}{mc^3 \beta_s^3 \gamma_s^3} \left[ \cos(\phi_s + \Delta \phi) - \cos \phi_s \right] = 0; \quad (6.23)$$

considering the second term as a derivative, $dU / d\Delta \phi$, and multiplying all by $d\Delta \phi / ds$ one gets

$$\frac{d}{ds} \left[ \frac{1}{2} \left( \frac{d\Delta \phi}{ds} \right)^2 + U(\Delta \phi) \right] = 0. \quad (6.24)$$

The expression in the bracket is constant and represents the total energy of oscillation, $H$. The potential $U(\Delta \phi)$ has two points where $dU / d\Delta \phi = 0$; one is a minimum (stable point, for $\Delta \phi = 0$), and the other a maximum (unstable point, for $\Delta \phi = -2\phi_s$), see Fig.27.b. At both points, the velocity in the phase plane is zero, and therefore they are called fixed points. By using (6.13) and requiring that $U(0) = 0$, the total energy $H$ can be written as

$$H = \frac{\omega}{mc^3 \beta_s^3 \gamma_s^3} \left\{ \frac{\omega}{2mc^3 \beta_s^3 \gamma_s^3} (\Delta W)^2 + qE_0 T \left[ \sin(\phi_s + \Delta \phi) - \Delta \phi \cos \phi_s - \sin \phi_s \right] \right\} = H. \quad (6.25)$$

For each $H$, there is a trajectory in the $\Delta \phi, \Delta W$ phase plane. For $H = H_{\text{max}} = U(-2\phi_s)$ one gets the separatrix, dividing the stable from the unstable region:

$$\frac{\omega}{2mc^3 \beta_s^3 \gamma_s^3} (\Delta W)^2 + qE_0 T \left[ \sin(\phi_s + \Delta \phi) + \sin \phi_s - (2\phi_s + \Delta \phi) \cos \phi_s \right] = 0. \quad (6.26)$$

The separatrix extends from $\Delta \phi \geq -2\phi_s$ to $\Delta \phi = +\phi_s$, hence over $3|\phi_s|$. The separatrix and other trajectories in the $\Delta \phi, \Delta W$ phase plane are presented in Fig.27.c. The area inside the separatrix is called the bucket. For $\Delta \phi = 0$, one has the maximum allowed $\Delta W$ on the separatrix:

$$\Delta \hat{W}_{\text{max}} = \pm \frac{\sqrt{\frac{qmc^3 \beta_s^3 \gamma_s^3 E_0 T (\phi_s \cos \phi_s - \sin \phi_s)}{\omega}}}{} \quad (6.27)$$

Fig.27 a) $E_z$ field in an RF gap; b) potential well of synchrotron oscillations; c) trajectories in the $\Delta \phi, \Delta W$ phase plane with the separatrix.
When deriving the formulae for the bucket (separatrix), $\beta_s$ and $\gamma_s \equiv \text{const}$ have been assumed and, in this way, a stationary bucket has been obtained. Had the energy variation in a linac been taken into account, the bucket would have opened up in the form of a golf club, a form particularly pronounced at low energies, as at injection, where the energy gain per cell corresponds to several percent of the total kinetic energy; see Fig.28.

![Fig.28 Dynamic separatrix of the form of a "golf club" at low energies.](image)

### 6.2 Longitudinal beam dynamics in electron linacs

Electron accelerators are usually TW, and it is common practice to count the phase from the zero crossing of the wave, and not from its crest. The set of first order synchrotron equations are written in the form

$$\frac{d\gamma}{ds} = -\frac{eE_0}{mc^2} \sin \varphi, \quad \frac{d\varphi}{ds} = \frac{\omega}{c} \left[ \frac{1}{\beta^\text{ph}} \frac{1}{\beta} \right] = \frac{\omega}{c} \left[ \frac{1}{\beta^\text{ph}} - \frac{\gamma}{(\gamma^2 - 1)^{1/2}} \right]; \quad (6.28)$$

note that $\beta^\text{ph} c$ is the phase velocity of the wave, $\beta c$ the electron velocity and $E_0$ the relevant space harmonic. Eliminating $ds$ as before

$$\frac{\omega}{c} \left[ \frac{1}{\beta^\text{ph}} - \frac{\gamma}{(\gamma^2 - 1)^{1/2}} \right] d\gamma = -\frac{eE_0}{mc^2} \sin \varphi \cdot d\varphi; \quad (6.29)$$

the extremum of phase $\varphi$ ($\varphi_{\text{max}}$) is obtained by putting

$$\frac{d\varphi}{d\gamma} = 0 \quad \Rightarrow \quad \frac{1}{\beta^\text{ph}} = \frac{\gamma}{(\gamma^2 - 1)^{1/2}} = \frac{1}{\beta}. \quad (6.30)$$

Integrating (6.29)

$$\cos \varphi = \frac{\omega \beta}{eE_0} \left( \frac{\gamma}{\beta^\text{ph}} - \frac{\beta^\gamma}{\beta} \right) + C \quad \text{and} \quad C = \cos \varphi_{\text{max}} - \frac{\omega \beta}{eE_0} \left( \frac{1 - \beta^2}{\beta^\text{ph}} \right)^{1/2} \quad (6.31)$$

one gets finally:

$$\cos \varphi - \cos \varphi_{\text{max}} = \frac{\omega \beta}{eE_0} \beta^\text{ph} \left( \gamma (1 - \beta\beta^\text{ph}) - (1 - \beta^2)^{1/2} \right). \quad (6.32)$$

For $\nu^\text{ph} < c$, one gets from (6.32) the Fig.29; the curves are centred around $\varphi_s = 0$, where there is no acceleration. For $\nu^\text{ph} = c$, equation (6.32) becomes

$$\cos \varphi - \cos \varphi_{\text{max}} = \frac{\omega \beta}{eE_0} \gamma (1 - \beta). \quad (6.33)$$
Electrons at first lag behind the wave, but as they are accelerated and approach $c$, they also approach $\varphi_{\text{max}}$ asymptotically; the best is to have $\varphi_{\text{max}} = -\pi/2$, and if electrons are injected at $\varphi = 0$ with an energy $\gamma_i$, the amplitude of the accelerating wave must be

$$E_0 = \frac{\omega mc}{\gamma_i (1 - \beta_i)} ,$$

(6.34)
in order the electrons remain trapped by the crest of the wave, see Fig.30. To have a reasonable energy spread at the output, the electrons should have a small phase spread around $\varphi_{\text{max}} = -\pi/2$.

![Fig.29 Trajectories in the longitudinal phase plane; $W = (\gamma - 1)mc^2$.](image)

![Fig.30 Trajectories of ultra-relativistic electrons in a linac with $v_{ph} = c$.](image)

### 6.3 Transverse beam dynamics in ion linacs

The relevant field components, $E_r$ and $B_{\theta}$, for the transverse beam dynamics in a TM$_{010}$ standing wave accelerator are obtained from the longitudinal field $E_z$ (see formulae (6.2) to (6.7))

$$E_z(r, z, t) = \frac{E_0 l}{2\pi} \int T(k) I_0(k, r) \cos kz \cdot \cos(\omega t + \varphi) \cdot dk$$

(6.35)

by applying Maxwell’s equation

$$\text{div}\vec{E} = \frac{1}{r} \frac{\partial}{\partial r}(r E_r) + \frac{\partial E_z}{\partial z} = 0 \quad \rightarrow \quad E_r(r, z) = -\frac{1}{r} \int \frac{\partial E_z}{\partial z} r \frac{\partial}{\partial r} ,$$

(6.36)

and making use of a property of Bessel functions $\int x I_0(x) dx = x I_1(x)$:

$$E_r(r, z, t) = \frac{E_0 l}{2\pi} \int_{-\infty}^{\infty} T(k) \frac{k}{k_r} I_1(k, r) \sin kz \cdot \cos(\omega t + \varphi) \cdot dk .$$

(6.37)
Note that the longitudinal stability of motion requires \( \frac{\partial E_z}{\partial z} < 0 \) (longitudinal focusing), hence \( E_z > 0 \), i.e., there is a transverse defocusing. For \( \frac{\partial E_z}{\partial z} = \text{const} \), \( E_z \) is a linear function of \( r \):

\[
E_z(r, z) = -\frac{\text{const}}{2} r \quad \text{or} \quad \frac{\partial E_z}{\partial r} = -\frac{1}{2} \frac{\partial E_z}{\partial z}.
\]

With linearized forces, the transverse defocusing gradient is half of the longitudinal focusing gradient.

To compute \( B_\varphi \) one uses

\[
\text{rot}\vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \rightarrow \frac{\partial B_\varphi}{\partial z} = -\frac{1}{c^2} \frac{\partial E_r}{\partial t},
\]

and obtains in a way analogous to \( E_z(r, z, t) \)

\[
B_\varphi(r, z, t) = \frac{E_0 l_0 \omega}{2\pi c^2} \int T(k) \frac{1}{k_r} I_1(k, r) \cos kz \cdot \sin(\omega t + \varphi) \cdot dk.
\]

The transverse force equation is

\[
\frac{d(\gamma mv_r)}{dt} = q(E_r - v_z B_\varphi).
\]

To compute the change of the radial momentum \( \gamma mv_r \) of a particle in an RF gap, one integrates the action of the fields \( E_r \) and \( B_\varphi \) over a cell, assuming \( v_z = v_p = \text{constant} \)

\[
\int_{z/2}^{z/2} E_r(r, z) \cos(\omega t + \varphi) \frac{dz}{v_p} - \frac{1}{v_p} E_0 l T(k) I_1(k, r) \frac{k}{k_r} \sin \varphi, \quad \text{and}
\]

\[
\int_{z/2}^{z/2} B_\varphi(r, z) \sin(\omega t + \varphi) \frac{dz}{v_p} = -\frac{1}{v_p} E_0 l T(k) I_1(k, r) \frac{\omega}{k_r} \sin \varphi;
\]

the right hand expressions were obtained by inversion of Fourier integrals. Finally:

\[
\delta(\gamma mv_r) = -\frac{1}{v_z} qE_0 l T(k) I_1(k, r) \frac{k}{k_r} \sin \varphi.
\]

To find the change of \( \gamma \), use

\[
\frac{\delta(\gamma mv_r)}{\gamma} = \delta \gamma + \frac{1}{\gamma} \frac{\delta \gamma}{\gamma} \rightarrow \delta \gamma' \approx \frac{\delta(\gamma mv_r)}{\gamma},
\]

In a gap, \( \gamma \) does not change much, making it possible to approximate

\[
\frac{\delta \gamma}{\gamma} \approx \frac{\delta v_z}{v_z} = \frac{\delta W}{2W},
\]

and get finally:

\[
\delta \gamma' = -\frac{qE_0 l}{2W} \left[ T(k) I_1(k, r) \frac{k}{k_r} \sin \varphi + T(k) I_0(k, r) \gamma' \cos \varphi \right].
\]

In the longitudinal dynamics, one has corrected the phase \( \varphi \) at mid-gap; similarly, in the transverse dynamics, one has to correct \( r \) at mid-gap, to be Liouvillean to first order:

\[
\delta r = -\frac{qE_0 l}{2W} \left( \frac{\partial}{\partial k} \left[ T(k) I_1(k, r) \frac{k}{k_r} \right] \cos \varphi + \frac{\partial}{\partial k} \left[ T(k) I_0(k, r) \right] \gamma' \sin \varphi \right).
\]
A discussion of radial correction terms is presented in Appendix 1. Note that the variables \( r, r' \) are not conjugate; conjugate to \( r \) is \( \gamma m v_r \).

**New variables for the transverse dynamics**

Introducing the *reduced radius* \( R \) defined by

\[
R = r \sqrt{\beta' \gamma} .
\]  

(6.49)

one obtains for the paraxial motion \( (k r r' \ll 1) \) the equation:

\[
\frac{d^2 R}{ds^2} = \left[ \frac{q}{2mc^3 \beta^3 \gamma^2} \frac{\partial E_z}{\partial \beta'} - \frac{q^2 \gamma^2 + 2}{(2mc^2)^2} (\beta' \gamma')^2 E_z \right] R .
\]  

(6.50)

The variables \( R, R' = dR/ds \) satisfy Liouville’s theorem. This can be deduced from the above equation, which does not contain \( dR/ds \). The first term in the bracket is the usual defocusing term, while the second one is always focusing and is called the “electron lens”, because it is large for electrons. For ions, the first term is largely predominant. Rewriting the second order differential equation in rectangular coordinates, one obtains two similar equations in *reduced coordinates* \( X \) and \( Y \). A discussion of the reduced variables and the derivation of various equations is presented in Appendix 2.

**External alternating gradient (AG) focusing in ion linacs; smooth variables**

Due to a non-zero transverse beam emittance and RF defocusing in accelerators, an external transverse beam focusing is required. The focusing forces are usually periodic and alternating in sign (AG focusing). We are often more interested in the *average focusing effect*, and only then in the additional wiggle caused by the AG focusing system. A function \( x(s) \) can be represented as

\[
x(s) = X(s) \left[ 1 + \varepsilon(s) \right],
\]  

(6.51)

where \( X(s) \) is the smooth, slowly (harmonically) varying part of a transverse beam coordinate \( x(s) \), and \( \varepsilon(s) \) is the wiggle. Note that in the longitudinal phase plane, the smooth synchrotron phase advance \( k_s(s) \) has already been calculated for an average longitudinal focusing force, see Section 6.1.

When dealing with linearized forces, one can define in the transverse phase plane \( \sigma_\beta \), the *smooth phase advance of betatron oscillations per focusing period* \( N\beta, \lambda \)

\[
\sigma_\beta = k_\beta N\beta, \lambda = 2\pi \frac{N\beta, \lambda}{\lambda_\beta} = \frac{B}{2\sqrt{2\pi}},
\]  

(6.52)

where \( k_\beta \) is the smooth *betatron phase advance per unit length*, \( N \) is the number of \( \beta, \lambda \) in the focusing period, \( \lambda_\beta \) is the betatron wavelength, and \( B \) is a focusing parameter given by

\[
B = \frac{\lambda^2}{c} \frac{q}{\gamma \gamma'} N^2 \chi \beta_s \cdot G ;
\]  

(6.53)

the meaning of the factors in the formula are:

\[ G \rightarrow \text{magnetic quadrupole gradient, [T/m]} \; ; \]

\[ \chi \rightarrow \text{normalised amplitude of the fundamental Fourier harmonic (the “stepwise” AG focusing is approximated by a Fourier series)} \; ; \]
for +− (N=2) focusing \[ \chi = \frac{4}{\pi} \sin\left(\frac{\pi}{2} \Gamma \right), \]
for ++−− (N=4) focusing \[ \chi = \frac{8}{\sqrt{2}\pi} \sin\left(\frac{\pi}{4} \Gamma \right), \]
for +++−−− (N=6) focusing \[ \chi = \frac{8}{\pi} \sin\left(\frac{\pi}{6} \Gamma \right); \]
\( \Gamma \) is the quadrupole filling factor (quadrupole length relative to cell length, \( l_q / \beta_s \lambda_s \)).

The derivation and discussion of smooth variables is presented in Appendix 3. Note that one can express \( k_\beta \) and \( A_\beta \) as
\[
 k_\beta = \frac{B}{2\sqrt{2}\pi N\beta_s \lambda_s} = \frac{q\lambda N \chi G}{2\sqrt{2}\pi mc \gamma_s'}, \tag{6.54}
\]
\[
 A_\beta = \frac{2\pi}{k_\beta} = \frac{4\sqrt{2}\pi^2 N\beta_s \lambda_s}{4\sqrt{2}\pi^2 mc \gamma_s'} = \frac{q\lambda N \chi G}{\lambda q N \chi G}. \tag{6.55}
\]
When the RF defocusing in the accelerator is included (the forces are linearized), the expression for the smooth transverse phase advance per focusing period becomes
\[
 \sigma_{0T} = \sqrt{\sigma_\beta^2 - \frac{1}{2} \sigma_{0L}^2} = \sqrt{\frac{B^2}{8\pi^2} - \frac{1}{2} \sigma_{0L}^2}; \tag{6.56}
\]
\( \sigma_{0L} \) is the smooth phase advance of synchrotron oscillations per transverse focusing period; as seen in Section 6.3, the transverse defocusing force gradient of the linearized RF field equals half of the longitudinal focusing force gradient. The \( \sigma_{0L} \) is given by
\[
 \sigma_{0L}^2 = -\frac{2\pi q N^2 \lambda E_0 T(k) \sin \phi_s}{mc^2 \beta_s \gamma_s'}, \tag{6.57}
\]
see formula (6.21); the meaning of the symbols is the usual one.

**Mathieu equation: parametric resonances**

From the formula of \( \sigma_{0L}^2 \) one sees that the RF defocusing depends on the phase angle \( \phi_s \). When particles perform phase oscillations, the value of \( \sigma_{0L} \) changes and so does \( \sigma_{0T} \). Betatron and synchrotron oscillations are parametrically linked, so parametric resonances must exist. To find them, consider the betatron phase advance of a test particle which performs phase oscillations around the synchronous particle with phase \( \phi_s \):
\[
 \sigma_{0T,s}^2 = \sigma_\beta^2 - \frac{1}{2} \sigma_{0L}^2 \frac{\sin(\phi_s + \Delta \phi)}{\sin \phi_s}; \tag{6.58}
\]
for small \( \Delta \phi \) one can write:
\[
 \sigma_{0T,s}^2 = \sigma_\beta^2 - \frac{1}{2} \sigma_{0L}^2 - \frac{1}{2} \sigma_{0L}^2 \cdot \cot \phi_s \cdot \Delta \phi = \sigma_{0T}^2 - \frac{1}{2} \sigma_{0L}^2 \cdot \cot \phi_s \cdot \Delta \phi \tag{6.59}
\]
\[ \Delta \phi = \Delta \phi' \cdot \sin(k_s \lambda_s). \]

The Mathieu-Hill differential equation for the smooth variable \( X(s) \) (smooth part of the transverse variable \( x(s) = X(s) [1 + \varepsilon (s)] \)), and with \( \eta = \frac{s}{N\beta_s \lambda_s} \) as independent variable is
\[
 \frac{d^2 X}{d\eta^2} + \left[ \sigma_{0T}^2 - \frac{1}{2} \sigma_{0L}^2 \Delta \phi' \cdot \cot \phi_s \cdot \sin(\sigma_{0L} \eta) \right] X = 0; \tag{6.60}
\]
introducing a new variable $\sigma_{0L} \eta = 2\pi \zeta$, the Mathieu equation is obtained in canonic form

$$\frac{d^2 X}{d\zeta^2} + \pi^2 \left[\frac{4\sigma_{0L}^2}{\sigma_{0L}^2} - 2\frac{\Delta \varphi \cdot \text{ctg} \varphi}{b} \cdot \sin 2\pi \zeta\right] X = 0 \quad \text{or}$$

$$\frac{d^2 X}{d\zeta^2} + \pi^2 [a - b \cdot \sin 2\pi \zeta] X = 0. \quad (6.61)$$

Parametric resonances occur for $a = n^2$, with $n = 1, 2, ...$

$$\sigma_{0T} = \frac{n}{2} \sigma_{0L} \quad (6.62)$$

The strongest resonance is for $n = 1$, i. e.

$$\sigma_{0T} = \frac{1}{2} \sigma_{0L} \quad (6.63)$$

and it must be avoided. Other resonances are usually unimportant.

### 6.4 Transverse beam dynamics in electron linacs

In electron linacs, the relativistic factor $\beta$ approaches unity very quickly and $\gamma$ takes on large values. Due to this, the transverse beam dynamics in electron linacs has the particularity that the accelerating field and the longitudinal motion have practically no influence on the transverse one. Apart from the case of very strong space charge forces, where a phenomenon called beam break up can occur (excitation of spurious transverse modes in cavities), the design of the transverse focusing requires only the knowledge of the beam energy and emittance. However, for very low energies, the transverse dynamics in electron linacs can be treated in a similar way to that in ion linacs.

### 6.5 Beam dynamics in RFQs

The electrodes of an RFQ (vanes or rods) form a well-defined boundary along the beam axis. Due to symmetry, the magnetic field on axis is zero, and nearly zero in the close neighbourhood. Therefore, in this region the wave equation can be replaced by the simpler Laplace equation in cylindrical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2} = 0, \quad (6.64)$$

with $U(r, \varphi, z)$ being the electric field potential. A general solution of the Laplace equation, where RFQ symmetries have been taken into account is

$$U(r, \varphi, z) = \frac{V}{2} \left[ \sum_n A_{0n} r^{2n} \cos 2n \varphi + \sum_n \sum_l A_{ln} I_{2n}(lkr) \cos 2n \varphi \cos klz \right], \quad \text{with} \quad (6.65)$$

\begin{align*}
l+n &= 2p + 1, \quad p=0,1,2,... \quad , \\
\pm V/2 \quad &\ldots \quad \text{electrode potential with respect to axis}, \\
I_{2n} \quad &\ldots \quad \text{modified Bessel function of order } 2n, \\
k &= 2\pi/\beta \lambda.
\end{align*}

The general solution contains all harmonics in infinite series; usually a few harmonics suffice. The lowest order solution has only two terms, one out of each infinite series:
The constants $A_{01}$ and $A_{10}$ are determined by taking the RFQ symmetries into account, see Fig. 31:

$$A_{10} = \frac{m^*}{m^*I_0(ka) + I_0(m^*ka)} \quad \text{...... acceleration parameter} \quad (6.67)$$

$$A_{01} = \frac{1}{a^2} \left[ 1 - A_{10} I_0(ka) \right] = \frac{\gamma}{a^2} \quad \text{...... focusing parameter} \quad (6.68)$$

The meaning of $m^*$, the vane modulation factor, is also evident from Fig. 31: it indicates the ratio between maximum and minimum aperture radii.

The field components from the lowest order potential function are:

$$E_r = -\frac{\partial U}{\partial r} = -\frac{V}{2} \left[ 2A_{01} r \cos 2\theta + kA_{10} I_1(kr) \cos kz \right], \quad (6.69)$$

$$E_\theta = -\frac{1}{r} \frac{\partial U}{\partial \theta} = VA_{01} r \sin 2\theta, \quad (6.70)$$

$$E_z = -\frac{\partial U}{\partial z} = \frac{V}{2} kA_{10} I_0(kr) \sin kz. \quad (6.71)$$

To produce the above fields, the surface of electrode tips must satisfy the equation:

$$S(r, \phi, z) = A_{01} r^2 \cos 2\phi + A_{10} I_0(kr) \cos kz = \pm 1. \quad (6.72)$$

---

**Fig. 31** Modulated electrodes of an RFQ (schematic).

**Longitudinal dynamics**

The energy gain on axis in cell $i$ is obtained by including the time factor $\sin(\omega t + \phi)$ into $E_z$ (note that contrary to Alvarez linacs, the phase $\phi$ refers in RFQs to the input of a cell, and is counted from the zero-crossing of the accelerating field and not from its crest) and integrating over the cell length, which is $\beta \Delta t/2$, typical for accelerators operating in $\pi$ mode; the result is

$$W_i - W_{i-1} = \left( \frac{\pi}{4} qA_{10} V \cos \phi \right). \quad (6.73)$$

Proceeding analogously to section 6.1, but putting $\gamma = 1$ (the RFQ works at low energies), one finds for the amplitude and phase advance of small synchrotron oscillations...
\[
\Delta \hat{\phi}(s) \propto \left[ \frac{-mc^2}{\pi^2 qA_{10}V\beta_s^2 \sin \varphi_s} \right]^{\frac{1}{2}} \quad \text{and} \quad (6.74)
\]
\[
k_s(s) = \frac{2\pi}{\Lambda_s} = \left[ \frac{-\pi^2 qA_{10}V \sin \varphi_s}{mc^2 \lambda^2 \beta_s^4} \right]^{\frac{1}{2}} \quad ; \quad (6.75)
\]

The variation of the above formulae with energy (non relativistic) is
\[
\Delta \hat{\phi} \propto \beta^{-\frac{1}{2}} \quad ; \quad k_s \propto \beta^{-2} \quad \text{and} \quad \frac{\Delta W}{\Delta \phi} \propto \beta \quad . \quad (6.76)
\]

The last expression has been obtained by considering the longitudinal emittance an invariant of motion.

**Transverse dynamics**

For the transverse dynamics one uses \( E_x \) and \( E_y \), rather than \( E_r \) and \( E_\phi \). From the lowest order potential function \( U \) one obtains
\[
E_x = -\frac{\partial U}{\partial x} \quad ; \quad (6.77)
\]
linearizing and simplifying one obtains the equation of motion close to the axis
\[
\frac{d^2 x}{ds^2} + \left[ B^* \sin ks + \Delta_{sf}^* \right] \cdot x = 0 \quad , \quad (6.78)
\]
with
\[
B^* = \frac{q\lambda V}{mc^2 \beta_s^2 a^2} \quad \text{and} \quad \Delta_{sf}^* = \frac{q\pi^2 qA_{10}V \sin \varphi_s}{2mc^2 \lambda^2 \beta_s^4} \quad ; \quad (6.79)
\]
\( B^* \) and \( \Delta_{sf}^* \) have the dimension \([m^{-2}]\). Multiplying them with \( \beta_s^2 \lambda^2 \), one obtains \( B \) and \( \Delta_{sf} \) which are dimensionless and appear in the equation of motion with respect to the independent variable \( \eta = \frac{s}{\beta_s \lambda} \) :
\[
\frac{d^2 x}{d\eta^2} + \left[ B \sin ks + \Delta_{sf} \right] \cdot x = 0 \quad . \quad (6.80)
\]

The formula for the smooth betatron phase advance per transverse focusing period \( \beta_0 \lambda \) is
\[
\sigma_{ot}^2 = \frac{B^2}{8\pi^2} + \Delta_{sf} \quad . \quad (6.81)
\]
Note that \( \Delta_{sf} = -\frac{1}{2} \sigma_{ot}^2 \), as already mentioned before. Note also that
\[
k_s(s) \beta_s \lambda = \sigma_{ot} \quad . \quad (6.82)
\]

The beam dynamics formulae for the RFQ are derived in Appendix 4.
7. HIGH INTENSITY EFFECTS

When the intensity of the beam increases, one is confronted with two distinct effects in a linac: the repulsion of particles being close together in a bunch (space charge effect) and the excitation of fields on the walls of the cavity (image effects for non relativistic beams and wake fields and beam break up for relativistic beams, in particular in electron linacs).

7.1 Space charge: Linear effects

The importance of space charge effects on the beam dynamics decreases \( \propto (\beta^{-2} \gamma^{-3}) \) and is therefore of less importance for electrons, which become relativistic very rapidly. On the contrary, space charge in intense ion accelerators is of such an importance, that it cannot be treated as a perturbation, but must be included in the design process already. The following concentrates therefore on ions, and to simplify the writing, derives formulae for protons as already done in Chapters 6.1 and 6.3.

To determine the parameters of a proton linear accelerator, space charge forces are at first linearized and rms values of particle coordinates are considered. The importance of rms values lies in the fact that their behaviour under space charge conditions is practically independent of the detailed shape of space charge forces, and depends rather on their linearized form. This is valid for all “ellipsoidal distributions” (equidensity “surfaces” are concentric ellipsoids), and under the condition that the rms beam emittance is a constant of motion (Lapostolle, Sacherer). Hence, in the accelerator design process, the “real beam” can be replaced by an “equivalent beam”, having the same rms values, but being of uniform space charge density distribution and having thus linear self forces. The importance of non-linear effects is later checked by beam simulation programs.

Rms values and space charge distributions

The rms emittance in a phase plane is defined by

\[
\varepsilon_{\text{rms}} = \sqrt{\langle x^2 \rangle - \langle xx' \rangle^2},
\]

and analogous expressions apply for \( \varepsilon_{\text{rms}}(y,y') \) and \( \varepsilon_{\text{rms}}(z,z') \). The quadratic mean values \( \langle x^2 \rangle \) and \( \langle xx' \rangle \) are calculated as

\[
\langle x^2 \rangle = \int (x - \bar{x})^2 f(x,x',y,y',z,z') dx dx' dy dy' dz dz',
\]

and analogously for \( \langle xx' \rangle \); \( \bar{x} \) is the mean value \( \langle x \rangle \) and \( f(x,x',y,y',z,z') \) is the density distribution function, defined here in a 6-dimensional \( (n=6) \) phase space. A distribution of special historical importance is the non-realistic Kapchinskij-Vladimirskij distribution (K-V), which is a “boundary distribution”, where particles are placed uniformly on the boundary of an \( n \)-dimensional volume. Originally the K-V distribution was specified with \( n=4 \) \( (x,x',y,y') \), giving a uniform distribution when projected on a 2-dimensional plane \( (x,y) \), and hence linear space charge forces. Between boundary values of coordinates (envelopes, as e.g. \( e_x \)) and their rms values (rms envelopes, as e.g. \( \tilde{e}_x = \sqrt{\varepsilon_x} \)) exist relations, which depend on the type of the density distribution and the number of dimensions \( n \) in which this distribution is specified. Corresponding relations exist also between total and rms emittances (as e.g. \( \varepsilon_x \) and \( \varepsilon_{\text{rms}} \)):

- **K-V** distribution \( \rightarrow \) \( e_x = \sqrt{n} \cdot \tilde{e}_x \); \( e_x = n e_{\text{rms}} \)
- **uniform** \( \rightarrow \) \( e_x = \sqrt{n+2} \cdot \tilde{e}_x \); \( e_x = (n+2) e_{\text{rms}} \)
- **parabolic** \( \rightarrow \) \( e_x = \sqrt{n+4} \cdot \tilde{e}_x \); \( e_x = (n+4) e_{\text{rms}} \).
Projecting an \( n \)-dimensional distribution into an \((n-2)\) dimensional space results in another type of the distribution, as already mentioned (e.g. a K-V changes into uniform, or a uniform into parabolic). Only Gaussian distributions have the property of remaining Gaussian in all their projections. However, this statement does not apply exactly to truncated Gaussian distributions (Gaussian distributions extend to infinity, and to represent a beam they have to be truncated, usually at about three rms values or standard deviations).

**Envelope equations; matched rms envelope equations**

The solution of Hill’s equation with a periodic coefficient \( K_x(s) = K_x(s + L) \),

\[
x''(s) + K_x(s)x = 0,
\]

(7.2)
is of the form \( x(s) = w(s) \exp(i\psi(s)) \); separating the real and imaginary part of the solution one gets two equations

\[
\psi''(s) = \frac{1}{w^2} \quad \text{and} \quad w''(s) + K_x(s)w - \frac{1}{w^3} = 0.
\]

(7.3)

(7.4)
The second equation is an envelope equation; multiplying it by \( \sqrt{\epsilon_x} \) (\( \epsilon_x \) is the beam emittance), and putting \( w = \sqrt{\beta_x} \) (\( \beta_x \) is the Twiss parameter), one obtains the zero space charge envelope equation

\[
r_x'' + K_x(s) \cdot r_x - \frac{\epsilon_x^2}{r_x} = 0, \quad \text{with} \quad r_x = \sqrt{\beta_x \epsilon_x};
\]

(7.5)
in an analogous way one obtains envelope equations for \( r_y \) and \( r_z \).

In the presence of space charge, the solution of the equation of motion becomes more complicated, even for linear space charge forces, because one has to know the beam envelopes. Kapchinskij and Vladimirskij have derived their famous envelope equations for continuous, unbunched beams (surface distribution in 4-D, uniform distribution in 2-D):

\[
\frac{d^2 r_x}{ds^2} + K_x(s) r_x - \frac{\epsilon_x^2}{r_x^3} - \frac{qI}{\pi \epsilon_0 m r^3 \sqrt{r_x + r_y}} = 0
\]

(7.6)

\[
\frac{d^2 r_y}{ds^2} + K_y(s) r_y - \frac{\epsilon_y^2}{r_y^3} - \frac{qI}{\pi \epsilon_0 m r^3 \sqrt{r_x + r_y}} = 0.
\]

(7.7)

In addition to the emittance term, a space charge term also appears in the equations.

In analysing space charge problems, we are often more interested in smooth beam envelopes, not containing the wiggle of the alternating gradient focusing. The important smooth envelope equations for a bunched beam with a uniform space charge distribution in three dimensions are

\[
\frac{d^2 \epsilon_x}{ds^2} + \bar{k} \epsilon_x - \frac{\epsilon_x^2}{\epsilon_x^3} - \frac{k_x I}{\epsilon_x} \cdot \left[ 1 - f \left( \frac{e_x}{\epsilon_x} \right) \right] = 0,
\]

(7.8)

\[
\frac{d^2 \epsilon_z}{ds^2} + \bar{k} \epsilon_z - \frac{\epsilon_z^2}{\epsilon_z^3} - \frac{k_z I}{3} \cdot f \left( \frac{e_z}{\epsilon_z} \right) = 0,
\]

(7.9)
with $\bar{k}_x^2$ and $\bar{k}_z^2$ representing the smooth focusing in the $x$ and $z$ direction, respectively and $k_{sc}$ being a space charge factor defined as

$$k_{sc} = \frac{q \beta \gamma}{4 \pi e_0 m c^3 \beta_x \gamma_x^2}; \quad (7.10)$$

$I$ is the beam current, $s$ is the independent longitudinal variable, and $f\left(\frac{e_z}{e_x}\right)$ is a function called the form factor of the ellipsoidal beam bunch, see Fig.32. In the equations (7.8) and (7.9) one has assumed $k_x = k_y$, $e_x = e_y$ and $\varepsilon_x = \varepsilon_y$.

From Fig.32 one sees that $f\left(\frac{e_z}{e_x}\right) \approx \frac{1}{5} \frac{e_x}{e_z}$, in the range $0.2 < \frac{e_x}{e_z} < 1.25$.

To obtain the matched smooth beam envelopes, one solves the envelope equations, where one has put the second derivatives $e_x'' = e_z'' = 0$:

$$\bar{k}_x^2 e_x - \frac{e_x^2}{e_x^3} - k_{sc} I g\left(\frac{e_z}{e_x}\right) = 0,$$

$$\bar{k}_z^2 e_z - \frac{e_z^2}{e_z^3} - k_{sc} I = 0.$$  \hspace{1cm} (7.11)

The above equations are valid only for uniform density (in real space) beams. By introducing rms envelopes and rms emittances, the equations become more generally valid (provided the distributions are of the ellipsoidal type) and many ambiguities are avoided. The matched rms envelope equations with space charge are

$$\frac{-k_x^2 \tilde{x}}{\tilde{x}^3} - \frac{\varepsilon_{rms}^2}{\tilde{x}^3} - \frac{1}{5\sqrt{5}} \frac{k_{sc} I}{\tilde{x} z} g\left(\frac{\tilde{z}}{\tilde{x}}\right) = 0,$$

$$\frac{-k_z^2 \tilde{z}}{\tilde{z}^3} - \frac{\varepsilon_{rms}^2}{\tilde{z}^3} - \frac{1}{5\sqrt{5}} \frac{k_{sc} I}{\tilde{x} z} = 0;$$  \hspace{1cm} (7.13)

$$\frac{-k_x^2 \tilde{x}}{\tilde{x}^3} - \frac{\varepsilon_{rms}^2}{\tilde{x}^3} - \frac{1}{5\sqrt{5}} \frac{k_{sc} I}{\tilde{x} z} g\left(\frac{\tilde{z}}{\tilde{x}}\right) = 0,$$

$$\frac{-k_z^2 \tilde{z}}{\tilde{z}^3} - \frac{\varepsilon_{rms}^2}{\tilde{z}^3} - \frac{1}{5\sqrt{5}} \frac{k_{sc} I}{\tilde{x} z} = 0;$$  \hspace{1cm} (7.14)

the factor $\frac{1}{5\sqrt{5}}$ comes from the fact that the total current $I$ is maintained in the formulae, instead of taking only the part linked with rms values. If the third term on the left hand side of the equations exceeds the second one, it is said that the beam is space charge dominated.

The matched rms envelopes without space charge are
\[ \tilde{x}_0 = \sqrt{\frac{\varepsilon_{x_{\text{rms}}}}{k_x}} \quad \text{and} \quad \tilde{z}_0 = \sqrt{\frac{\varepsilon_{z_{\text{rms}}}}{k_z}} ; \quad (7.15) \]

using these zero-current matched envelopes, one can **normalize** the matched rms envelope equations with space charge and get

\[ \frac{\tilde{x}}{x_0} - \frac{1}{(\frac{\tilde{x}}{x_0})^3} - \frac{\delta_x}{x_0} = 0 \quad \text{and} \quad (7.16) \]

\[ \frac{\tilde{z}}{z_0} - \frac{1}{(\frac{\tilde{z}}{z_0})^3} - \frac{\delta_z}{z_0} = 0 ; \quad (7.17) \]

\( \delta_x \) and \( \delta_z \) are **space charge factors** defined by

\[ \delta_x = \frac{q l}{10\sqrt{5} \varepsilon_0 mc^3 \beta_y \gamma_y \cdot \bar{k}_x \varepsilon_{x_{\text{rms}}} \cdot \Delta \tilde{\phi}} \cdot \left( \frac{z_0}{x_0} \right) \quad \text{and} \quad (7.18) \]

\[ \delta_z = \frac{q l}{10\sqrt{5} \varepsilon_0 mc^3 \beta_y \gamma_y \cdot \bar{k}_z \varepsilon_{z_{\text{rms}}} \cdot \Delta \tilde{\phi}} \cdot \left( \frac{z_0}{x_0} \right), \quad (7.19) \]

where \( \Delta \tilde{\phi} \) is the extension in phase corresponding to \( \tilde{z} \). In the above relations, the fact that the axis ratio of the beam bunch remains about the same with and without space charge, has been used. The normalized matched rms envelope equations can be solved giving

\[ \frac{\tilde{x}}{x_0} = \sqrt{\frac{\delta_x}{2} + \frac{\delta_x^2}{4} + 1} \approx 1 + \frac{\delta_x}{4} \quad \text{and analogously for} \quad \frac{\tilde{z}}{z_0}. \quad (7.20) \]

As \( \frac{\tilde{x}}{x_0} = \sqrt{\frac{\sigma_{xT}}{\sigma_T}} \), where \( \sigma_T \) is the phase advance per period with space charge (note that in general \( \tilde{x} = \sqrt{\frac{L \varepsilon_{x_{\text{rms}}}}{\sigma_T}} \), \( L \) being the focusing period), the following relation is found easily

\[ \frac{\sigma_{xT}}{\sigma_T} - \frac{\sigma_T}{\sigma_{0T}} = \delta_x, \quad \text{and analogously for the longitudinal phase plane.} \quad (7.21) \]

Usually the reduction in phase advance due to space charge is taken into account by introducing a **space charge parameter** \( \mu \) in the expression

\[ \sigma_T^2 = \sigma_{0T}^2 (1 - \mu_T); \quad (7.22) \]

with some straightforward calculations one finds that \( \delta_x \) and \( \mu_T \) are linked by

\[ \delta_x = \frac{\mu_T}{\sqrt{1 - \mu_T}} \quad \text{and} \quad \mu_T = \delta_x \frac{\sigma_T}{\sigma_{0T}} = \frac{\delta_x}{2} \left[ \sqrt{\delta_x^2 + 4} - \delta_x \right]; \quad (7.23) \]

an analogous expression links \( \mu_L \) and \( \delta_z \).

If the beam envelopes are not matched, one must solve the envelope equations containing the second derivatives. For a continuous beam, the general solution is composed of two modes, the zero and \( \pi \) mode, where envelope oscillations in the two phase planes are in **phase** or in **phase opposition**, respectively.
7.2 Space charge: Non-linear effects

Equations with linearized space charge forces and rms values are essential for the design of linear accelerators, but bear in mind that they are only approximations. For example, a continuous beam with a uniform density distribution in the real 2-D space, requires a non-realistic surface distribution in the 4-D phase space. For a bunched beam with a uniform density in 3-D space, there does not even exist a corresponding 6-D phase space density. Uniform density distributions are not stationary, i.e., they change with time. Stationary distributions are solutions of a system of equations composed of the Vlasov equation, Poisson equation and the equation of motion. For very low beam intensities, gaussian density distributions are stationary. For very high beam intensities, it is interesting that a stationary solution is an almost uniform density distribution, which falls to zero on the beam boundary, over a distance of the order of a Debye length \( \lambda_D = \sqrt{\frac{e_0 k_B T}{n q^2}} \), \( e_0 \) being the vacuum permittivity, \( k_B \) the Boltzmann constant, \( T \) the absolute temperature and \( n \) the number of particles per \( m^3 \); note that \( k_B T \) is linked to the kinetic energy of particles in the rest frame of the beam.

The density inside the beam is such that space charge forces cancel the external focusing forces, and particles move freely inside the beam, bouncing back on beam edges. The 2-D phase space projections and particle trajectories do not have an elliptical, but a more rectangular shape and the density is non-uniform. Often emittance and beam dimensions increase, according to various processes.

**Emittance increase: halo formation**

High-intensity, high-energy linear accelerators require an analysis of non-linear space charge effects. Such an effect is the beam emittance growth, occurring even if the equivalent beam is rms matched to the accelerator acceptance; with a mismatch, this effect is reinforced. The main reasons for the emittance increase are:

---Transformation of the space charge potential energy into kinetic energy, when the distribution changes to a stationary one (minimum of potential energy); this change occurs very rapidly, during a quarter of the plasma oscillation \( \omega_p (\omega_p = \sqrt{\frac{q^2 n}{e_0 \gamma^2 m}}) \).

---Primary effect of a mismatch: it is observed that envelope oscillations, symmetric or antisymmetric (“breathing” in phase or phase opposition), are damped in a non-linear regime, but the final beam radius is always larger than the original matched one. If the beam radius oscillates between \( \bar{x}_{\max} \) and \( \bar{x}_{\min} \), the final radius will be

\[
\bar{x}_f = \sqrt{\frac{\bar{x}_{\max}^2 + \bar{x}_{\min}^2}{2}}
\]

leading to a larger emittance. The effect is however small for small mismatches.

---Secondary effect of a mismatch (halo formation): in a uniform density beam oscillating under space charge conditions, individual particles always oscillate slower than half the core frequency and there is no resonance between these two oscillations; in a real beam, tail particles outside the core are more subject to the external focusing field and resonances can occur. In such a case, computations show that oscillations with amplitudes 8 to 10 times the rms radius can occur. A large tail, called halo, can appear around the beam, the size of which does not strongly depend on the mismatch amplitude nor the tune depression. Several effects can lead to the halo formation, like misalignment, focusing misadjustments and other, non-adiabatic changes. Computational studies show that, at the edge of the beam core, very small coordinate changes can change a core trajectory into a halo one. Such a situation
is sometime called *chaos*. In order to avoid beam loss, the aperture has to be quite large. In the longitudinal direction, the presence of non-linear focusing terms (for longer bunches) and the existence of the bucket complicate additionally the situation; the bunches, if possible, should be kept small.

--- Energy transfer between phase planes during the process of *equipartitioning* (thermalisation); equipartitioning means that the kinetic energies of particles in each of the phase plane in the beam rest frame tend to equalise; this thermalisation occurs for high space charge densities ($\delta \gg 1$). The condition for equipartitioning, expressed in the laboratory frame is:

$$\frac{\varepsilon_{\text{rms}}^2}{x^2} = \frac{\varepsilon_{\text{rms}}^2}{y^2} = \frac{\varepsilon_{\text{rms}}^2}{z^2}.$$  

Even if the conditions under which equipartitioning takes place are not clear, it seems safe to try to satisfy the above conditions, in particular at low energies, where the density is usually the highest; in the course of acceleration, one can depart from these conditions progressively in order to avoid having too large transverse dimensions.

--- Scattering due to collisions: this effect is possibly responsible for the above thermalisation and may be important at very high intensities; however, the effect is still under study.

### 7.3 Image effects

The derivations in Section 7.1 were made for a beam propagating in free space. In practice, the beam is surrounded by RF cavities or drift tubes, and the beam self fields are changed in order to satisfy the boundary conditions. For low velocity beams such changes are very rapid compared to the motion of particles and the situation does not differ much from an electrostatic case. It is then convenient to introduce the concept of *image effects*. Theoretically such a concept can be applied only for plane boundaries or for a continuous beam in a cylindrical pipe. Detailed calculations show, however, that the method is quite good also for bunched beams; in such cases image effects produce some transverse defocusing and longitudinal focusing, but the effect is weak (except for very large and misaligned beams) and can usually be neglected. The only place where such effects are sometimes considered is in RFQs having a small aperture and being used for very low particle velocities: considering the electrodes as cylinders parallel to the beam axis, the image forces create an octupolar transverse field which affects the focusing and can distort the beam.

### 7.4 Wake fields and beam break up (BBU)

With relativistic particles, charges moving inside a metallic environment excite wake fields, which lag behind them. For not too short bunches, such fields, initiated by the head of the bunch being slightly off axis, may affect the bunch tail transversally, displacing it further away from the axis. This is called *head-tail* effect, which can become important by repeating itself along longer linacs. To prevent this, a method has been suggested by Balakin et al. (BNS damping) consisting in displacing the RF accelerating phase 15 to 20° before the crest at the beginning of the linac, and creating thus different energy spreads and oscillation frequencies between the head and tail of the beam; at the end of the linac, the phase is changed again to restore a small energy spread. Of course, a good beam alignment and proper focusing reduce the head-tail effect also.

In a linac cavity one usually has EM waves of the TM$_0$ type and mode for acceleration of particles. Apart from these waves, a slightly misaligned train of bunches can excite transverse TEM waves which are often of the backward type and for some frequency travel at the velocity of light (≡ beam velocity). These waves can excite the following bunches and lead, when a threshold value depending on RF losses is crossed, to an exponential growth of beam dimensions along the pulse and resulting in
a transverse beam loss. This effect is called regenerative BBU. In cavities designed with a constant accelerating field gradient, the wave velocity of TEM waves is not constant along the structure and such cavities are less subject to BBU than the constant impedance ones.

A linear accelerator is usually composed of a succession of cavities; if they are all equal as is often the case in an electron linac, the beam, slightly affected by the first cavity, will excite the following ones, more and more, and lead to a cumulative BBU. A stronger focusing can reduce this effect. Also slight differences in linac cavities, which cause a change of the frequency of the synchronous transverse wave, reduce the BBU effect. Whenever possible, one should try to damp the transverse modes, in particular in superconducting linacs.

7.5 Space charge simulation methods

A real beam bunch, composed of tens or hundreds of millions of particles is replaced in simulation by thousands, tens of thousands and sometime hundreds of thousands of macroparticles with the same statistical distribution, but with a larger charge in order to produce the same circulating current. Several types of space charge calculations can be used.

**Point to point interaction (PPI codes)**

Each particle in a beam is repelled by all the others with a force expressed in the frame moving with the beam by

\[ F_r = \frac{q_1 q_2}{4 \pi \varepsilon_0 r^2}, \]

where \( q_1 \) and \( q_2 \) are the charges of any two particles in the beam and \( r \) is their distance; the force acts along \( r \). In the PPI method, the charge of the macroparticles is put in the above formula, and when two macroparticles come too close together, anomalous repulsion forces would result. To avoid such effects, one introduces a distance \( R \), correlated to the average distance between macroparticles, and modifies the above formula whenever \( r < R \) as

\[ F_r = \frac{q_1 q_2 r}{4 \pi \varepsilon_0 R^3}, \]

where the charge \( q_2 \) acting on the test particle \( q_1 \) has been replaced by a uniformly filled sphere of radius \( R \) and the force reduced by \( \left( \frac{r}{R} \right)^3 \). It is fortunate that the result of computations does not depend too critically on the choice of the value for \( R \). An inconvenience of this method is that the number of computations, and hence the computer time, increases as \( N^2 \).

**Ellipsoidal bunch distribution model**

The space charge field of an ellipsoidal distribution in a bunch can be integrated quickly; assuming a distribution of the form

\[ \rho(x, y, z) = f(u) \quad \text{with} \quad u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \]

and with \( a, b \) and \( c \) being the rms bunch dimensions, the function \( f(u) \) can be determined from the statistical distribution of the macroparticles and then the space charge field computed. The distributions in \( x \) and \( y \) are often similar, but not the one in \( z \) and therefore the method is only approximate. It can be improved by using two or three ellipsoids along the \( z \)-axis to describe the distribution in a bunch.
Series expansion method

If one does not want to assume any possible symmetry in the beam, one has to use a general 3-D expansion

\[ \rho(x, y, z) = \sum_i \sum_m \sum_n A_{lmn} f_i \left( \frac{x}{a} \right) f_m \left( \frac{y}{b} \right) f_n \left( \frac{z}{c} \right). \]

(7.25)

Fourier expansion is not very convenient because it requires a periodicity \( 2\pi k_a, 2\pi k_b, \) and \( 2\pi k_c \) in the \( x, y \) and \( z \) directions, respectively, and the distribution becomes repetitive. To reduce the effect of periodicity one can increase the period by increasing \( k \), but the number of terms necessary to represent the real distribution is proportional to \( k^3 \).

An interesting expansion requiring a small number of terms is the Hermite expansion

\[ \rho(x, y, z) = \sum_i \sum_m \sum_n A_{lmn} H_i \left( \frac{x}{a} \right) H_m \left( \frac{y}{b} \right) H_n \left( \frac{z}{c} \right) \cdot e^{-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}}, \]

(7.26)

where the \( H \) are Hermite polynomials and the \( A \) are obtained by simple polynomial summations over the macroparticle coordinates. With such an expansion, the first term \( A_{000} \) corresponds to a gaussian ellipsoidal distribution, corrected by other \( A_{lmn} \) terms, all of total charge zero. Relatively few terms are required to obtain a good representation: only a few tens of terms have usually an amplitude larger than 1/100 of \( A_{000} \). The field corresponding to various \( A_{lmn} \) terms is given by easy integrations and simple analytic correction terms; in the SCHERM routine, this allows the study, in not too long a time, of the effects of a lack of symmetry; most of the other methods assume symmetry.

Particle in cell method (PIC code)

The most common codes used nowadays make use of a rectangular mesh. The charges of macroparticles are distributed in such a mesh, with some smoothing, and the fields are computed at the nodes of the mesh and can then be interpolated everywhere. The mesh has to be large enough to include all or almost all macroparticles; in fact, usually a few outermost macroparticles (about 0.1%) are left out and their dynamics computed as if all the charge were concentrated in the mesh centre.

The PIC method can be applied to 3-D problems, usually with the help of Fast Fourier Transform (FFT). The most common code, SCHEFF, uses however a 2-D rotational approach. With

\[ \frac{r^2}{ab} = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \]

(7.27)

a rectangular mesh is constructed in the \( r_z \) plane and fields belonging to circular rings of space charge (each ring has a uniform charge distribution) are computed and interpolated. The fields in the transverse plane are obtained from the radial field \( E_r \)

\[ E_x = E_r \frac{2}{r \frac{a}{b} + 1} \quad \text{and} \quad E_y = E_r \frac{2}{r \frac{b}{a} + 1}, \]

(7.28)

which corresponds to expressions for transverse fields in a continuous uniform beam

\[ E_x = \frac{\rho}{\varepsilon_0} \frac{b}{a+b} \frac{b}{a} \quad \text{and} \quad E_y = \frac{\rho}{\varepsilon_0} \frac{a}{a+b} \frac{b}{a+1}. \]

(7.29)
Strictly speaking, the above expressions are not valid for non-uniform distributions and the approximation with rings is not very good for elliptical ratios $a/b$ (or $b/a$) larger than 1.5 and for beams which are far from an elliptical symmetry in the transverse plane. It turns out, however, that in practice the 2-D rotational PIC code SCHEFF is usually quite accurate and very fast; in doubtful situations, other routines can be used.

### 7.6 Intensity limits in linear accelerators

There are no general theoretical formulae for intensity limits in linear accelerators. The approximate formulae usually applied for ion linear accelerators are based on linear dynamics and have therefore a limited significance, but nevertheless they show the importance of various parameters. The linearization of external forces is less precise in the longitudinal phase plane due to the highly non-linear potential well. The separatrix length $\psi(\varphi_0)$ and the bucket shrink with space charge approximately as $(1 - \mu_L)$ and the maximum intensity is obtained for a $\mu_L$ which is the best compromise between a high beam density and a reasonable bucket length. Very often the intensity is constrained by requirements of limited emittance increases and small beam losses. It is recommended that whenever possible some simple prescriptions are followed such as avoiding sudden changes in acceleration rate or focusing conditions, achieving energy equipartitioning between transverse and longitudinal motions (in particular at low energies), keeping $\sigma_{\text{OL}} \leq 60^\circ$ and $\sigma_T / \sigma_{\text{OL}} \geq 0.4$ (which gives $\mu_{\text{Lmax}} = 0.84$ or $\delta_{\text{Lmax}} = 2$) and avoiding the resonance $\sigma_T = 1 / 2 \sigma_{\text{OL}}$. In the longitudinal plane, $\mu_{\text{Lopt}}$ obtained by the authors’ analytical model is 0.425 (some authors give 0.3) and it refers to a stationary bucket. With acceleration, the bucket length increases (expressed in [m] and not RF angles) and higher values of $\mu_L$, giving a higher intensity, can be used if the acceleration is rapid enough.

#### Approximate transverse current formula

From the expression for the space charge factor $\delta_x$ in Chapter 7.1

$$\delta_x = \mu_T \frac{\sigma_{\text{OL}}}{\sigma_T} = \frac{ql}{10 \sqrt{5} \varepsilon_0 m c^3 \beta_x^3 \gamma_x^3 \cdot k_x \varepsilon_{\text{rms}} \cdot \Delta \vec{\phi}} \cdot \frac{g}{\left(\frac{\varepsilon_0}{\varepsilon_{\text{rms}}}\right)} \cdot \left(\frac{z_0}{\bar{x}_0}\right), \quad (7.30)$$

and with

$$k_x = \frac{\sigma_{\text{OL}}}{N \beta \lambda}, \quad \frac{z_0}{\bar{x}_0} = \sqrt{\frac{\varepsilon_{\text{rms}} / \sigma_{\text{OL}}}{\varepsilon_{\text{rms}} / \sigma_{\text{OL}}} \quad \text{and} \quad \sigma_T = \sigma_{\text{OL}} \sqrt{1 - \mu_T} \quad (7.31)$$

the approximate current formula becomes

$$I_T = 10 \sqrt{5} \frac{m}{q} \frac{\varepsilon_0 c^3}{N \lambda} \beta_x^2 \gamma_x^3 \cdot \varepsilon_{\text{rms}} \Delta \vec{\phi} \cdot g^{z_1} \sqrt{\frac{\varepsilon_{\text{rms}} / \sigma_{\text{OL}}}{\varepsilon_{\text{rms}} / \sigma_{\text{OL}}}} \cdot \frac{\mu_T}{\sqrt{1 - \mu_T}} \sigma_{\text{OL}}. \quad (7.32)$$

Some substitutions have to be made in the above formula to get an expression that is more convenient for the transverse current limit. For example, the transverse emittance should be expressed via the aperture radius and the wiggle factor, and the parameters of the longitudinal motion adequately set. It will not be made in this paper.

#### Approximate longitudinal current formula

Proceeding in the same way as for the transverse plane, the approximate current formula is obtained as
\[ I_L = 10 \sqrt{5} \frac{m E_0 c^3}{q N \lambda^2} \beta_s \gamma_s^3 \cdot \sqrt{\sigma_{z\text{rms}} \sigma_{x\text{rms}}} \Delta \tilde{\varphi} \cdot \frac{\mu_L}{\sqrt{1 - \mu_L}} \frac{\sigma_{0L}}{\sigma_{0r}} \text{,} \quad (7.33) \]

where

\[ \bar{k} = \frac{\sigma_{0L}}{N \beta \lambda} \quad \text{and} \quad \sigma_L = \sigma_{0L} \sqrt{1 - \mu_L} \text{.} \quad (7.34) \]

The formulae for \( I_T \) and \( I_L \) can also be brought into other forms, which might correspond better to the specific problem one is considering.

Note that the intensity limit in electron linacs is usually given by the threshold of BBU (see Chapter 7.4).

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BIBLIOGRAPHY


[10] Proceedings of the Linear Accelerator Conferences:
    a)  Chalk River, Ontario, Canada, September 1976 (AECL-5677);
    b)  Montauk, N. Y., September 1979 (BNL 51134);
    c)  Bishop’s Lodge, Santa Fe, N. M., October 1981 (LA-9234-C);
    d)  Lufthansa-Schulungszentrum, Seeheim, Germany, May 1984 (GSI-84-11);
    e)  Stanford University, Stanford, California, June 1986 (SLAC-303);
    f)  Newport News, Virginia, October 1988 (CEBAF-Report-89-001);
    g)  Albuquerque, N. M., September 1990 (LA-12004-C);
    h)  Ottawa, Ontario, Canada, August 1992 (AECL-10728, 2 vols.);
    i)  Geneva, Switzerland, August 1996 (CERN 96-07, 2 vols.);
APPENDICES

A.1 Mid-gap corrections

Phase change at mid gap

In a drift space, the phase varies linearly with distance, as the energy remains constant. In a gap there is acceleration and some corrections have to be included to take this into account. The phase change in the first half of the gap is:

$$\delta \varphi_1 = \int_{-\frac{1}{2}}^{0} \delta k \frac{\partial k}{\partial \gamma} dz \int_{-\frac{1}{2}}^{0} \delta \gamma dz , \tag{A.1}$$

where $\delta k$ and $\delta \gamma$ correspond to the progressive changes in $k$ and $\gamma$ when entering a gap; note that

$$\delta \gamma = \int_{-\frac{1}{2}}^{z} \frac{qE_z(s,t)}{mc^2} ds . \tag{A.2}$$

Integrating $\delta \varphi_1$ by parts

$$\delta \varphi_1 = \int_{-\frac{1}{2}}^{0} \frac{\partial k}{\partial \gamma} \left[ z \delta \gamma \right]_{-\frac{1}{2}}^{0} \frac{\partial k}{\partial \gamma} dz \int_{-\frac{1}{2}}^{0} \frac{z qE_z(z,t)}{mc^2} dz ; \tag{A.3}$$

the first term is zero, so only the second one has to be evaluated. A similar computation can be performed for the second half of the gap, giving for the full gap

$$\delta \varphi = -\int_{-\frac{1}{2}}^{0} \frac{z qE_z(z,t)}{mc^2} dz . \tag{A.4}$$

Expressing $E_z(z)$ with a Fourier expansion and noting that

$$z \cos kz = \frac{\partial}{\partial k} \sin kz \tag{A.5}$$

giving eventually

$$\delta \varphi = \frac{qE_z l_z}{mc^2 \beta^2 \gamma^3} \frac{\partial}{\partial k} \left[I(k)I_0(kr)\right] \sin \varphi . \tag{A.6}$$

Note that all variables refer to mid-gap values.

Liouville’s theorem

An ensemble of particles is in general described by two families of conjugated variables, one indicating the position ($q$) and the other the momentum ($p$). In non-dissipative systems, the equations of motion in phase space can be obtained from a function called hamiltonian:

$$q' = \frac{\partial H}{\partial p} \quad \text{and} \quad p' = -\frac{\partial H}{\partial q} ; \tag{A.7}$$

the prime indicates the derivation with respect to the independent variable $s$ or $t$. The hamiltonian is in general a function of $p$, $q$ and $s$ or $t$. If at a given moment (or position) the particles occupy in phase space a volume $V(t)$, at the time $t+\Delta t$ they will occupy the volume $V(t+\Delta t)$, see Fig.A1. It is easy to show that these volumes are the same, and to do so, one uses the Gauss theorem (transformation of a surface integral into a volume integral) and the hamiltonian:
If the motion can be described by a hamiltonian \( H(q,p,t) \), the particles move in phase space like an incompressible fluid. This is Liouville’s theorem.

\[
\frac{dV(t)}{dt} = \int w d\vec{f} = \int (\nabla \vec{w}) dv = \int \left( \frac{\partial}{\partial q} q' + \frac{\partial}{\partial p} p' \right) dv = 0 \quad \text{(A.8)}
\]

Integrals: surface volume ; \[ \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial q \partial \tilde{p}} \]

\( \text{(Gauss theorem)} \quad \text{(Hamilton)} \)

\[ \vec{f} \quad \text{surface element} \]

\[ \vec{w} = \begin{pmatrix} q' \\ p' \end{pmatrix} \quad \text{phase space velocity of surface element} \]

\[ dv \quad \text{volume element} \]

\[ \text{Fig.A1 Volumes occupied by particles in phase space.} \]

To describe the motion in the longitudinal phase plane one can use the hamiltonian

\[ H = -qE_o I(k) \sin \phi = H^*(W) \sin \phi, \quad \text{(A.9)} \]

from which one can get the already mentioned mid gap kicks or changes by

\[ \delta W = -\frac{\partial H}{\partial \phi}; \quad \delta \phi = \frac{\partial H}{\partial W}. \quad \text{(A.10)} \]

These are hamiltonian-like relations, except that they apply to discrete and not differential changes. When moving from cell \( i \) to cell \( i+1 \), the transformation of the phase plane surface occupied by the particles is given by the jacobian

\[
J_{i,i+1} = \begin{vmatrix}
\frac{\partial W_{i+1}}{\partial W_i} & \frac{\partial \phi_{i+1}}{\partial \phi_i} \\
\frac{\partial W_{i+1}}{\partial \phi_i} & \frac{\partial \phi_{i+1}}{\partial \phi_i}
\end{vmatrix} = \begin{vmatrix}
1 - \frac{dH^*}{dW} \cos \phi & d^2 H^* \sin \phi \\
H^* \sin \phi & 1 + \frac{dH^*}{dW} \cos \phi
\end{vmatrix} \quad \text{(A.11)}
\]

The jacobian equals unity to first order in \( \frac{\partial W}{W} \), so Liouville’s theorem is satisfied to the same order; the error of the second order is very small.

**Computation of mid-gap values**

In the previously derived mid-gap changes or kicks, \( \delta \phi \) and \( \delta W \), appear as mid-gap values which are not known. They can be computed by a combined use of field integral factors \( T(k) \) and \( S(k) \), see Chapter 6.1 such as
the above expression is equal to the field amplitude \( E_z(z) \) in the first half of the cell \( i \) (for \( z < 0 \)), but is zero for the second half of the cell (for \( z > 0 \)). Using this expression for computing \( \delta W \) as has been done before, one obtains the energy gain up to mid gap:

\[
\delta W = \frac{qE_i l}{2} \left[ T(k) I_0(k, r) \cos \varphi + S(k) I_0(k, r) \sin \varphi \right];
\]  

(A.13)
a similar relation can be obtained for \( \delta \varphi_i \). From these relations, mid-gap values of \( W \) and \( \varphi \) are determined by an iteration process.

**Correcting terms to include radial slope of trajectories at mid-gap**

All the mid-gap kicks have been computed assuming particle trajectories parallel to the axis. In order to account for a slope, one has to replace, in the expression for \( E_z \), \( r \) by \( r + r' z \) and \( I_0(k, r) \) by \( I_0(k, r) + k r' z I_1(k, r) \). The derivation can proceed as before, leading to

\[
\delta W = qE_i l T(k) I_0(k, r) \cos \varphi + qE_i l \frac{\partial}{\partial k} \left[ T(k) k I_1(k, r) \right] r' \sin \varphi .
\]  

(A.14)
A similar derivation can be made for \( \delta \varphi \) and the radial motion \( \delta r \) and \( \delta r' \). In the case one omits some of these terms, it is safe to do it in a way that preserves the Liouvillian character.

### A.2 A new, more general beam dynamics computation method

**Longitudinal phase plane: computation of energy gain**

When a cavity (cell) contains several gaps and there are no symmetries around a mid-gap plane, another computation method is introduced. The new principle is the introduction of the concept of the equivalent field. To start, the origin of the coordinate system is placed at an arbitrary point outside the accelerating element and transit time coefficients are computed with the usual integrals over the whole length of the element:

\[
T(k) = \int_0^L E_z(z, 0) \cos(kz) dz , \quad \text{ (A.15)}
\]

\[
S(k) = \int_0^L E_z(z, 0) \sin(kz) dz ; \quad \text{ (A.16)}
\]

the energy gain is

\[
\Delta W(k, \varphi_0) = q \left[ T(k) \cos \varphi_0 - S(k) \sin \varphi_0 \right] , \quad \text{ (A.17)}
\]

where \( \varphi_0 \) is the phase of the particle at the origin of the coordinate system, \( \varphi = kz + \varphi_0 \), and \( k = \omega / v_p \), with \( v_p \) being the particle velocity.

The maximum gain is obtained for \( \varphi_0 = \varphi_c \) satisfying

\[
\tan(\varphi_c) = -\frac{S(k)}{T(k)}, \quad \text{ (A.18)}
\]

and it amounts to

\[
\Delta W_c(k) = q \sqrt{T^2(k) + S^2(k)} = q T_0(k) . \quad \text{ (A.19)}
\]

The coefficients \( T(k) \) and \( S(k) \) change when one moves the origin of the field calculations, but the maximum energy gain remains constant. \( T_0(k) \) corresponds to a transit time factor of an equivalent symmetric field with the origin in the middle of the symmetry. It is fully defined with \( T(k) \) and \( S(k) \) and their first and second derivatives. With these three equations, all the parameters in the following expression for the transit time factor are also defined:
with \( V_0 = E_0 L_e \), where \( E_0 \) is the constant amplitude of the equivalent accelerating field extending over an equivalent distance \( L_e \), and \( \partial = (k - \hat{k}) L_e \). Note that \( \hat{k} \) corresponds to a travelling wave velocity for which the energy gain is maximum (in a single gap case, this optimum velocity was infinite; here it is the velocity of the wave of constant amplitude \( E_0 \), travelling over a fixed length \( L_e \), and \( \partial \) is the corresponding phase slip). The distance \( l \) between the centre of symmetry of the equivalent field and the origin of the real field is found via

\[
\varphi_c = -lk + n\pi \rightarrow l = -\frac{\partial \varphi_c}{\partial k} = -\frac{S'(k)T(k) - S(k)T'(k)}{T^2(k) + S^2(k)}. \tag{A.21}
\]

The length \( L_e \) over which the equivalent travelling wave extends can correspond to several \( \beta \lambda_\gamma \), but the phase seen by a particle will not change much. In computations a linear phase evolution over \( L_e \) is assumed, and the position of this line in the \( \varphi, z \) plane obtained by requesting that its average value corresponds to the average phase of the real phase law. Without entering into the details of computations, the equivalent field method is quite accurate (when compared to the slow step by step integration) and it satisfies Liouville’s theorem numerically.

**Transverse phase plane: introduction of the reduced variables**

The reduced radius \( R \) is introduced via the Picht transformation

\[
R = r \sqrt{\beta} \gamma = r (\gamma^2 - 1)^{1/4} \quad \text{and} \quad R' = dR / ds. \tag{A.22}
\]

Furthermore

\[
r' = \frac{dr}{ds} = R'(\gamma^2 - 1)^{1/4} - \frac{R}{2} \gamma (\gamma^2 - 1)^{5/4} \frac{d\gamma}{ds}. \tag{A.23}
\]

From

\[
\frac{d(ymv_r)}{dt} = q(E_r - v_z B_\alpha) \tag{A.24}
\]

one gets

\[
d(r'\beta \gamma) = \frac{q}{mc} (E_r - \beta c B_\alpha) dt = K(s, r) dt. \tag{A.25}
\]

In the paraxial approximation

\[
K(s, r) = K(s) r \quad \text{and} \quad K(s) = \frac{q}{mc} \left( \frac{\partial E_r}{\partial r} - \beta c \frac{\partial B_\alpha}{\partial r} \right). \tag{A.26}
\]

With Maxwell’s equations

\[
\frac{\partial E_r}{\partial r} = -\frac{1}{2} \frac{\partial E_z}{\partial s} \quad \text{and} \quad \text{rot} \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}, \tag{A.27}
\]

and with

\[
mc^2 \frac{d^2 \gamma}{ds^2} = qE_z \tag{A.28}
\]

giving

\[
mc^2 \frac{d^2 \gamma}{ds^2} = q \frac{dE_z}{ds} = q \frac{\partial E_z}{\partial s} + q \frac{\partial E_z}{\partial t}, \tag{A.29}
\]

one obtains finally, after some calculations
This equation can be integrated in a gap and is advantageously used with the concept of equivalent field. The radius \( r \) may change rapidly, but \( R \) remains smooth and a numerical integration requires only a few steps and is more precise than the usual integration of \( r \) with a varying \( v_z \). As in the equation the term \( dR/ds \) does not appear, \( R \) and \( R' \) satisfy Liouville’s theorem, provided the longitudinal and transverse motions are uncoupled.

### A.3 Smooth variables for transverse focusing in ion linacs

The smooth solution of Hill’s equation

\[
\frac{d^2 x(s)}{ds^2} + K(s)x(s) = 0, \quad K(s + L) = K(s),
\]

with the period \( L \) much shorter than the betatron wavelength, is of the type

\[
x(s) = X(s)[1 + \varepsilon(s)],
\]

where \( X(s) \) is the slowly variable smooth coordinate and \( \varepsilon(s) \) is the wiggle due to the AG focusing. The smooth coordinate is expressed in the form

\[
X(s) = \text{const} \cdot e^{\frac{a}{k_B} s},
\]

and \( k_B \) is the smooth betatron phase advance per unit length. The following formulae to determine \( k_B \) and \( \varepsilon(s) \) are given without proof.

**Computation of \( k_B \):** it is given with the formula

\[
k_B^2 = \overline{K} + \left[ \int \left( K - \overline{K} \right) ds \right];
\]

the first term on the right hand side is the average value of \( K(s) \) over the period \( L \) (it is often \( =0 \)). To determine the second term, one first computes the indefinite integral, squares the result and then finds the average over the period \( L \). The most convenient way is to express the integrand and the integral in analytic form. For example, the usual “rectangles” of the AG focusing, see Fig.A2.a, can be approximated by a Fourier series, of which one would retain only the fundamental frequency term. However, some precaution is required: correct results are obtained only if the integral starts at a point of the AG period which gives it, when integrated over the period \( L \), a value zero. The procedure as shown in Fig.A2.b is wrong, while in Fig.A2.c it is correct.

**Fig.A2 a)** Rectangles representing the AG focusing; b) The area under the dotted line represents the integral of the AG focusing; it is \( \neq 0 \); c) Initial point chosen so as to give the integral = 0.
Computation of $\alpha(s)$: The formula giving $\alpha(s)$ is

$$\epsilon(s) = \text{const} - \int ds \left( K - \bar{K} \right) ds,$$

(A.35)

where the $\text{const}$ is chosen so to give $\bar{\epsilon} = 0$. The integrand and the integrals should again be expressed in analytical form.

To illustrate the above method, $\sigma_\beta = N\beta_s h \cdot k_\beta$ is computed for AG focusing with magnetic quadrupoles having gradients $\pm G$. In the quadrupoles, $K(s)$ of equation (A.31) is given with

$$K(s) = \frac{\pm qG}{mc\beta_s \gamma_s},$$

(A.36)

whilst outside the quadrupoles $K(s) = 0$. To simplify the expressions, the independent variable

$$\eta = \frac{s}{N\beta_s \lambda},$$

is introduced, as done on page 30. The equation (A.31) becomes

$$\frac{d^2 x(\eta)}{d\eta^2} + K(\eta)x(\eta) = 0,$$

(A.37)

with

$$K(\eta) = N^2 \beta_s^2 \lambda^2 K(s).$$

(A.38)

The solution for the smooth variable is

$$X(\eta) = \text{const} \cdot e^{\sigma_{\eta} \eta},$$

(A.39)

and the formula (A.34) gives directly $\sigma_{\eta}^2$ provided $K(s)$ is replaced by $K(\eta)$. Note that for AG focusing the average values $\bar{K}(s) = \bar{K}(\eta) = 0$. According to Fig.A2.c, the fundamental Fourier harmonic is

$$\chi \cdot K(\eta) \cdot \cos 2\pi \eta,$$

(A.40)

with $\chi$ being defined on page 29. The indefinite integral of the formula (A.34) is

$$I = \chi \cdot K(\eta) \int \cos 2\pi \eta \cdot d\eta = \frac{\chi \cdot K(\eta)}{2\pi} \sin 2\pi \eta.$$

(A.41)

The average value of $I^2$ over the focusing period is

$$I^2 = \frac{\chi^2 K^2(\eta)}{4\pi^2} \int_0^1 \sin^2 2\pi \eta \cdot d\eta = \frac{\chi^2 K^2(\eta)}{8\pi^2} = \sigma_{\eta}^2.$$

(A.42)

Introducing the focusing parameter $B$, see formula (6.53), gives as in (6.52)

$$\sigma_{\beta}^2 = \frac{B^2}{8\pi^2},$$

a result which has been used in formula (6.56).
A.4 Beam dynamics in RFQs

**Longitudinal dynamics**

The energy gain of a particle in a cell is computed with the longitudinal RF field

\[ E_L(x, y, z, t) = -\frac{\partial U}{\partial z} = \frac{V}{2} kA_{10} I_0(kr) \sin kz \sin(\omega t + \phi), \]  

which integrated over the cell \( i \) gives for the energy gain on axis:

\[ W_i - W_{i-1} = \int_0^{\beta_i/2} qE_L(0, 0, z, t)dz = \frac{V}{2} kA_{10} \int_0^{\beta_i/2} \sin kz \sin(kz + \phi)dz = \frac{\pi}{4} qV A_{10} \cos \phi. \]  

The synchrotron motion is obtained from the coupled first-order synchrotron equations

\[ \frac{d\Delta \phi}{ds} = -\frac{\omega \Delta W}{mc^3 \beta_s^3} \quad \text{and} \quad \frac{d\Delta W}{ds} = \frac{\pi qVA_{10}}{2\beta_s \lambda} [\cos(\phi_s + \Delta \phi) - \cos \phi_s]; \]  

proceeding as in chapter 6.1 one obtains \( \Delta \phi(s) \) and \( k_s(s) \). For the maximum allowed energy spread one gets

\[ \Delta \phi_{\text{max}} = \sqrt{qmc^2 \beta_s^3 V A_{10} (\phi_s \cos \phi_s - \sin \phi_s)}. \]  

**Transverse dynamics**

The field component in the transverse plane is computed from the lowest order potential function

\[ E_s = -\frac{\partial U(x, y, z, t)}{\partial x} = -\frac{\partial}{\partial x} \left\{ \frac{V}{2} \left[ A_{01} (x^2 - y^2) + A_{10} I_0(kr) \cos kz \right] \sin(\omega t + \phi) \right\} \]  

giving

\[ E_s = -\frac{V}{2} \left\{ 2A_{01} x + A_{10} I_1(kr) k \frac{dr}{dx} \cos kz \right\} \sin(kz + \phi). \]  

The first term on the right hand side represents the quadrupole focusing (alternating in time); the second one is the perturbation of the quadrupole field due to vane modulation. Using \( I_1(kr) \approx kr/2 \) (valid for small arguments), \( dr/dx = x/r \) (from the differentiation of \( r^2 = x^2 + y^2 \)), and \( \cos(kz)\sin(kz + \phi) = 1/2[\sin(2kz + \phi) + \sin \phi] \), one obtains

\[ E_s = -\frac{V}{2} \left\{ 2A_{01} \sin(kz + \phi) + \frac{1}{4} A_{10} k^2 \left[ \sin(2kz + \phi) + \sin \phi \right] \right\} x. \]  

In the equation of motion, usually only the first harmonic, \( \sin(kz) \) is kept; introducing \( B^* \) and \( \Delta_{srf} \), and differentiating with respect to \( s \) gives finally

\[ \frac{d^2 x}{ds^2} + \left[ B^* \sin ks + \Delta_{srf} \right] x = 0. \]
**Conditions on RFQ parameters**

Usually RFQs operate at low energies and with intense beams; therefore their inventor Kapchinskij proposed that “sound” conditions be respected: keep the external as well as the space-charge forces constant. Constant external forces mean constant $\sigma_{0T}$ and $\sigma_{0L}$; constant space charge forces mean constant beam radius (smooth) and bunch length. The following equations have to be satisfied:

\[
\frac{\chi}{a^2} = \frac{1 - A_{10} I_0(ka)}{a^2} = \text{const} ; \\
\frac{A_{10} \sin \varphi_s}{\beta_s^2} = \text{const} ; \\
\psi(\varphi_s) \beta_s \lambda = \text{const}
\]

where $\psi(\varphi_s)$ is the separatrix length in radians. The RFQ is usually divided into several sections, each one of which performs a specific task. Particularly important is the first section, called the radial matching section, which matches the input beam transport line (containing focusing elements disposed in space) to the RFQ focusing (focusing forces alternate in time).