Tully–Fisher Relation and its Implications for Halo Density Profile and Self-interacting Dark Matter

H. J. Mo$^{1\star}$ and Shude Mao$^{2†}$

$^1$Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Straße 1, 85740 Garching, Germany
$^2$University of Manchester, Jodrell Bank Observatory, Macclesfield, Cheshire SK11 9DL, UK

ABSTRACT

We show that the Tully-Fisher relation observed for spiral galaxies can be explained in the current scenario of galaxy formation without invoking subtle assumptions, provided that galactic-sized dark haloes have shallow, core-like central profiles, with a core radius proportional to halo circular velocity. In such a system, both disk and halo may contribute significantly to the maximum rotation of the disk, and the interaction between the disk and halo components acts to reduce the scatter in the Tully-Fisher relation. With model parameters (such as the ratio between disk and halo mass, the specific angular momentum of disk material, disk formation time) chosen in plausible ranges, the model can well accommodate the zero-point, slope, and scatter of the observed Tully-Fisher relation, as well as the observed large range of disk surface densities and sizes. In particular, the model predicts that low surface-brightness disk galaxies obey a Tully-Fisher relation very similar to that of normal disks, if disk mass-to-light ratio is properly taken into account. About half of the gravitational force at maximum rotation comes from the disk component for normal disks, while the disk contribution is lower for galaxies with lower surface density.

The halo profile required by the Tully-Fisher relation is as shallow as that required by the observed rotation curves of faint disks, but much shallower than that predicted by conventional CDM models. We discuss the implication of such profiles for structure formation in the universe and for the properties of dark matter. Our results cannot be explained by some of the recent proposals for resolving the conflict between conventional CDM models and the observed rotation-curve shapes of faint galaxies. If dark matter self-interaction (either scattering or annihilation) is responsible for the shallow profile, the observed Tully-Fisher relation requires the interaction cross section $\sigma_X$ to satisfy $\langle |\sigma_X|v|/m_X \rangle \sim 10^{-16} \text{cm}^3 \text{s}^{-1} \text{GeV}^{-1}$, where $m_X$ is the mass of a dark matter particle.

Key words: galaxies: formation - galaxies: structure - galaxies: spiral - cosmology: theory - dark matter

1 INTRODUCTION

It has been known since the work of Tully and Fisher (1977) that there are systematic relations between the luminosity and rotation velocity of disk galaxies. Since the rotation curves of relatively bright disk galaxies are quite flat at large radii, a characteristic rotation velocity $V_c$ (e.g. the maximum rotation velocity $V_{\text{max}}$) can be defined for each disk. When luminosity $L$ is plotted against $V_{\text{max}}$ for a sample of spiral galaxies, it is found that most galaxies lie close to a power-law relation (the Tully-Fisher relation):

$\star$ e-mail: hom@mpa-garching.mpg.de
$†$ e-mail: smao@jb.man.ac.uk
where \( L \) is the slope, and \( \Lambda \) is the ‘zero-point’. The observed value of \( \alpha \) ranges from about 2.5 to about 4; both \( \alpha \) and \( \Lambda \) may depend on the waveband (Strauss & Willick 1995 and references therein). In the I-band, recent determination by Giovanelli et al. (1997) gives

\[
M_I - 5 \log h = -21.00 - 7.68 (\log W - 2.5)
\]

where \( W \approx 2V_{\text{max}} \) is the 21 cm hydrogen line width, and the corresponding Tully-Fisher slope is \( \alpha = 7.68/2.5 \approx 3.1 \). The observed Tully-Fisher relation is quite tight: for a fixed \( V_{\text{max}} \) the spread in the absolute magnitude is smaller than 1 magnitude. It is possible that the small scatter is caused by observational errors and that the intrinsic scatter could be even smaller. Because of its tightness, the Tully-Fisher relation can be used as a distance indicator to spiral galaxies. But the tightness also gives stringent constraints on the formation and evolution of disk galaxies. Since the typical rotation velocity (such as \( V_{\text{max}} \)) is related to the total gravitational mass, while \( L \) is related to the total number of stars, the position of a particular galaxy in the \( L-V_{\text{c}} \) plane depends both on the efficiency for gas to be converted into stars and on the mass distribution in the galaxy.

Real galaxy disks are observed to cover two orders of magnitude in luminosity and in surface luminosity density, and with a variety of rotation curves. The question arises whether the observed Tully-Fisher relation requires the conspiracy of some very special initial conditions for disk formation or it has more generic origin. Whatever the origin, a successful model should explain both the tight Tully-Fisher relation and the diversity of the disk population.

The current standard scenario of disk formation assumes that galaxy disks form as gas cools in dark matter haloes (e.g. Fall & Efstathiou 1980; Dalcanton, Spergel & Summers 1997; Mo, Mao & White 1998, hereafter MMW). In this scenario, the mass of a disk is determined by the mass of its host halo together with the efficiency with which the halo gas settles into a disk; the size of a disk is determined by the angular momentum the disk material initially acquired from the tidal field of the cosmic density field together with the processes which can change disk angular momentum; and the rotation curve of a disk is determined by the density profile of its host halo and the interaction between the halo and disk components. Detailed modelling shows that this scenario of disk formation, combined with current theory of cosmogony, is quite powerful in interpreting observational data of galaxy disks (e.g. Dalcanton et al. 1997, MMW, van den Bosch 1998; Avila-Reese, Firmani & Hernandez 1998; Mao, Mo & White 1998; Heavens & Jimenez 1999; Mo, Mao & White 1999; Syer, Mao & Mo 1999). In particular, MMW showed that disk formation in current CDM models can explain many observations on galaxy disks, provided that the following conditions are fulfilled: (1) dark haloes are not very concentrated; (2) only part of the gas in a dark halo settles into a disk; (3) disk material does not lose much of its angular momentum when it settles into a disk; (4) present disks form quite late, at redshifts \( z \approx 1 \).

Numerical simulations have also been used to study disk formation in CDM models (e.g. Weil, Eke & Efstathiou 1998; Sommer-Larsen, Gelato & Vedel 1999; Kodama, Sofue & Wada 1999; Navarro & Steinmetz 2000). One might hope to use simulations to check if the assumptions made in the standard model are plausible. However, there is a long-standing unresolved problem. In these simulations, it is found that protogalactic gas is mostly in dense clumps. As such clumps sink towards the halo centre to make a disk, they lose much of their angular momentum to the dark halo due to dynamical friction, and the resulting disk is much too small to match any realistic galaxy disk. One possibility to solve this problem is to assume that some feedback effects can keep the protogalactic gas in a diffused form so that the effect of dynamical friction is reduced. Such process has not been very successfully implemented in current models of disk formation, but is required in order to make any progress in the standard framework. Recent high-resolution N-body simulations of dark haloes (Moore et al. 1999; Jing & Suto 2000; Springel et al. 2000) also expose disk formation in CDM model to another problem: CDM haloes are too concentrated to match the observed rotation curves of disk galaxies (e.g., Navarro 1998). This mismatch suggests that some modifications on the CDM cosmogony may also be needed in order to have a successful model of disk formation.

In this paper we take the attitude that the main idea of the standard scenario for disk formation is correct but some modifications on earlier models are required. As discussed above, disk formation in this scenario involves the following important aspects: (1) density profiles of dark haloes, (2) the fraction of halo gas that can settle into a disk, (3) the specific angular momentum of disk material relative to halo material, and (4) the assembly times of present disks. We will take an empirical approach towards all these different aspects. Our goal is to find a simple model within the standard scenario, so that the most important properties of the disk population, such as the Tully-Fisher relation, can be explained without invoking subtle assumptions. The formalism to be used is very similar to that in MMW, but with substantially relaxed model assumptions.

The main result of the paper is that the halo profile required for explaining the Tully-Fisher relation is much shallower than that predicted by conventional CDM models and that the required concentration of galactic haloes does not depend strongly on halo mass. This gives strong constraints on the properties of dark matter. In fact, our results cannot be explained by some of the recent proposals for resolving the conflict between conventional CDM models and the halo profile of faint galaxies. Assuming that the shallow profile is due to dark matter self-interaction (either scattering or annihilation), we discuss the implication of our results for the mass and cross section of the dark matter particles.
2 SIMPLE CONSIDERATION

2.1 Self-Gravitating Disks

To start with, let us consider a simple model where galaxy disks are self-gravitating and have exponential surface-density profiles:

$$\Sigma(R) = \Sigma_0 \exp(-R/R_d),$$

where the central surface density $\Sigma_0$ and the disk scale-length $R_d$ are related to the disk mass by $M_d = 2\pi\Sigma_0 R_d^2$. The rotation curve of such a disk is given by

$$V_d^2(R) = 4\pi G \Sigma_0 R_d y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)], \quad y \equiv \frac{R}{2R_d},$$

where $I_n$ and $K_n$ are modified Bessel functions of the first and second kinds (Freeman 1970). This rotation curve peaks at $R \approx 2.16R_d$, with a maximum rotation velocity given by $V_{\text{max}}^2 \approx 2.5G\Sigma_0 R_d$. Assuming a disk mass-to-light ratio $\Upsilon_d \equiv L_d/M_d$, this relation can be written

$$L_d = B \left(\frac{V_{\text{max}}}{200 \text{ km s}^{-1}}\right)^4 \left(\frac{I_0}{100 \Lsol \text{ pc}^{-2}}\right)^{-1} \left(\frac{\Upsilon_d}{\Upsilon_\odot h}\right)^{-2},$$

where $\Upsilon_\odot \equiv M_\odot/L_\odot$, $I_0 \equiv \Sigma_0/\Upsilon_d$ is the disk central luminosity density, and

$$B \approx 8.5 \times 10^{11} h^{-2} L_\odot.$$  

Equation (5) looks quite similar to the observed relation (1). But the amplitude is too high compared with the observed Tully-Fisher relation. To see this, we use equation (2) to obtain

$$L_d(V_{\text{max}} = 200 \text{ km s}^{-1}) \approx 2.2 \times 10^{10} h^{-2} L_\odot.$$  

The typical scale-length of normal galaxies with $V_{\text{max}} = 200 \text{ km s}^{-1}$ is about $3.5 h^{-1} \text{kpc}$ (Courteau 1996, 1997; de Jong 1996). The implied typical disk surface brightness is therefore $I_0 \approx 300 \Lsol \text{ pc}^{-2}$. This surface luminosity density is slightly higher than the median value of the Freeman disk, $\mu_B = 21.7$ (Freeman 1970), with a colour correction $(B-I) = 1.8$ (de Jong 1996). Since the observed colours of disk galaxies are quite uniform, the stellar mass-to-light ratio should not vary significantly among normal disks with similar luminosities. Based on various considerations — stellar population synthesis (e.g. de Jong 1996), stellar counts and kinematics in the solar neighborhood (Kuijken & Gilmore 1989; Gould, Bahcall & Flynn 1996), and kinematics of external galaxies (Bottema 1997) — the stellar mass-to-light ratio of galaxy disks in the $I$-band is about 2$h$. Since the masses of normal disks are dominated by stars, we may write $\Upsilon_d \approx 2h$ in the $I$-band. Inserting these values of $I_0$ and $\Upsilon_d$ into equation (5), we see that the predicted luminosity at $V_{\text{max}} = 200 \text{ km s}^{-1}$ is about 3 times as high as that observed. There are two possibilities to explain this factor of 3. First, it might be that the value of $\Upsilon_d$ is more close to $3h$ than to $2h$. Given the uncertainty in $\Upsilon_d$, such a moderate increase may be allowed. In this case, the observed Tully-Fisher zero-point would be consistent with the assumption that galaxy disks are self-gravitating at the radii of maximum rotation. The second possibility is that the value of $\Upsilon_d$ is close to $2h$, but about half of the gravitational force responsible to the maximum rotation actually comes from an extra mass component, i.e., a dark halo. The second possibility is much more likely, not only because dark haloes are required to explain the flat rotation curves of spiral galaxies and to stabilize a thin disk (e.g. Efstathiou, Lake & Negroponte 1982), but also because a model with disks dominating $V_{\text{max}}$ cannot be made consistent with the small Tully-Fisher scatter. This can be seen from equation (5): if the mass-to-light ratio $\Upsilon_d$ is assumed not to change significantly among galaxies with similar $V_{\text{max}}$ (probably a good assumption given that their colours are quite uniform), the scatter in $L_d$ is the same as that in $I_0$ for given $V_{\text{max}}$. The observed range in surface brightness is about 1.5 magnitudes for normal galaxies and is even larger if low surface-brightness galaxies are included (see Bothun, Impey & McGaugh 1997). The implied scatter is therefore much larger than that observed (see also Courteau & Rix 1999). Thus, the observed Tully-Fisher relation is not consistent with the assumption that disk rotation is dominated by disk gravity at the radius of maximum rotation.

2.2 The Effects of Dark Haloes

If dark halo contributes significantly to $V_{\text{max}}$ in a typical disk galaxy, we want to know whether the presence of dark haloes can also help to explain the zero-point and small scatter in the Tully-Fisher relation, given that disks have large spread in their surface brightness. Taking into account the gravity of dark halo (assumed to be spherically symmetric), we should replace equation (5) by

$$L_d = B \left(\frac{V_{\text{max}}}{200 \text{ km s}^{-1}}\right)^4 \left(\frac{\Upsilon_d}{\Upsilon_\odot h}\right)^{-2} \left(\frac{I_0}{100 \Lsol \text{ pc}^{-2}}\right)^{-1} \left(\frac{V_d^2}{V_{\text{max}}^2}\right)^2,$$
where the total rotation velocity at the peak of the rotation curve is a sum in quadrature of contributions from the disk and the halo:

\[ V_{\text{max}}^2 = V_d^2 + V_h^2. \]  

(9)

For the halo density profiles to be considered in the following, the rotation curves typically reach a maximum near 3\(R_d\) where the disk contribution is also near its peak. In what follows, we will identify the rotation velocity at 3\(R_d\) to be the one in the Tully-Fisher relation. For simplicity, we will still denote this velocity by \(V_{\text{max}}\). Inserting the values of \(I_0\) and \(\Upsilon_d\) inferred in the last subsection into equation (8), we find that the ratio between the predicted and observed Tully-Fisher zero-points is

\[ \mathcal{F}(V_{\text{max}} = 200 \text{ km s}^{-1}) \equiv \frac{I_{d, \text{predicted}}(V_{\text{max}} = 200 \text{ km s}^{-1})}{I_{d, \text{observed}}(V_{\text{max}} = 200 \text{ km s}^{-1})} \approx 1.0 \times \left( \frac{\Upsilon_d}{2h\Upsilon_\odot} \right)^{-2} \left( \frac{I_0}{300 L_\odot \text{ pc}^{-2}} \right)^{-1} \left[ \frac{(V_d/V_{\text{max}})^2}{0.5} \right]^2. \]  

(10)

We see that \(V_d^2/V_{\text{max}}^2 \approx 0.5\) is required for the model prediction to match the observation. In other words, dark halo must contribute about half of the gravitational force at the radius of maximum rotation for a typical disk galaxy with \(V_{\text{max}} \approx 200\) km s\(^{-1}\).

To see if the presence of halo also help to reduce the scatter, consider a ‘typical’ disk with mass \(M_d\) and surface density \(I_0\). As before we assume \(\Upsilon_d\) to be invariant from galaxy to galaxy, so that \(M_d\) is equivalent to \(L_d\) and \(I_0\) is equivalent to \(\Sigma_0\).

Now suppose we increase the disk mass from \(M_d\) to \((1 + \epsilon)M_d\) in such a way that disk scale-length is fixed, i.e. we increase disk mass without changing disk concentration in the halo. In this case, both \(I_0\) and \(V_d^2\) are increased by a factor of \((1 + \epsilon)\).

Suppose such an increase in \(M_d\) induces a change \(\delta V_h^2\) in halo contribution, the \(\mathcal{F}\) factor defined in equation (10) becomes

\[ \mathcal{F}' = \frac{1 + \epsilon}{(1 + \epsilon')^2} \mathcal{F} \quad \text{with} \quad \epsilon' \equiv \frac{\epsilon V_d^2 + \delta V_h^2}{V_{\text{max}}^2}. \]  

(11)

If maximum rotation is dominated by the disk, then \(\epsilon' \approx \epsilon\) and \(\mathcal{F}' \approx \mathcal{F}/(1 + \epsilon)\), i.e. the Tully-Fisher zero-point is reduced by a factor of \((1 + \epsilon)\). In this case, we recover the result for self-gravitating disks. If, on the other hand, disk gravity is negligible, then \(\epsilon' \approx 0\) and \(\mathcal{F}' \approx \mathcal{F}(1 + \epsilon)\), i.e. the Tully-Fisher zero-point is enhanced by a factor of \((1 + \epsilon)\). In this halo-dominated case, disk rotation curve is independent of disk mass, and so the Tully-Fisher zero-point is directly proportional to the disk mass. Thus somewhere between disk-dominating and halo-dominating, the Tully-Fisher zero-point will not be altered by the change of \(M_d\). In fact, if both \(\epsilon\) and \(\epsilon'\) are small, and if \(\delta V_h^2\) is neglected, then \(\mathcal{F}' = \mathcal{F}\) for \((V_d/V_{\text{max}})^2 \approx 0.5\), i.e. the Tully-Fisher zero-point does not depend on \(M_d\) if disk contributes about half of the gravitational force at the maximum rotation.

As another example, we increase \(I_0\) (or \(\Sigma_0\)) by a factor of \((1 + \epsilon)\) but keep \(M_d\) unchanged, i.e. we increase disk concentration in the halo without changing the mass. In this case, we have for small \(\epsilon\) that

\[ \mathcal{F}' = \frac{1}{(1 + \epsilon')^2} \mathcal{F} \quad \text{with} \quad \epsilon' \equiv \frac{(\epsilon/2)V_d^2 + \delta V_h^2}{V_{\text{max}}^2}. \]  

(12)

So, \(\mathcal{F}' \approx \mathcal{F}\) for halo-dominating and \(\mathcal{F}' \approx \mathcal{F}/(1 + \epsilon)\) for disk-dominating. The above simple arguments suggest that the presence of dark haloes can help to reduce the Tully-Fisher scatter. However, the details depend on \(\delta V_h^2\), the halo response to the change in the disk, which we will study in the following.

### 3 Detailed Modelling

In order to examine in more detail the constraints provided by the Tully-Fisher relation on disk formation, we consider exponential disks in realistic dark haloes. Our modelling here follows the disk formation model of MMW. The reader is referred to that paper for details; here we only repeat the essentials of the model. Briefly, after the initial protogalactic collapse the gas and dark matter are assumed to be uniformly mixed in a virialized object. As a result of dissipative and radiative processes, the gas component gradually settles into a disk. We assume that the mass of this disk is a fraction \(m_d\) of the halo mass, and that its specific angular momentum is \(j_d\) times that of the dark halo. If the mass profile of the disk is taken to be exponential, and if dark halo responds to the growth of disk adiabatically (see Barnes & White 1984 and Navarro & Steinmetz 2000 for tests of the validity of this assumption), then \(\Sigma_0\), \(R_d\) and the galaxy’s rotation curve are determined uniquely. Specifically,

\[ R_d = \frac{1}{\sqrt{2}} \left( \frac{j_d}{m_d} \right) \lambda R_{hF}, \quad \Sigma_0 = \frac{m_d M_h}{2\pi R_h^2}, \quad V_{\text{max}} = V_h F_V, \]  

(13)

where \(\lambda\) is the spin parameter of the halo, \(M_h\), \(V_h\) and \(r_h\) are the mass, circular velocity and virial radius of the halo before responding to disk gravity, \(F_R\) and \(F_V\) are factors which depend on the halo profile and disk action. For a given cosmology, \(M_h\), \(V_h\) and \(r_h\) are related by
\[ r_h = \frac{V_h}{10H(z)} , \quad M_h = \frac{V_h^3}{10GH(z)} , \]

where \( H(z) \) is the Hubble constant at redshift \( z \) (see MMW for details).

Let us consider a case where the halo density profiles have the form

\[
\rho(r) = \frac{\rho_0}{(r + r_c)^3} = \frac{V_h^2}{4\pi Gr^2} \frac{c}{\ln(1+c) - c(1 + 3c/2)(1 + c)^3} \frac{r^2/r_s^2}{(r/r_s + 1)^3} ,
\]

where \( r_c = r_h/c \) is a core radius and the quantity \( c \) is known as the halo concentration factor. The above profile can be compared with the NFW profile:

\[
\rho(r) = \frac{\rho_0}{r(r + r_s)^2} = \frac{V_h^2}{4\pi Gr^2} \frac{c_{\text{NFW}}}{\ln(1 + c_{\text{NFW}}) - c_{\text{NFW}}/(1 + c_{\text{NFW}})} \frac{r/r_s}{(r/r_s + 1)^2} ,
\]

where \( r_s = r_h/c_{\text{NFW}} \) is a scale radius (Navarro, Frenk & White, 1997, hereafter NFW), and \( c_{\text{NFW}} \equiv r_h/r_s \) describes the halo concentration in this profile. Although profiles (15) and (16) have different behaviour at small radii, the two can be made to have similar global properties by choosing the values of \( c \) and \( c_{\text{NFW}} \). For example, galactic-sized haloes with \( V_h \sim 200 \text{ km s}^{-1} \) in CDM models have \( c_{\text{NFW}} \approx 20 \) (as quoted in Navarro & Steinmetz 1999). The corresponding profile looks similar to profile (15) with \( c = 40 \) over a large range of radii. In this paper we will present our results based on profile (15) with \( c \) as a free parameter, but we will also refer to the results obtained for the NFW profile.

Disk formation in this kind of dark haloes is described in considerable detail in MMW, and the procedure outlined there can be used to calculate \( F_V \) and \( F_R \) as functions of \( c, j_d, \lambda \) and \( m_d \) for a given profile. From these we can calculate the effects of changing model parameters on the Tully-Fisher relation and disk size (or surface density). Figures 1 and 2 show the results of such calculations. Several important conclusions can be reached from the results. As the value of \( m_d \) changes from 0.01 to 0.1, the central surface density \( \rho_0 \) changes by a factor of about 30, but the Tully-Fisher zero-point [described by the factor \( \mathcal{F} \) defined in equation (10)] changes only by a factor of about 3 for given \( \lambda \). The value of \( m_d \) must be smaller than the overall baryon fraction in the universe, \( \Omega_{B,0}/\Omega_0 \) (which is, according to cosmic nucleosynthesis, about 0.05 for an Einstein-de Sitter universe with \( h = 0.5 \), and about 0.1 for a low-density universe with \( \Omega_0 = 0.3 \) and \( h = 0.7 \)). Also \( m_d \) should not be much smaller than 0.01, because such disks, if exit, may not be able to form many enough stars due to the failure to meet the Toomre (1964) instability criterion] to be included in current Tully-Fisher samples. Therefore, any reasonable change in \( m_d \) will not introduce large Tully-Fisher scatter. This is somewhat contrary to intuition. If disk gravity were negligible, an increase of \( m_d \) by a factor of 10 would increase the Tully-Fisher zero-point by a factor of about 10; if, on the other hand, halo gravity were negligible, an increase of \( m_d \) by a factor of 10 would decrease the Tully-Fisher zero-point by a factor similar to that in \( I_0 \), i.e. of about 30. The reason for the insensitivity of the Tully-Fisher zero-point to the change of \( m_d \) is that the interaction between disk and halo acts to reduce the scatter, as is demonstrated in Section 2.2 by simple analytic arguments. Indeed, the Tully-Fisher zero-point is almost independent of \( m_d \) in the range where \( V_d^2 < V_{\text{max}}^2/2 \); it increases with \( m_d \) at \( V_d^2 > V_{\text{max}}^2/2 \). This behaviour is exactly what is expected from the simple model discussed in Section 2.2. Similar effect is found in the simulations of Navarro & Steinmetz (2000). However, the simulated disks are very compact due to angular momentum transfer from disk material to dark haloes. It is unclear if the small scatter in the Tully-Fisher relation for the simulated disks is compatible with the observed large ranges of disk sizes and surface densities. Indeed, as shown by equation (8), if disks are compact enough to dominate the maximum rotation velocity, the scatter in the Tully-Fisher zero-point should be as large as that in the disk central surface densities.

The distribution in the spin parameter \( \lambda \) also introduces scatters in the Tully-Fisher zero-point. Since changing \( \lambda \) is equivalent to changing the disk concentration in a halo, the effect is smaller for smaller \( m_d \), as implied in equation (12). However, the scatter induced by the \( \lambda \)-distribution is not expected to be large. From N-body simulations we know that the spin parameters of dark haloes have log-normal distribution with median value \( \bar{\lambda} \approx 0.05 \) and dispersion \( \sigma_{\ln \lambda} \approx 0.5 \) (see eq. [15] in MMW). If disks have similar specific angular momenta as their dark haloes, then only 10 percent of the systems have \( \lambda \geq 0.1 \) and 10 percent of them have \( \lambda \leq 0.025 \). Such a spread in \( \lambda \) does not induce too large scatter in the Tully-Fisher relation, as shown in MMW and as can also be inspected from Figures 1 and 2. However, if disks could lose a large fraction of their angular momenta so that their effective spins are much lower than 0.05, they would be too compact and the Tully-Fisher zero-point would be too low. This is why numerical simulations of disk formation generally give too low Tully-Fisher zero points, because in such simulations, disk material loses much of its angular momentum (e.g. Navarro & Steinmetz 2000). Notice that disks with large \( m_d \) and small \( \lambda \), which are predicted to have systematically low Tully-Fisher zero-point, may be unstable, and so the scatter induced by the \( \lambda \)-distribution may be reduced if such systems do not form real disks. But significant loss of angular momentum would then lead to too few stable disks. Another important result shown in Figure 1
Figure 1. (a) The change in Tully-Fisher zero-point for haloes with $V_h = 200 \text{ km s}^{-1}$ and $c = 40$. [Notice that profile (15) with $c = 40$ matches the NFW profile with $c_{NFW} \approx 20$.] Curves from top down have $\lambda = 0.1$ (long-dashed curve), 0.05 (solid curve) and 0.03 (short-dashed). The two horizontal lines bracket the range of $\mathcal{F}$ required to match the observed Tully-Fisher zero-points among different galaxies at $V_{\text{max}} \sim 200 \text{ km s}^{-1}$. Panel (b) is the predicted central luminosity density. Curves from bottom up have $\lambda = 0.1$, 0.05 and 0.03. The two horizontal lines indicate roughly the observed range of $I_0$ in the $I$-band for normal galaxies (the median value is taken to be $I_0 = 300 \text{ L}_\odot \text{ pc}^{-2}$ and the range of surface brightness is taken to be $\pm 1$ magnitude to represent approximately the observed range of Freeman disks). Panel (c) shows the disk contribution to the total rotation. Curves from bottom up have $\lambda = 0.1$, 0.05, and 0.03. The horizontal line separate disk-dominating from halo dominating. Panel (d) shows the disk scale-length. Curves from bottom up have $\lambda = 0.03$, 0.05, and 0.1.

and 2 is that low surface-brightness disks (formed in systems with high $\lambda$ and low $m_d$) have a Tully-Fisher zero-point very similar to that of ‘normal’ disks with surface-brightness close to that of a Freeman disk. Thus, the observational fact that low surface-brightness galaxies obey a Tully-Fisher relation similar to that of normal spiral galaxies (Zwaan et al. 1995, but also see O’Neil, Bothun, & Schombert 1999) is explained here without invoking any subtle assumptions. In reality, however, some offset in the Tully-Fisher zero-point is expected for the low surface-brightness population. Some low surface-brightness disks may contain more cold gas (because of low star formation efficiency) than normal galaxies (e.g. McGaugh & de Blok 1997; O’Neil, Bothun, & Schombert 1999), and so their Tully-Fisher zero-point is expected to be lower because of their higher disk mass-to-light ratios. Our results suggest that low-surface brightness galaxies should obey the same Tully-Fisher relation as high surface-brightness galaxies, if the variation of gas fraction is properly taken into account. This is in agreement with the observational results that the relation between rotation velocity and disk mass is tighter than the relation between rotation velocity and luminosity (e.g. McGaugh 2000).

The change of halo concentration $c$ has quite a large effect on the predicted zero-point. To match with observations, the value of $c$ is required to be low, $c \sim 7$. Similar match can be obtained for the NFW profile with $c_{NFW} \sim 3$. The implied halo concentration is therefore much lower than the value ($c_{NFW} \sim 20$) obtained from N-body simulations of CDM models for
galactic-sized haloes. Since a lower $c$ value means lower concentration of haloes, the result suggests that real galaxy haloes must be much less concentrated than CDM haloes. This low value of $c$ is in fact required by the rotation-curve shapes of galaxies with low luminosity and low surface brightness (e.g. Navarro 1998). The result here suggests that haloes with the same low concentrations are also needed for normal galaxies.

Another source that can cause scatter in the Tully-Fisher relation is the redshift distribution of disk assembly. As one can see from equation (14), for a given halo circular velocity $V_h$, haloes at redshift $z$ are lighter by a factor of $H(z)/H(0)$ than at $z = 0$. If other parameters (i.e. $m_d$, $\lambda$, $c$, and $\eta_d$) are kept the same, disks in haloes with the same $V_h$ have the same $V_{\text{max}}$ without depending on redshift, and so disks assembled at higher redshifts would have lower luminosity. The effect could be quite large. At $z = 1$, $H(z)/H(0)$ is about 2.8 for an Einstein-de Sitter universe, and about 1.8 for a flat universe with $\Omega_0 = 0.3$, $\Lambda = 0.7$. This factor becomes 5.2 and 3.0 at $z = 2$ for these two cosmologies. Thus, unless present disks have quite uniform formation time, the induced scatter would be too large. This problem was noticed by MMW, and they solve it by assuming most of the present disks to be assembled at $z1$. In this case, disk formation in a flat universe with $\Omega_0 = 0.3$, $\Lambda = 0.7$ is compatible with the observed Tully-Fisher zero-point and scatter$^\dagger$.

Here we suggest another possibility. If haloes at higher redshift are less concentrated, we can inspect from Figure 2 that the redshift effect on Tully-Fisher zero-point is reduced. In fact, if halo concentration decreases with redshift $z$ as $[H(z)]^{-\beta}$, with $\beta \sim 0.5 - 1$, then the redshift effect can be removed almost completely for $z2$. This is shown in Figure 3.

$^\dagger$ Notice that the halo concentration used in MMW is lower than that given by recent high-resolution N-body simulations. The disk instability criteria used there also act to increase the Tully-Fisher zero-point.
Figure 3. The same as Figure 1, except that disks here are assumed to be assembled at $z = 2$ in haloes with $V_h = 200 \text{ km s}^{-1}$ and $c = 7[H_0/H(z)]^{1/2}$, $z = 2$. A low-density cosmology, with $\Omega = 0.3$ and $\Lambda = 0.7$, is used in the calculation. Notice that the curves are very similar to those shown in Figure 2, except that disk scale-length is smaller by a factor of about $H(z = 2)/H_0$.

In the model where the observed Tully-Fisher zero-point is reproduced, the distribution in $I_0$ is broader than that of Freeman disks. In particular, the model predicts the existence of low-surface brightness galaxies in systems with high $\Lambda$ and small $m_d$. This is consistent with the fact that galaxy disks with surface brightness lower than that of a Freeman disk are observed in deep photometric observations (Bothun et al. 1997). As one can also see from Figure 2, the contribution of the disk component to the maximum rotation varies significantly. Generally, the maximum rotation velocity is dominated by the disk component in systems with high disk surface brightness, and becomes halo dominated in low surface-brightness systems. There is still intense debate whether observed disk galaxies are maximal or not (e.g., Bottema 1997; Courteau & Rix 1999; Debattista & Sellwood 1998; Englmaier & Gerhard 1999; Bosma 2000). For our own Galaxy, the observed disk scale-length is about 3.5 kpc. The dark matter mass within a radius 10 kpc (which is about 3$R_d$) is about $5 \times 10^{10}$ M$_\odot$ (e.g. Dehnen & Binney 1998). Using a rotation velocity of 220 km s$^{-1}$ at this radius, we have $(V_d/V_{\text{max}})^2 \approx 0.5$, in good agreement with our prediction (see the bottom left panel in Figure 5). If the dark halo of the Milky Way were as concentrated as that predicted by CDM models, then the predicted value of $(V_d/V_{\text{max}})^2$ would be much smaller. Navarro & Steinmetz (1999) used this to argue against CDM models.

Figure 4 shows the ratio between $V_{\text{max}}$ and halo circular velocity $V_h$. As one can see, the boost in the velocity is substantial in systems with high $c$, high $m_d$ and low $\lambda$. For $c = 7$, significant boost occurs for $m_d > 0.05$ and $\lambda < 0.03$, while for $c = 40$ the boost is significant for all values of $m_d$ and $\lambda$ because of the concentrated halo profile.

Our discussion so far has been based on systems with halo circular velocity $V_h = 200 \text{ km s}^{-1}$. Since the model described above also reproduced the Tully-Fisher slope (see MMW), the discussion is also valid for other $V_h$. As a summary, we show in Figure 5 disk properties versus $V_{\text{max}}$ for a Monte-Carlo sample, where the halo circular velocity changes uniformly between 50
Figure 4. The ratio between $V_{\text{max}}$ (disk rotation velocity at $3R_d$) and halo circular velocity $V_h$ in models with (a) $c = 7$ and (b) $c = 40$. Short-dashed, solid, and long-dashed curves show results for $\lambda = 0.03$, 0.05, and 0.1, respectively.

$V_{\text{max}}/V_h$ vs. $m_d$ for $c = 7$ and $c = 40$. The ratio $V_{\text{max}}/V_h$ increases with increasing $m_d$ for both values of $c$. The solid curve represents $\lambda = 0.03$, the short-dashed curve $\lambda = 0.05$, and the long-dashed curve $\lambda = 0.1$.

and 250 km s$^{-1}$, $m_d$ has a uniform distribution from 0.01 to 0.1, $\lambda$ has the log-normal distribution discussed above but with a lower cutoff at 0.03 (the small number of systems with $\lambda < 0.03$ may produce disks that are too compact to be globally stable, see MMW for a discussion), disk assembly redshift has uniform distribution between $z = 0$ and $z = 2$, and the halo concentration $c$ changes with redshift as $c = 7[H_0/H(z)]^{1/2}$. From Figure 5, it is clear that although $m_d$ and formation redshift are allowed to vary in substantial ranges, the scatter around the Tully-Fisher relation is still compatible with observations. The predicted central luminosity density (top right) and disk sizes (bottom right) are also consistent with observations (Courteau 1996, 1997; de Jong 1996). The bottom left panel in Figure 5 shows the disk contribution to $V_{\text{max}}$; it is quite clear that some systems are disk dominated while others are not. Disk domination of $V_{\text{max}}$ preferentially occurs in systems with high $m_d$ and low $\lambda$.

As mentioned above, although the results presented above are based on the special functional form (15) for the halo density profile, our conclusions are not altered if other reasonable forms are used. This is not surprising, because the quantities we are interested in here are global properties of the halo/disk systems. Thus, the most important requirement is that dark haloes have shallow profiles in the inner region, while the exact form of the halo profile is not stringently constrained by the global properties considered here. Detailed modelling of disk rotation curves shows that model (15) may fare better than model (16) (Mao et al. 2000 in preparation).
Figure 5. Various disk properties (obtained by Monte Carlo simulations) plotted vs. $V_{\text{max}}$ (defined to be the circular velocity at three disk scale-lengths). The redshift of disk assembly is assumed to have uniform distribution over 0 to 2, $m_d$ is assumed to have uniform distribution from 0.01 to 0.1, and the spin parameter $\lambda$ follows the log-normal distribution with a lower-cutoff of 0.03. The concentration parameter decreases with redshift as $7[H_0/H(z)]^{1/2}$. A low-density cosmology, with $\Omega_0 = 0.3$ and $\Lambda = 0.7$, is used in the calculation.

The top left panel shows the Tully-Fisher relation, where the solid line indicates the observed relation by Giovanelli et al. (1997). The top right panel shows the central surface luminosity density. The bottom left panel shows the disk contribution to the circular velocity at three disk scale-lengths while the bottom right panel shows the disk scale-length vs. $V_{\text{max}}$.

4 DISCUSSION

In this paper, we have applied the same formalism as in MMW, but with substantially relaxed assumptions, to study the properties of disk galaxies. We find that even if we allow the disk mass and formation redshift to vary substantially, the observed properties (including the Tully-Fisher relation, disk sizes and central luminosities) can still be reproduced, provided that the dark halo profiles are shallow. The requirement of the shallow profile is consistent with direct modelling of rotation curves, particularly for low surface-brightness galaxies; the low central density of dark matter can also explain why bars in galaxies seem to rotate rapidly (Debattista & Sellwood 1998). The required profile is, however, much shallower than those found in numerical simulations (NFW; Jing & Suto 1999; Moore et al. 1999; Springel et al. 2000).

The shallow halo profiles required by disk galaxies have many important observational consequences. For example, the lensing properties of disk galaxies may be different from those in models where concentrated profiles are assumed (e.g. Bartelmann & Loeb 1998). There are already a number of lenses that appear to be caused by disk galaxies, such as 2237+0305 (Huchra et al. 1984), 1600+4344 (Jackson et al. 1995) and 0218+357 (Patnaik et al. 1993). None of these systems shows a central image, which implies that the core radii for these lensing disks may be fairly small. It would be very interesting to use these systems to put quantitative constraints on the core radius. Similarly, the lack of central images in elliptical lenses also suggests that the total density profile (baryons plus dark matter) is near singular in the central region, i.e. the core radius must be small (e.g. Kochanek 1996). An important question is whether the halo profiles of elliptical galaxies have similar core...
radii as disk galaxies. It is possible that the dark halo profiles in elliptical galaxies are significantly modified by dynamical processes during formation (e.g., by merging).

Recent high-resolution N-body simulations of the formation of cold dark matter haloes show that such haloes generally contain too many subclumps to match the number of dwarf galaxies observed in the Local Group. This happens because the CDM particles which form a dark halo were generally in progenitors with high central densities. As such progenitors merge to form a larger halo, their central parts can survive as subclumps (Moore et al. 1999; Klypin et al. 1999; Springel et al. 2000). However, if the progenitors have shallow profiles in their central regions, they are more likely to be destroyed by the merging process, and the number of subclumps in dark haloes can be reduced. Also, if the core radius of halo does not change much with redshifts, we would expect a limiting redshift beyond which dark haloes may not be able to form in large number. This may help to alleviate the overcooling problem in CDM models where too much gas can cool in small haloes at high redshift, but is constrained by the fact that large number of star forming galaxies are observed at redshift $z \geq 3$ (e.g. Steidel et al. 1998). Clearly, many of these issues need thorough investigation before any definite conclusions can be drawn.

An equally important issue is the origin of the shallow profiles. Since cuspy profiles are quite generic in conventional CDM models, the formation of haloes with shallow profiles may require some modification in the properties of the dark matter particle (e.g. Spergel & Steinhardt 1999; Hogan & Dalcanton 2000) or some change in the power spectrum of initial perturbations (Kamionkowski & Liddle 1999).

In the proposal of Spergel and Steinhardt, dark matter is assumed to be self-interacting. The initial hope was that the collisions between dark matter particles can heat up the low-entropy material and thereby produce a shallower density profile. However, recent numerical simulations show that the inner density profiles of collisional haloes are even steeper than their collisionless counterparts (Moore et al. 2000; Yoshida et al. 2000; Burkert 2000). Collisional dark halo might also be too spherical to match clusters like MS21137-23 (Miralda-Escude 2000). In the proposal of Hogan and Dalcanton, galactic haloes are assumed to be dominated by warm dark matter with initial velocity dispersion. As such progenitors merge to contain too many subclumps to match the number of dwarf galaxies observed in the Local Group. This happens because the characteristic radius which decreases with halo circular velocity as $r_c \propto V_h^{-1/2}$. This is not favored by our results. The predicted relation between $r_c$ and $V_h$ implies that the halo concentration factor scales as $r_c/V_h \propto V_h^{1/2}$. Thus, at a given redshift the concentration for a halo with $V_h = 250 \text{ km s}^{-1}$ is about 6 time larger than that for a halo with $V_h = 75 \text{ km s}^{-1}$. From the results shown in Fig. 1a and Fig. 2a we see that the resulting Tully-Fisher relation is much too shallow.

If dark matter self-interaction (either scattering or annihilation) is indeed responsible for the shallow profile of galactic haloes, the Tully-Fisher relation can be used to constrain the mass and cross section of dark matter particles. Denote the mass and cross section by $m_X$ and $(\sigma_X |v|)$, respectively. The collision rate per particle is $\Gamma = n_X \langle \sigma_X |v| \rangle$. Collision is effective only in systems where $\Gamma^{-1} < \text{Hubble time}$. This defines a critical density $n_{\text{crit}} = \frac{H(z)}{\langle \sigma_X |v| \rangle}$, above which the effect of self-interaction is important. The above critical density defines a characteristic radius ($r_c$) in a dark halo: $\rho(r_c) = m_X n_{\text{crit}}$, where $\rho(r)$ is the halo density profile. Suppose that the halo profile before modification by self-interaction is $\rho(r) = V_h^2/(4\pi Gr^2)$ near $r_c$, the characteristic radius can be written

$$r_c = \frac{V_h}{\left(4\pi G m_X n_{\text{crit}}\right)^{1/2}} = \frac{V_h}{\left[4\pi G H(z)\right]^{1/2}} \left(\frac{\langle \sigma_X |v| \rangle}{m_X}\right)^{1/2}.$$  \hfill (18)

If self-interaction of dark matter particles is to reduce the local density of dark matter particles, the characteristic radius $r_c$ may be identified as a ‘core’ radius. The halo concentration is then

$$c \equiv \frac{r_h}{r_c} = \left[\frac{\pi G}{25 H(z)}\right]^{1/2} \left(\frac{\langle \sigma_X |v| \rangle}{m_X}\right)^{-1/2}.$$  \hfill (19)

Thus, if $\langle \sigma_X |v| \rangle$ is velocity independent in the velocity range relevant for galactic haloes, halo concentration $c$ is proportional to $1/\sqrt{H(z)}$, independent of $V_h$. This is just the profile we want to explain the Tully-Fisher relation. In order to have a concentration $c = 7$ [assuming profile (15)], the mass and cross section should satisfy

$$\frac{\langle \sigma_X |v| \rangle}{m_X} \sim 10^{-16} \text{ cm}^{-1} \text{ s}^{-1} \text{ GeV}^{-1}.$$  \hfill (20)

The value of $\sigma_X$ implied is consistent with that obtained by Spergel & Steinhardt (1999) and Firmani et al. (2000) based on different arguments. Much work remains to be done to see if a consistent model can be found to fulfill the requirement.

Acknowledgments

We thank Gerhard Börner, Ian Browne, Karsten Jedamzik, Peter Wilkinson for helpful discussions.
REFERENCES

Avila-Reese, V., Firmani C., Hernandez, X. 1998, 505, 37
Bosma A., 2000, in F. Combes et al. eds, XVth IAP Meeting Dynamics of Galaxies. ASP Conference Series Vol. 197, p91
Heavens A.F., Jimenez R., 1999, 305, 770
Springel V. et al., 2000, in preparation

This paper has been produced using the Blackwell Scientific Publications \LaTeX\ style file.