Asymptotic Dynamics in Quantum Field Theory
When does the coupling switch off?

Robin Horan\textsuperscript{2}, Martin Lavelle\textsuperscript{3} and David McMullan\textsuperscript{4}

School of Mathematics and Statistics
The University of Plymouth
Plymouth, PL4 8AA
UK

Abstract: We discuss the approach to asymptotic dynamics due to Kulish and Faddeev. We show that there are problems in applying this method to theories with four point interactions. The source of the difficulties is identified and a more general method is constructed. This is then applied to various theories including some where the coupling does switch off at large times and some where it does not.

Introduction:

In most textbook descriptions of scattering it is tacitly assumed that the coupling ‘switches off’ at large times. This is sometimes dignified with the name ‘adiabatic approximation’. The LSZ formalism, which forms the basis for the description of scattering processes in quantum field theory, thus assumes that at large times, particles obey a free dynamics. When this is not justified, we have to pay the price of infra-red divergences. However, there is a body of work concerned with investigating the asymptotic interaction Hamiltonian and whether or not the above assumption is justified. In this talk we will describe a general method for the study of the asymptotic interaction and apply it to some concrete examples.

The physical importance of this problem is clear. It is well known that in gauge theories, such as QED and QCD, this assumption is incorrect and that the physics is characterised by long range interactions. In QED, for example, the masslessness of the photon means that the potential between static charges falls off only as $1/r$. It is well known that perturbative calculations of the S-matrix for the unbroken gauge theories of the Standard Model have infra-red divergences.

It is important that we have a precise general description of the scattering process in quantum field theory so that when looking at theories such as QCD and other theories

\textsuperscript{1}Talk presented by R. Horan
\textsuperscript{2}email: rhoran@plymouth.ac.uk
\textsuperscript{3}email: mlavelle@plymouth.ac.uk
\textsuperscript{4}email: dmcmullan@plymouth.ac.uk
of the Standard Model, where our intuition is not so developed and greater reliance must be placed on the mathematics, we may have confidence in the correctness of our deductions. In order to gain this confidence, our methodology must first be tested against those theories that are well understood. Only then can we try to understand the asymptotic dynamics which determines hadronic spectra.

In [1], working in the context of non-relativistic Coulombic scattering, Dollard showed how the asymptotic dynamics could be described by replacing the interaction Hamiltonian for the theory with a different Hamiltonian, which we can think of as being the *asymptotic* Hamiltonian. Subsequently this approach was studied in a relativistic setting, (for QED) by Kulish and Faddeev [2] and this approach has been utilised by various other authors in QED and QCD, with varying degrees of success, for references see [3]

We shall start with the best approach to asymptotic dynamics previously available, that of Kulish and Faddeev [2]. We will give a brief outline of the method described in their article. We will then show that there are difficulties which arise when trying to apply this method to theories other than QED, which is the theory investigated in their article.

After this we shall describe our general approach to asymptotic dynamics [3] which is based upon that of Kulish and Faddeev but with two principal modifications: we work at the level of matrix elements rather than operators and we note that the asymptotic interaction can only be expected to vanish if the particles are at asymptotic times a long way away from each other.

The initial description of our approach to the study of asymptotic dynamics will be in the context of $\phi^4$ interaction theory, after which we shall look at some applications of our method, in particular to the three and four point interactions of *scalar* QED. Finally we shall draw some conclusions and discuss some possible further avenues of investigation.

**A Brief Outline of the Kulish-Faddeev Method**

Our starting point is the usual interaction Hamiltonian for QED

$$\mathcal{H}_{\text{int}}(t) = -e \int d^3 x \ A_\mu(t, x) J^\mu(t, x).$$

Working in the interaction picture we may insert the plane wave expansions

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \bigl\{ b(p, s) u^s(p)e^{-ip\cdot x} + d^s(p, s) \bar{v}^s(p)e^{ip\cdot x} \bigr\},$$

and

$$A_\mu(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} \bigl\{ a_\mu(k)e^{-ik\cdot x} + a_\mu^*(k)e^{ik\cdot x} \bigr\},$$

where $E_p = \sqrt{|p|^2 + m^2}$ and $\omega_k = |k|$ are the usual energy terms.

Following Kulish and Faddeev we now substitute the plane wave expansions of $\psi$ and $A_\mu$ into the interaction Hamiltonian and integrate out the spatial variable $x$,
obtaining a momentum \( \delta \)-function. After integrating out this \( \delta \)-function, the resulting integrals may be conveniently grouped according to the frequency components. Each integral will have a time dependence of the form \( e^{i\Phi t} \), where \( \Phi \) is made up of sums and differences of the energy terms.

The substance of the Kulish-Faddeev method is then contained in the following claims [2]:

- Those terms in which \( \Phi \) never vanish, such as the term of the form
  \[
  \Phi = E_{p+k} + E_p + \omega_k ,
  \]
  decrease sufficiently rapidly as \( t \to \pm \infty \) and the coupling may here be set to zero.

- Those terms which can vanish for some values of the momenta, e.g. the term
  \[
  \Phi = E_{p+k} - E_p - \omega_k ,
  \]
  which is zero for vanishing \( k \), determine the desired behaviour of \( \mathcal{H}_{\text{int}} \). If there are such terms then we cannot invoke the ‘adiabatic approximation’.

It is of course highly gratifying here to note that this term only vanishes for soft photon momenta – which one immediately identifies as the source of the infra-red divergences of the LSZ scheme in QED. In a minor extension of the above calculation, one can show that giving the photon a small mass would, according to Kulish and Faddeev let us put the coupling asymptotically to zero and indeed that step, as is well known, regulates infra-red divergences.

Before one can apply this to other gauge theories, and especially QCD, it seems reasonable to test these claims by applying the method to some toy models. The most straightforward examples of quantum field theories are massive scalar theories. They are textbook examples with no infra-red problems. Indeed for the simple case of \( \phi^3 \) theory, which as far as the above argument is concerned is similar to QED with a photon mass, the method would imply that the coupling may be asymptotically set to zero. This happy situation, though, breaks down when we go to a four point interaction.

**Application of the Kulish-Faddeev Method to \( \phi^4 \) Theory**

In this theory the interaction Hamiltonian is

\[
\mathcal{H}_{\text{int}} = \frac{\lambda}{4!} \int d^3x : \phi^4(x,t) :,
\]  

where the free field expansion for \( \phi \) is just

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} (a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x} ) .
\]

Following the above procedure in this case, results in 12 terms, one of which has the form

\[
\Phi = E_p + E_q - E_k - E_{p+q-k} .
\]
The Kulish-Faddeev method then implies that problems may occur if $\Phi = 0$ has any solutions. However, it is easy to show [3] that this equation has a continuation of solutions!

From this we conclude that the vanishing of $\Phi$ tells us nothing about the asymptotic behaviour of a scattering process. This should not be surprising when one recalls that the Kulish-Faddeev method is concerned with the operator convergence of the interaction Hamiltonian. It is well known (see p. 126 of [4] or Sect. 5-1-2 of [5]) that if the interaction Hamiltonian converges to zero in norm, then all the fields are free fields. We should therefore work at the level of matrix elements, and this we now proceed to do.

**General Approach to Asymptotic Dynamics**

We consider the matrix element of the $\phi^4$ scattering process which involves two incoming and two outgoing wave packets. The incoming and outgoing wave packets are given by the following expressions:

$$
\Psi_{\text{IN}} = \int d^3r d^3w f(r)g(w)a_\uparrow(r)a_\uparrow(w)|0>,
$$

and

$$
\Psi_{\text{OUT}} = \int d^3u d^3v h(u)i(v)a_\uparrow(u)a_\uparrow(v)|0>,
$$

where $f, g, h, i$ are test functions for the respective incoming/outgoing particles.

The matrix element $<\Psi_{\text{OUT}}^\dagger | H_{\text{int}}(t) | \Psi_{\text{IN}}>\) is then proportional to

$$
\int d^3p d^3q d^3k h(k)i(p + q - k)f(p)g(q)e^{-it\Phi},
$$

where the exponent $\Phi$ has the value

$$
\Phi = E_p + E_q - E_k - E_{p+q-k}.
$$

Note that this exponent is the one which caused a problem in our attempt to apply the Kulish-Faddeev method to $\phi^4$. Unlike that case, the expression is now a genuine integral, i.e. a c-number, so we may use elementary methods to investigate the large time behaviour. We shall apply the method of stationary phase [3].

This says that, provided there are no points in the region of integration at which all first order partial derivatives of $\Phi$ vanish, i.e. there are no stationary points, then the integral vanishes as $t \to \pm \infty$.

The partial derivatives of $\Phi$ in this case are

$$
\frac{\partial \Phi}{\partial p_i} = \frac{p_i}{E_p} - \frac{p_i + q_i - k_i}{E_{p+q-k}},
$$

$$
\frac{\partial \Phi}{\partial q_j} = \frac{q_j}{E_q} - \frac{p_j + q_j - k_j}{E_{p+q-k}},
$$

$$
\frac{\partial \Phi}{\partial k_n} = \frac{k_n}{E_k} - \frac{p_n + q_n - k_n}{E_{p+q-k}}.
$$
The simultaneous vanishing of these implies

\[
\frac{p}{E_p} = \frac{q}{E_q}.
\] (14)

Since there are no infra-red divergences in this theory, we need to see that all the points in the region of this integration for which this last equation holds can be excluded.

We now make the following observations:

- The test functions \( f, g \) for the incoming wave packet, have the arguments \( p, q \) respectively, in the expression \( \langle \Psi_{\text{OUT}}^* | \mathcal{H}_{\text{int}}(t) | \Psi_{\text{IN}} \rangle \)

- The expressions \( p/E_p \) and \( q/E_q \) represent the *velocities* of the particles in the incoming wave packet.

The desired asymptotic behaviour will thus be obtained if the supports of \( f, g \) are defined in such a way such that they exclude the possibility that \( p/E_p = q/E_q \). The precise statement for this is

The test functions \( f, g \) must have non-overlapping support in velocity space.

This condition on the test functions, of having non-overlapping support in velocity space, is, however, central to the LSZ-formalism and the construction of the S-Matrix [3]. We therefore see that the conditions required by LSZ fully suffice to guarantee vanishing asymptotic dynamics for massive \( \phi^4 \) theory. This is in complete accord with perturbative calculations in that theory being free of infra-red divergences.

However, we are discounting the following situation where although the incoming particles are separated, the outgoing particles are clearly not free at large times

\[
\begin{align*}
\text{Momentum} & : \quad p + q = 2k \\
\text{Energy} & : \quad E_p + E_q = 2E_k,
\end{align*}
\] (15)

A simple mathematical argument [3] may be used to show that these two conditions are incompatible with the separation of the incoming particles. Other matrix elements associated with this four-point interaction may be treated in a similar fashion. This approach shows that for massive \( \phi^4 \) theory we have vanishing asymptotic dynamics, and this is consistent with the theory having no infra-red problems.
Asymptotic Dynamics of Scalar QED

We shall now apply this approach [3] to scalar QED. We know that scalar QED has both three and four point interactions. Its perturbation theory tells us that it has the same infra-red structure as fermionic QED – since the infra-red divergences can be shown to be independent of the spin of the matter fields as long as these are massive. We would therefore expect its four point interaction term to have trivial asymptotic dynamics. This is reminiscent of what we have just seen, but now two of the fields (the $A^\mu$’s) are massless.

The interaction Hamiltonian for scalar QED is then

$$H_{\text{int}}(t) = -e \int d^3x \ : J^\mu(x) A_\mu(x) :$$

where the current $J^\mu$ is given by

$$J^\mu = i (\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi) - eg^{\mu \nu} A_\nu \phi^\dagger \phi .$$

The plane wave expansion for $\phi$ is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (a(p)e^{-ip\cdot x} + b^\dagger(p)e^{ip\cdot x}) .$$

We now want to consider (scalar) electron-photon scattering as shown in the diagram.

We shall therefore examine the asymptotic dynamics of the matrix element corresponding to the following wave packets:

$$\Psi_{\text{IN}} = \int d^3r d^3w \ f(r)b^\dagger(r)c^\prime(w)a^\dagger_\nu(w)|0> ,$$

$$\Psi_{\text{OUT}} = \int d^3u d^3v \ g(u)b^\dagger(u)h^\mu(v)a^\dagger_\mu(v)|0> .$$

Here $f, c^\prime, g$ and $h^\mu$ are the respective test functions. The matrix element under consideration $<\Psi^\dagger_{\text{OUT}}|H_{\text{int}}(t)|\Psi_{\text{IN}}>\text{ is then proportional to}$

$$\int d^3p d^3q d^3k c^\dagger(p)h_\nu(q)g(k)f(k + q - p)e^{i\Phi} ,$$

where the phase function becomes

$$\Phi = \omega_p + E_{k+q-p} - E_k - \omega_q .$$
We now make the following deductions concerning this matrix element: physically the incoming and outgoing charged fields must be separated; the test functions $c_i, h_i$ must then automatically have disjoint supports; therefore at least one of them, say $c_i$, will then \emph{not have zero in its support}. With this condition the function

\[ \Phi = \omega_p + E_{k+q-p} - E_k - \omega_q, \]

(23)

now has continuous partial derivatives in both the $p$ and $k$ variables.

We can therefore apply the method of stationary phase to the integral (but now with respect to the $p$ and $k$ variables only).

The partial derivatives of $\Phi$ are

\[ \frac{\partial \Phi}{\partial p_i} = \frac{p_i}{\omega_p} - \frac{k_i + q_i - p_i}{E_{k+q-p}}, \]

\[ \frac{\partial \Phi}{\partial k_j} = \frac{k_j + q_j - p_j}{E_{k+q-p}} - \frac{k_j}{E_k}. \]

(24)

If there is a point at which \emph{all} of these vanish then we would require:

\[ \frac{p}{\omega_p} = \frac{k}{E_k}. \]

(25)

This is, however, \emph{impossible}, since the first vector has unit length but the second does not, so the integral must indeed vanish as $t \to \pm \infty$. (Other matrix elements associated with the four point interaction term may, we note, be treated in exactly the same manner.)

This shows us that in scalar QED the four point interaction does not survive asymptotically. As mentioned above, this is completely consistent with our knowledge of perturbation theory for scalar QED since loops involving this vertex do not generate infra-red divergences.

\textbf{Scalar QED: the Infra-Red Approximation}

Having seen which vertices do \emph{not} survive asymptotically, it is time to turn to those that do. Infra-red divergences in perturbative calculations indicate when the coupling does not switch off. In such cases, we would expect this to arise from our formalism.

Since perturbative calculations tell us that the infra-red problem in the scalar theory is the same as that for \emph{fermionic} QED, we would expect the asymptotic properties to be the same. We now look at the asymptotic behaviour of the scattering process associated with the emission of a soft photon.
The wave packets for this scattering process are taken to be

\[ \Psi_{\text{IN}} = \int d^3y \ f(y) b^\dagger(y) |0\rangle, \]  
\[ \Psi_{\text{OUT}} = \int d^3u d^3v \ g(u) b^\dagger(u) h^\mu(v) a^\dagger_\nu(v) |0\rangle. \]

The associated matrix element is then \(<\Psi_{\text{OUT}}^\dagger | \mathcal{H}_{\text{int}}(t) | \Psi_{\text{IN}}\rangle\) and when the appropriate calculations are made, it will be found that this expression is

\[ -e \int d^3q \ d^3k f(q + k) g(q) q^\mu h^\mu(k) e^{-i\Phi t}, \]

where \( \Phi = E_{q+k} - E_q - \omega_k \).

Rather than trying to see if this vanishes, we will show that the difference between this interaction and another, simpler one does. Consider then the interaction in which, instead of \( J^\mu(x) \), we take the asymptotic current given by the expression

\[ i \int \frac{d^3q}{(2\pi)^3} \left( \frac{1}{2E_q} \right)^2 b^\dagger(q) b(q) q^\mu \delta^3(x - \frac{q}{E_q} t), \]

This current has a much simpler dynamics and a simple classical interpretation [2].

The Hamiltonian associated with this asymptotic current is

\[ \mathcal{H}_{\text{int}}^\text{as} = -e \int d^3x \ J^\mu_{\text{as}}(x) A^\mu(x), \]

and the corresponding matrix element

\[ <\Psi_{\text{OUT}}^\dagger | \mathcal{H}_{\text{int}}^\text{as}(t) | \Psi_{\text{IN}}\rangle, \]

is given by

\[ -e \int d^3q \ d^3k f(q) g(q) q^\mu h^\mu(k) e^{-i\Phi't}, \]

where \( \Phi' = q \cdot k / E_q - \omega_k \).

We have proven the following [3]:

**Theorem:**

The matrix element

\[ <\Psi_{\text{OUT}}^\dagger | \mathcal{H}_{\text{int}}(t) - \mathcal{H}_{\text{int}}^\text{as}(t) | \Psi_{\text{IN}}\rangle, \]

vanishes as \( t \to \pm \infty \).

The interaction Hamiltonian therefore has, asymptotically, a simpler form. This theorem, together with the previous result, tells us that in scalar QED only the three point interaction has non-vanishing asymptotic dynamics and its weak limit is exactly the same as that for spinorial QED, which is indeed what perturbation theory tells us.
Conclusions

The Kulish-Faddeev approach [2] to the study of asymptotic dynamics can be made [3] precise and applicable to theories with four point interactions but we must work at the level of matrix elements with appropriate non-overlap between incoming and outgoing states. This is exactly consistent with the LSZ formalism, where the separation is a precondition of the theory.

We have seen that there is no need to make the ‘naive adiabatic approximation’. It is possible to determine, from the formalism, if the interaction Hamiltonian switches off. This was argued by Kulish and Faddeev [2] for QED, where physical intuition tells us that we should not expect to be able to ‘switch off’ the coupling in the asymptotic region. This is because the Coulomb interaction of QED in $d = 4$ has a slow $1/r$ fall-off. However, the Kulish-Faddeev argument for when the asymptotic interaction Hamiltonian vanishes is flawed: in particular we have seen that it predicts that we may not switch off the coupling in massive $\phi^4$ theory! However, we know that both the S-matrix and the on-shell Green’s functions of that theory are well defined.

The source of this problem is, we have shown, that one should not demand such strong restrictions on the operators. Rather we need to consider matrix elements. A second lacuna in the Kulish-Faddeev approach is that they do not require, at large times, the separation between particles to become large.

We were able to show that with these requirements, we could explain the asymptotic behaviour of several theories (abelian gauge theories and massive scalar theories).

In those theories where the asymptotic interaction Hamiltonian does not vanish, the form of the asymptotic limit is in general hard to obtain. For QED we employed the Ansatz which was given by Kulish and Faddeev and were able to show rigorously, using our approach, that the weak limit form of their result is correct.

I recall that the non-vanishing of the coupling at large times means that gauge transformations remain non-trivial. The physical fields are therefore much richer than the basic Lagrangian fields. It has been shown in D. McMullan’s talk that using the correct physical fields the coupling does effectively switch off. It was then shown in M. Lavelle’s talk that the use of these fields removes infra-red divergences in QED.

Two natural theories where this method ought to be applied are massless QED and QCD. The former is known to have a richer asymptotic dynamics which, unlike massive QED, is spin dependent. In particular it has collinear divergences which are also a prominent feature of QCD. The Kulish-Faddeev approach has been applied to massless QED [6,7] and the resulting Ansatz for the asymptotic interaction Hamiltonian can be inserted into our method. For QCD there is still much more work to be done, but, since this asymptotic dynamics is responsible for holding the hadrons together as they approach particle detectors, it is clear that finding its exact form would be of huge importance.

Acknowledgements: I thank the Royal Society for a travel grant to attend this workshop and the local organisers for their hospitality.
References


