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ASSOCIATED TO A WORD HYPERBOLIC GROUP

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Symbolic coding for the geodesic flow associated to a word hyperbolic group

by

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Abstract.— Let Γ be a word hyperbolic group. Gromov has constructed a compact space $\overline{G} = \overline{G}(\Gamma)$ equipped with a flow which is defined up to orbit-equivalence and which is called the geodesic flow of Γ . In the special case where Γ is the fundamental group of a Riemannian manifold of negative sectional curvature, $\overline{G}$ is the unit tangent bundle of the manifold equipped with the usual geodesic flow.

In this paper, we construct, for every hyperbolic group Γ , a subshift of finite type and a continuous map from the suspension of this subshift onto $\overline{G}(\Gamma)$, which is uniformly bounded-to-one and which sends each orbit of the suspension flow onto an orbit of the geodesic flow.

0.— Introduction and statement of results.

The geodesic flow associated to a Riemannian manifold M is the flow $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ on the space of (local) geodesics $g : \mathbb{R} \rightarrow M$ defined by $\mathcal{F}_t g(u) = g(t + u)$ for all t and $u \in \mathbb{R}$. In the case where M is a compact surface with negative curvature, the discovery of symbolic codings for the geodesics in M , that is, codings by bi-infinite sequences of symbols belonging to a finite alphabet, has led to a remarkable progress in the understanding of the behaviour of this flow. Such symbolic codings have been introduced by J. Hadamard

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in the paper [Had], which was published just one century ago. M. Morse used symbolic codings for geodesic flows in two famous papers ([Mor1] and [Mor2], see also [Bot] and [Ser] for a survey of the results of Morse).

One of the results of Morse is that the geodesic flow on a pair of pants equipped with a metric of negative curvature with geodesic boundary is orbit-equivalent to the suspension flow on the mapping torus of a two-sided subshift of finite type. More precisely, let S be a compact surface homeomorphic to a pair of pants, that is, to a sphere with three disjoint open disks removed (see Figure 1), and let S be equipped with a Riemannian metric of negative curvature with geodesic boundary. Consider a finite set (called alphabet) consisting of four symbols, $A = \{a_1, a_2, a_1^{-1}, a_2^{-1}\}$ and let $\Sigma \subset A^{\mathbb{Z}}$ be the set of sequences $(x_n)_{n \in \mathbb{Z}}$ such that for all $n \in \mathbb{Z}$, we have $x_n \in A$ and $\{x_n, x_{n+1}\} \neq \{a_i, a_i^{-1}\}$, $i = 1, 2$. We equip the set Σ with the topology of pointwise convergence. The shift map $T : \Sigma \rightarrow \Sigma$ defined by $(x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$ is a homeomorphism. The dynamical system (Σ, T) is a *subshift of finite type* (we shall recall the definition of such an object in Section 3 below). Let $\text{Susp}(\Sigma, T)$ denote the suspension (or mapping torus) of the map $T : \Sigma \rightarrow \Sigma$. In other words, $\text{Susp}(\Sigma, T) = \Sigma \times \mathbb{R} / \mathbb{Z}$, where $\mathbb{Z}$ acts on $\Sigma \times \mathbb{R}$ by the rule $n(x, u) = (T^n x, u - n)$. The space $\text{Susp}(\Sigma, T)$ is equipped with a natural flow $(\varphi_t)_{t \in \mathbb{R}}$, defined as the quotient flow of the product flow on $\Sigma \times \mathbb{R}$ which acts by the identity on the first factor and by translations on the second factor. The flow φ_t on $\text{Susp}(\Sigma, T)$ is called the *suspension flow*. We note by the way that this suspension flow has an interpretation as the geodesic flow on the “figure 8” graph (see Figure 2), that is, the graph having one vertex and two edges of length one attached to this vertex (we shall see this fact in detail in Section 5 below). The result of Morse which we referred to above states that there is a homeomorphism between $\text{Susp}(\Sigma, T)$ and the space of the geodesic flow on S , which is an *orbit equivalence*, that is, a homeomorphism which sends bijectively each orbit of the suspension flow on $\text{Susp}(\Sigma, T)$ onto an orbit of the geodesic flow.

Figure 1

An important step in the theory of codings by suspensions of subshifts of finite type for geodesic flows on compact negatively curved Riemannian manifolds of arbitrary dimension is based on the work done by D. Anosov in the 1960s (see [Ano]). In fact, this work applies

Figure 2

to a wider class of flows which have been called since then *Anosov flows*. R. Bowen used Markov partitions to construct symbolic codings by suspensions of subshifts of finite type for Anosov flows and therefore, in particular, for geodesic flows on Riemannian manifolds of negative curvature (see [Bow]).

Morse's result implies that the geodesic flow on our pair of pants S is, up to orbit equivalence, independent of the choice of the Riemannian metric of negative curvature and geodesic boundary. This result is in fact a special case of a rigidity theorem which is due to M. Gromov and which is contained in the paper [Gro1]. In this paper, Gromov proves that the geodesic flow on a compact manifold of arbitrary dimension equipped with a Riemannian metric of negative curvature depends only, up to orbit equivalence, on the fundamental group of the manifold. Thus, the geodesic flow is, up to orbit equivalence, a topological invariant for groups which can be realized as fundamental groups of compact manifolds of negative curvature. This result of Gromov is also contained in [Gro2] (Corollary 8.3.E). More generally, in the paper [Gro2], Gromov introduces the notion of a word hyperbolic group, and he associates to each such group Γ a space $\overline{G}(\Gamma)$ equipped with a flow $\overline{\mathcal{F}}$, which is defined up to orbit equivalence, and which Gromov calls the *geodesic flow* associated to Γ . In the case where Γ is the fundamental group of a compact Riemannian manifold M of negative curvature, then Γ is word hyperbolic and $(\overline{G}(\Gamma), \overline{\mathcal{F}})$ is, up to orbit equivalence, the geodesic flow on the tangent bundle of M .

In what follows, we study, following the ideas sketched by Gromov in Section 8.5.Q of [Gro2], a symbolic coding for the geodesic flow $(\overline{G}(\Gamma), \overline{\mathcal{F}})$.

The main result that we prove in this paper is the following (see Corollary 5.9 below)

Theorem 0.1.— Let Γ be a word hyperbolic group. Then, there exists a two-sided subshift of finite type (Σ, T) and a continuous map from the suspension $\text{Susp}(\Sigma, T)$ of this subshift onto $\overline{G}(\Gamma)$, which is uniformly bounded-to-one and which sends each orbit of the suspension flow onto an orbit of the geodesic flow.

The two-sided subshift of finite type (Σ, T) which we construct to prove Theorem 0.1 is a geometric object associated to the Cayley graph X of the group Γ with respect to a finite ordered symmetric set of generators. The elements of the subshift consist in $\mathbb{Z}$ -orbits of equivalence classes of horofunctions on X (two horofunctions being equivalent whenever they differ by a constant). To prove that $\text{Susp}(\Sigma, T)$ has the required properties, we use the work that we did in [CP2], where we proved that the space of equivalence classes of horofunctions has a natural structure of a one-sided subshift of finite type.

The plan of this paper is the following :

Section 1 contains background material on hyperbolic groups, horofunctions and their associated cocycles.

In Section 2, we use the space Φ of cocycles to construct three invertible dynamical systems. $(\bar{\Phi}, \bar{\alpha})$, $(\tilde{\Phi}/\Gamma, \alpha^*)$ and $(\tilde{\Phi}_0, \tilde{\alpha}_0)$ which are topologically conjugate.

In Section 3, we prove that the three dynamical systems defined in Section 2 are topologically conjugate to a two-sided subshift of finite type.

In Section 4, we introduce the space $\tilde{\Phi}_{\mathbb{R}}$ which is equipped with a Γ -action and a flow. We prove that there exists a natural topological conjugacy between the quotient space $\bar{\Phi}_{\mathbb{R}} = \tilde{\Phi}_{\mathbb{R}}/\Gamma$ equipped with its quotient flow $\mathcal{F}^*$ and the suspension of the two-sided subshift of finite type $(\bar{\Phi}, \bar{\alpha})$ equipped with its suspension flow.

In Section 5, we define a map $\bar{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \bar{G}$ and we prove that this map is continuous, surjective, uniformly finite-to-one and that it sends each $\mathcal{F}^*$ -orbit of $\tilde{\Phi}_{\mathbb{R}}/\Gamma$ onto an $\bar{\mathcal{F}}$ -orbit of $\bar{G}$. As a corollary, we obtain Theorem 0.1 stated above. This section ends with a detailed study of two particular cases of the coding, namely the case where Γ is a finitely generated free group and the case where $\Gamma = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Since there are many notations in this paper, we have included at the end, to help the reader, and index of all the maps that we define.

1.— Horofunctions and cocycles on hyperbolic groups.

Let us recall a few basic notions about hyperbolic groups. For the details, we refer the reader to [Gro2], [CDP] and [GH].

We start with the definition of a δ -hyperbolic metric space. Let E be a metric space. The distance between two points x and $y \in E$ will be denoted by $|x - y|$. Let x_0 be a basepoint of E . For every x and $y \in E$, the *Gromov product of x and y with respect to x_0* , denoted by $(x \cdot y)_{x_0}$, is defined by the formula

$$(x \cdot y)_{x_0} = \frac{1}{2}(|x - x_0| + |y - x_0| - |x - y|).$$

Let δ be a nonnegative real number. The space E is said to be *hyperbolic* (or δ -hyperbolic) if for every choice of a basepoint x_0 and for every x, y and z in E , we have

$$(x \cdot y)_{x_0} \geq \min((x \cdot z)_{x_0}, (y \cdot z)_{x_0}) - \delta.$$

Let Γ be a finitely generated group and let A be a finite set of generators for Γ . We assume that A is symmetric, that is, for each $a \in A$, we have $a^{-1} \in A$. We assume also that the identity element $Id \in \Gamma$ does not belong to A . The *Cayley graph* X associated to (Γ, A) is then the 1-dimensional simplicial complex with set of vertices $X^0 = \Gamma$ and set of edges X^1 consisting of the subsets of the form $\{x, y\} \subset \Gamma$, where x and y satisfy $y = xa$ for some element $a \in A$. The *label* of an oriented edge (x, y) is the element a of A such that

$y = xa$. The Cayley graph X has a natural structure of a geodesic metric space, obtained by making each edge isometric to the interval $[0, 1]$ and defining then the distance between two arbitrary points in X as the length of a shortest curve joining them.

The group Γ is said to be δ -hyperbolic with respect to A if the Cayley graph X is a δ -hyperbolic metric space. A *word hyperbolic group*, or simply a *hyperbolic group*, is a group which is δ -hyperbolic for some $\delta \geq 0$ with respect to some finite generating set A . We recall that if Γ is hyperbolic with respect to some finite generating set, then it is hyperbolic with respect to any other choice of a finite generating set (but with a different constant δ).

We recall now a few definitions and results from the paper [CP2] on horofunctions, gradient rays and related matters.

Definition 1.1 (Horofunctions).—A *horofunction* on X is a function $h : X \rightarrow \mathbb{R}$ satisfying the following three properties:

(i) “Quasi-convexity property”: There exists $\epsilon \geq 0$ such that h is ϵ -convex. In other words, for every geodesic segment $[x_0, x_1] \subset X$ and for every $t \in [0, 1]$, we have

$$f(x_t) \leq (1 - t)f(x_0) + tf(x_1) + \epsilon.$$

where x_t is the point on $[x_0, x_1]$ satisfying $|x_0 - x_t| = t|x_0 - x_1|$.

(ii) “Distance-like property”: For every $x \in X$ and for every λ satisfying $h(x) \geq \lambda$, we have

$$h(x) = \lambda + \text{dist}(x, h^{-1}(\lambda)).$$

The level set $h^{-1}(\lambda)$ is called the *horosphere of radius λ* associated to the horofunction h , and Property (ii) says in other words that for every $\lambda_1 \geq \lambda_2$, every point on the horosphere of radius λ_1 is at distance $\lambda_1 - \lambda_2$ from the horosphere of radius λ_2 .

(iii) The function h takes integral values on the vertices of X , that is, $h(X^0) \subset \mathbb{Z}$. (This condition implies that all the horofunctions that we consider here are *integral horofunctions* in the sense of [CP1] and [CP2].)

Definition 1.2 (Cocycles).—A function $\varphi : X \times X \rightarrow \mathbb{R}$ is called a *cocycle* if there exists a horofunction $h : X \rightarrow \mathbb{R}$ such that for every x and $y \in X$, we have $\varphi(x, y) = h(x) - h(y)$. We call such a function h a *primitive* for φ , and we say that φ is the *cocycle of h* .

Definition 1.3 (Gradient arcs).—Let φ be a cocycle on X . A φ -*gradient arc* is a path $g : I \rightarrow X$ parametrized by arclength and satisfying $\varphi(g(t), g(t')) = t' - t$ for every t and t' in I . In the case where $I = \mathbb{R}$, we say that g is a *gradient line*. In the case where $I = [0, \infty[$, we say that g is a *gradient ray*. If g is a gradient ray and $x = g(0)$, then we say that g *starts at x* .

Let Φ be the set of cocycles on X , equipped with the topology of uniform convergence on compact sets. The space Φ is compact and metrizable and the group Γ acts continuously on Φ by the rule $(\gamma, \varphi) \mapsto \gamma\varphi$, where $\varphi \in \Phi$, $\gamma \in \Gamma$ and where $\gamma\varphi \in \Phi$ is defined by

$$\gamma\varphi(x, y) = \varphi(\gamma^{-1}x, \gamma^{-1}y)$$

for x and $y \in X$.

We recall that every φ -gradient arc is in particular a geodesic arc and that for every $\varphi \in \Phi$, each φ -gradient ray $g : [0, \infty[\rightarrow X$ converges to a point in the boundary at infinity $\partial X = \partial \Gamma$ which is independent of the choice of g . This point is called the *point at infinity of φ* , and is denoted by $\varphi(\infty)$.

We fix for all the rest of this paper a total order relation on the generating set A .

Definition 1.4 (The map $\alpha : \Phi \rightarrow \Phi$).— We define the map $\alpha : \Phi \rightarrow \Phi$ by setting $\alpha(\varphi) = a^{-1}\varphi$ for $\varphi \in \Phi$, where $a = a(\varphi)$ is the smallest element in A satisfying $\varphi(Id, a) = 1$.

The order relation on A induces a natural order relation on the set $A^{\mathbb{N}}$ (the lexicographic order). Let $x \in X^0 = \Gamma$. A geodesic ray starting at x is completely determined by its associated infinite sequence of labels of consecutive edges. Thus, the lexicographic order on $A^{\mathbb{N}}$ induces an order relation on the set of φ -gradient rays starting at x . The set of φ -gradient rays starting at x has a smallest element, which is called *the smallest φ -gradient ray starting at x* .

In this paper, we shall make use of several basic properties of φ -gradient rays and smallest φ -gradient rays, which are all explained in the paper [CP2].

2.— The dynamical system $(\overline{\Phi}, \overline{\alpha})$.

In the rest of this paper, Γ is a word hyperbolic group equipped with a finite symmetric ordered set of generators A , and X is the corresponding Cayley graph.

In this section, we describe three homeomorphisms $\overline{\alpha} : \overline{\Phi} \rightarrow \overline{\Phi}$, $\alpha^* : \tilde{\Phi}/\Gamma \rightarrow \tilde{\Phi}/\Gamma$ and $\tilde{\alpha}_0 : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}_0$ which are associated to the group Γ together with its ordered set of generators A , and we show that these three homeomorphisms are topologically conjugate in a natural way. These homeomorphisms are derived from the map $\alpha : \Phi \rightarrow \Phi$ defined on the space Φ of cocycles (but recall that $\alpha : \Phi \rightarrow \Phi$, unlike our three homeomorphisms, is generally not invertible). In Section 3 below, we shall show that the three homeomorphisms $\overline{\alpha}$, α^* and $\tilde{\alpha}_0$ are topologically conjugate to a two-sided subshift of finite type. This subshift will be used in Section 5 for the coding of the geodesic flow associated to the group Γ .

We start with a general definition. Let E be a geodesic metric space which is proper (that is, every closed ball in E is compact).

Definition 2.1 (The geodesic flow associated to a metric space).— Let GE denote the set of geodesic lines $\ell : \mathbb{R} \rightarrow E$, equipped with the topology of uniform convergence on compact sets. The space GE is locally compact and metrizable. Its topology is induced for instance by the metric

$$(2.1.1) \quad |\ell_1 - \ell_2| = \int_{-\infty}^{\infty} |\ell_1(t) - \ell_2(t)| e^{-|t|} dt.$$

In particular, GE is a Hausdorff topological space. We call it the *space of geodesics* of E . There is a natural flow $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ on GE , defined by the rule $\mathcal{F}_t \ell(u) = \ell(t + u)$. This flow is called the *geodesic flow on GE* .

The group $Isom(E)$ of isometries of E acts naturally on GE , and this action preserves the metric defined in (2.1.1). Furthermore, there is a natural involution $i_E : GE \rightarrow GE$, defined by $\ell \mapsto \ell'$, with $\ell'(t) = \ell(-t)$ for every $\ell \in GX$. The involution i_E preserves the metric defined by (2.1.1). It is easy to see that the actions of $Isom(E)$, i_E and $\mathcal{F}$ on E satisfy the following compatibility properties: for every $\gamma \in Isom(E)$, for every $\ell \in GE$ and for every $t \in \mathbb{R}$, we have $\gamma \mathcal{F}_t(\ell) = \mathcal{F}_t(\gamma \ell)$, $i_E(\mathcal{F}_t \ell) = \mathcal{F}_{-t} i_E(\ell)$ and $\gamma i(\ell) = i_E(\gamma \ell)$.

We recall that if E is a topological space equipped with a left action of a group G , then we say that G *acts properly discontinuously on E* if for every compact subset $K \subset E$, the set $\{\gamma \in G : \gamma K \cap K \neq \emptyset\}$ is finite. We recall also that if E is a locally compact Hausdorff topological space and if G acts properly discontinuously on E , then the quotient space E/G is Hausdorff.

Since $\Gamma \subset Isom(X)$, the group Γ acts on X and we have the following

Proposition 2.2.— *The action of Γ on GX is free and properly discontinuous.*

PROOF. To prove that the action of Γ is free, let $\ell : \mathbb{R} \rightarrow X$ be an element of GX that is fixed by $\gamma \in \Gamma$. Consider an arbitrary vertex $v \in X^0 = \Gamma$ which belongs to $\ell(\mathbb{R})$. Then, we have $\gamma v = v$, which implies that $\gamma = Id$. Thus, the action is free.

To prove now that the action is properly discontinuous, consider the map $\epsilon : GX \rightarrow X$ defined by $\epsilon(\ell) = \ell(0)$, for every $\ell \in GX$. This map is continuous, and it commutes with the actions of Γ on the spaces GX and X . As the action of Γ on X is properly discontinuous, we deduce easily, using this map ϵ , that the action of Γ on GX is also properly discontinuous. ■

Definition 2.3 (The space $\bar{\Phi}$ and the map $\bar{\alpha}$).— Let $\bar{\Phi}$ be the set of sequences $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$, $\varphi_n \in \Phi$ and $\varphi_{n+1} = \alpha(\varphi_n)$. Equivalently, $\bar{\Phi}$ is the projective limit of the projective system of maps $(E_n, \pi_n)_{n \in \mathbb{Z}}$ where for every $n \in \mathbb{Z}$, $E_n = \Phi$ and $\pi_n : E_n \rightarrow E_{n-1}$ is the map α . We equip $\bar{\Phi}$ with the topology induced by the canonical inclusion $\bar{\Phi} \subset \Phi^{\mathbb{Z}}$ (which is also its topology as a projective limit of topological spaces). Since Φ is compact and metrizable, $\bar{\Phi}$ is also compact and metrizable, as a closed subset of $\Phi^{\mathbb{Z}}$. We define finally the homeomorphism $\bar{\alpha} : \bar{\Phi} \rightarrow \bar{\Phi}$ by setting, for $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}}$,

$$\bar{\alpha}(\bar{\varphi}) = (\varphi'_n)_{n \in \mathbb{Z}},$$

where $\varphi'_n = \varphi_{n+1}$ for every $n \in \mathbb{Z}$.

Definition 2.4 (Primitive gradient line).— Let $\varphi \in \Phi$ and let $\ell : \mathbb{R} \rightarrow X$ be a φ -gradient line. We say that ℓ is *primitive* if the following two equivalent conditions are satisfied :

(2.4.1) $\ell(\mathbb{Z}) \subset \Gamma$ and for every $n \in \mathbb{Z}$, the element $\ell(n+1)$ is obtained from $\ell(n)$ by right multiplication by the smallest element $a \in A$ satisfying $\varphi(\ell(n), \ell(n)a) = 1$.

(2.4.2) $\ell(\mathbb{Z}) \subset \Gamma$ and for every $n \in \mathbb{Z}$, the ray defined by $t \mapsto \ell(t+n)$ for $t \geq 0$ is the smallest φ -gradient ray starting at $\ell(n)$.

The fact that these two conditions are equivalent follows easily from the definition of a smallest φ -gradient ray (see [CP2], Proposition 6.2).

Lemma 2.5.— *Let $\varphi \in \Phi$, let $\gamma \in \Gamma$ and let ℓ be a primitive φ -gradient line. Then $\gamma\ell$ is a primitive $\gamma\varphi$ -gradient line.*

PROOF. Let us show that condition (2.4.2) is satisfied for the map $\gamma\ell : \mathbb{R} \rightarrow X$. Since $\ell(\mathbb{Z}) \subset \Gamma$, we have $\gamma(\ell(\mathbb{Z})) \subset \Gamma$. Let $n \in \mathbb{Z}$. Since the geodesic ray defined by $t \mapsto \ell(t+n)$ for $t \geq 0$ is the smallest φ -gradient line starting at $\ell(n)$, Proposition 6.4 of [CP2] implies that the geodesic ray defined by $t \mapsto \gamma\ell(t+n)$ for $t \geq 0$ is the smallest $\gamma\varphi$ -gradient line starting at $\gamma\ell(n)$. This proves that $\gamma\ell$ is a primitive $\gamma\varphi$ -gradient line. ■

Definition 2.6 (The space $\tilde{\Phi}$ and the homeomorphism $\tilde{\alpha} : \tilde{\Phi} \rightarrow \tilde{\Phi}$).— Let $\tilde{\Phi} \subset \Phi \times GX$ be the set of pairs (φ, ℓ) , where $\varphi \in \Phi$ and where ℓ is a primitive φ -gradient line. We equip $\tilde{\Phi}$ with the topology induced by the product topology on $\Phi \times GX$. We note that $\tilde{\Phi}$ is locally compact and metrizable. In particular it is a Hausdorff space. We define the map $\tilde{\alpha} : \tilde{\Phi} \rightarrow \tilde{\Phi}$ by setting

$$\tilde{\alpha}(\varphi, \ell) = (\varphi, \mathcal{F}_1\ell)$$

for every $(\varphi, \ell) \in \tilde{\Phi}$. (It is clear from Definition 2.4 that if ℓ is a primitive φ -gradient line, then so is $\mathcal{F}_1\ell$.) The map $\tilde{\alpha}$ is clearly a homeomorphism, whose inverse is defined by $(\varphi, \ell) \mapsto (\varphi, \mathcal{F}_{-1}\ell)$.

Definition 2.7 (The action of Γ on $\tilde{\Phi}$).— We consider the continuous action of Γ on $\tilde{\Phi}$ defined by

$$\gamma(\varphi, \ell) = (\gamma\varphi, \gamma\ell)$$

for every $\gamma \in \Gamma$ and $(\varphi, \ell) \in \tilde{\Phi}$. We note that Lemma 2.5 shows that $(\gamma\varphi, \gamma\ell)$ is an element of $\tilde{\Phi}$, as required.

Proposition 2.8.— *The action of Γ on $\tilde{\Phi}$ is free and properly discontinuous.*

PROOF. As Γ acts freely and properly discontinuously on GX (Proposition 2.2), it acts also freely and properly discontinuously on the product space $\Phi \times GX$, and consequently on the subset $\tilde{\Phi} \subset \Phi \times GX$. ■

Corollary 2.9.— *The space $\tilde{\Phi}/\Gamma$, equipped with the quotient topology, is Hausdorff.* ■

Lemma 2.10.— *The homeomorphism $\tilde{\alpha} : \tilde{\Phi} \rightarrow \tilde{\Phi}$ commutes with the action of Γ on $\tilde{\Phi}$. In other words, for every $\tilde{\varphi} \in \tilde{\Phi}$ and for every $\gamma \in \Gamma$, we have $\tilde{\alpha}(\gamma\tilde{\varphi}) = \gamma\tilde{\alpha}(\tilde{\varphi})$.*

PROOF. Let $\tilde{\varphi} = (\varphi, \ell) \in \tilde{\Phi}$. We have, from the definitions,

$$\begin{aligned} \tilde{\alpha}(\gamma\tilde{\varphi}) &= \tilde{\alpha}(\gamma(\varphi, \ell)) = \tilde{\alpha}(\gamma\varphi, \gamma\ell) = (\gamma\varphi, \mathcal{F}_1(\gamma\ell)) = (\gamma\varphi, \gamma\mathcal{F}_1\ell) \\ &= \gamma(\varphi, \mathcal{F}_1\ell) = \gamma\tilde{\alpha}(\varphi, \ell) = \gamma\tilde{\alpha}(\tilde{\varphi}). \end{aligned}$$

This proves Lemma 2.10. ■

Definition 2.11 (The homeomorphism $\alpha^* : \tilde{\Phi}/\Gamma \rightarrow \tilde{\Phi}/\Gamma$).— By using Lemma 2.10, we define $\alpha^* : \tilde{\Phi}/\Gamma \rightarrow \tilde{\Phi}/\Gamma$ as the quotient map induced by $\tilde{\alpha}$. In other words, $\alpha^* : \tilde{\Phi}/\Gamma \rightarrow \tilde{\Phi}/\Gamma$

is the map satisfying $\alpha^* \circ \tilde{p} = \tilde{p} \circ \tilde{\alpha}$, where $\tilde{p} : \tilde{\Phi} \rightarrow \tilde{\Phi}/\Gamma$ is the canonical projection. Since the space $\tilde{\Phi}/\Gamma$ is equipped with the quotient topology, the map α^* is, like $\tilde{\alpha}$, a homeomorphism.

Definition 2.12 (The space $\tilde{\Phi}_0$ and the map $p : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}/\Gamma$).— We denote by $\tilde{\Phi}_0$ the subset of $\tilde{\Phi}$ consisting in the pairs (φ, ℓ) satisfying $\ell(0) = Id$, and by $p : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}/\Gamma$ the restriction to $\tilde{\Phi}_0$ of the canonical map $\tilde{p} : \tilde{\Phi} \rightarrow \tilde{\Phi}/\Gamma$.

Definition 2.13 (The map $\tilde{\alpha}_0 : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}_0$).— We define the map $\tilde{\alpha}_0 : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}_0$ by

$$\tilde{\alpha}_0(\varphi, \ell) = \ell(1)^{-1}(\varphi, \mathcal{F}_1 \ell).$$

We note that we have $\ell(1)^{-1}(\varphi, \mathcal{F}_1 \ell) = (\ell(1)^{-1}\varphi, \ell(1)^{-1}\mathcal{F}_1 \ell)$ and that the $\ell(1)^{-1}\varphi$ -primitive gradient line $\ell(1)^{-1}\mathcal{F}_1 \ell$ satisfies $\ell(1)^{-1}\mathcal{F}_1 \ell(0) = \ell(1)^{-1}\ell(1) = Id$, as required. ■

The next Lemma shows in particular that $\tilde{\alpha}_0$ is a homeomorphism, with inverse defined by $(\varphi, \ell) \mapsto \ell(-1)^{-1}(\varphi, \mathcal{F}_{-1} \ell)$.

Lemma 2.14.— *The map $\tilde{\alpha}_0$ is a homeomorphism and we have, for every $n \in \mathbb{Z}$ and for every $(\varphi, \ell) \in \tilde{\Phi}_0$,*

$$\tilde{\alpha}_0^n(\varphi, \ell) = \ell(n)^{-1}(\varphi, \mathcal{F}_n \ell).$$

PROOF. For every $n \in \mathbb{Z}$, let $T_n : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}_0$ denote the map defined by $T_n(\varphi, \ell) = \ell(n)^{-1}(\varphi, \mathcal{F}_n \ell)$. Let us prove first that for all n and $m \in \mathbb{Z}$, we have $T_{m+n} = T_m \circ T_n$.

For $(\varphi, \ell) \in \tilde{\Phi}_0$, we have

$$T_n(\varphi, \ell) = \ell(n)^{-1}(\varphi, \mathcal{F}_n \ell) = (\ell(n)^{-1}\varphi, \ell(n)^{-1}\mathcal{F}_n \ell).$$

Thus, we have

$$\begin{aligned} T_m \circ T_n(\varphi, \ell) &= (\ell(n)^{-1}\mathcal{F}_n \ell(m))^{-1}(\ell(n)^{-1}\varphi, \mathcal{F}_m \ell(n)^{-1}\mathcal{F}_n \ell) \\ &= (\ell(n)^{-1}\ell(m+n))^{-1}(\ell(n)^{-1}\varphi, \ell(n)^{-1}\mathcal{F}_{m+n} \ell) \\ &= \ell(m+n)^{-1}\ell(n)(\ell(n)^{-1}\varphi, \ell(n)^{-1}\mathcal{F}_{m+n} \ell) \\ &= \ell(m+n)^{-1}(\varphi, \mathcal{F}_{m+n} \ell). \end{aligned}$$

This proves that $T_{m+n} = T_m \circ T_n$. Now since $T_0 = Id$ and $T_1 = \tilde{\alpha}_0$, we conclude that $\tilde{\alpha}_0^n = T_n$ for every $n \in \mathbb{Z}$. This proves Lemma 2.14. ■

Let us recall now that if E and E' are topological spaces and if $f : E \rightarrow E$ and $f' : E' \rightarrow E'$ are continuous maps, then the dynamical systems (E, f) and (E', f') are said to be *topologically conjugate* if there exists a homeomorphism $h : E \rightarrow E'$ satisfying $h \circ f = f' \circ h$. Such a homeomorphism h is called a *topological conjugacy* between (E, f) and (E', f') .

Proposition 2.15.— *The map $p : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}/\Gamma$ is a topological conjugacy between the dynamical systems $(\tilde{\Phi}_0, \tilde{\alpha}_0)$ and $(\tilde{\Phi}/\Gamma, \alpha^*)$. In other words, p is a homeomorphism satisfying $p \circ \tilde{\alpha}_0 = \alpha^* \circ p$.*

PROOF. For every $(\varphi, \ell) \in \tilde{\Phi}$, we clearly have $\ell(0)^{-1}(\varphi, \ell) \in \tilde{\Phi}_0$. Let $r : \tilde{\Phi} \rightarrow \tilde{\Phi}_0$ be the map defined by

$$r(\varphi, \ell) = \ell(0)^{-1}(\varphi, \ell).$$

It is clear that r is a retraction, that is, r is continuous and satisfies $r(\varphi, \ell) = (\varphi, \ell)$ for every $(\varphi, \ell) \in \tilde{\Phi}_0$. Furthermore, for every $(\varphi, \ell) \in \tilde{\Phi}$ and for every $\gamma \in \Gamma$, we have

$$\begin{aligned} r(\gamma(\varphi, \ell)) &= r(\gamma\varphi, \gamma\ell) = (\gamma\ell(0))^{-1}(\gamma\varphi, \gamma\ell) \\ &= \ell(0)^{-1}\gamma^{-1}(\gamma\varphi, \gamma\ell) = \ell(0)^{-1}(\varphi, \ell) = r(\varphi, \ell). \end{aligned}$$

Thus, r is Γ -invariant, and it induces therefore a map $\bar{r} : \tilde{\Phi}/\Gamma \rightarrow \tilde{\Phi}_0$.

Let us prove now that $\bar{r}$ is the inverse of p . From the fact that $r(\varphi, \ell) = (\varphi, \ell)$, we see that $\bar{r} \circ p = Id$. On the other hand, for $(\varphi, \ell) \in \tilde{\Phi}$, the element $r(\varphi, \ell)$ is in the Γ -orbit of (φ, ℓ) , which shows that $p \circ \bar{r} = Id$. Thus, $\bar{r}$ is the inverse of p . This proves that p is a homeomorphism.

It remains to see that $p \circ \tilde{\alpha}_0 = \alpha^* \circ p$. Let $(\varphi, \ell) \in \tilde{\Phi}_0$. We have

$$\begin{aligned} \alpha^*(p(\varphi, \ell)) &= \tilde{p}(\tilde{\alpha}(\varphi, \ell)) \quad (\text{see Definition 2.11}) \\ &= \tilde{p}(\varphi, \mathcal{F}_1\ell) \\ &= \tilde{p}(\ell(1)^{-1}(\varphi, \mathcal{F}_1\ell)) \\ &= \tilde{p}(\tilde{\alpha}_0(\varphi, \ell)) \\ &= p(\tilde{\alpha}_0(\varphi, \ell)) \quad (\text{since } p \text{ is the restriction of } \tilde{p} \text{ to } \tilde{\Phi}_0). \end{aligned}$$

This completes the proof of Proposition 2.15. ■

Proposition 2.16.— *Let $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}}$ be an element of $\overline{\Phi}$ and for all $n \in \mathbb{Z}$, let $g_n : [0, \infty[\rightarrow X$ be the smallest φ_n -gradient ray starting at Id . Then, there exists a unique primitive φ_0 -gradient line $\ell = \ell(\bar{\varphi}) : \mathbb{R} \rightarrow X$ having the following property : For every $n \geq 0$ and for every $t \geq -n$ we have $\ell(t) = g_{-n}(n)^{-1}g_{-n}(t+n)$.*

PROOF. For every $n \geq 0$, let $\ell_n : [-n, \infty[\rightarrow X$ be the map defined by

$$\ell_n(t) = g_{-n}(n)^{-1}g_{-n}(t+n).$$

Let us fix an integer $n \geq 0$. Let us prove that the maps $\ell_n : [-n, \infty[\rightarrow X$ and $\ell_{n+1} : [-n-1, \infty[\rightarrow X$ coincide on $[-n, \infty[$. By definition, we have $\ell_{n+1}(t) = g_{-n-1}(n+1)^{-1}g_{-n-1}(t+n+1)$ for every $t \geq -n-1$. Thus, we have to prove that for every $t \geq -n$, we have

$$(2.16.1) \quad g_{-n}(n)^{-1}g_{-n}(t+n) = g_{-n-1}(n+1)^{-1}g_{-n-1}(t+n+1).$$

Since $\alpha(\varphi_{-n-1}) = \varphi_{-n}$, we have the following relation between φ_{-n} -gradient rays and φ_{-n-1} -gradient rays starting at Id (Proposition 6.7 (ii) of [CP2]): For every $t \geq 0$, we have

$$(2.16.2) \quad g_{-n}(t) = g_{-n-1}(1)^{-1}g_{-n-1}(t+1).$$

By taking $t = n$, we obtain

$$g_{-n}(n) = g_{-n-1}(1)^{-1}g_{-n-1}(n+1).$$

that is,

$$g_{-n}(n)^{-1} = g_{-n-1}(n+1)^{-1}g_{-n-1}(1).$$

Replacing in (2.16.2), we obtain, for every $t \geq 0$,

$$g_{-n}(t) = g_{-n}(n)g_{-n-1}(n+1)^{-1}g_{-n-1}(t+1),$$

that is,

$$g_{-n}(n)^{-1}g_{-n}(t) = g_{-n-1}(n+1)^{-1}g_{-n-1}(t+1)$$

for every $t \geq 0$, or, equivalently, (2.16.1) for $t \geq -n$. This proves that $\ell_n : [-n, \infty[\rightarrow X$ and $\ell_{n+1} : [-n-1, \infty[\rightarrow X$ coincide on $[-n, \infty[$. Let $\ell : \mathbb{R} \rightarrow X$ be the map defined by $\ell|_{[-n, \infty[} = \ell_n$ for every $n \geq 0$.

It remains to prove that $\ell : \mathbb{R} \rightarrow X$ is a primitive φ_0 -gradient line. Let $n \geq 0$. We show that the ray defined by $t \mapsto \ell(t-n)$ for $t \geq 0$ is the smallest φ_0 -gradient ray starting at $\ell(-n)$. Since $\alpha^n(\varphi_{-n}) = \varphi_0$, we have $\varphi_0 = g_{-n}(n)^{-1}\varphi_{-n}$. Now since $g_{-n} : [0, \infty[\rightarrow X$ is the smallest φ_{-n} -gradient ray starting at Id , Proposition 6.4 (ii) of [CP2] shows that the map $g_{-n}(n)^{-1}g_{-n} : [0, \infty[\rightarrow X$ is the smallest φ_0 -gradient ray starting at $g_{-n}(n)^{-1} = \ell(-n)$. This completes the proof of Proposition 2.16. $\blacksquare$

We shall need the following

Lemma 2.17.— *Let $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$ and let $\ell = \ell(\bar{\varphi}) : \mathbb{R} \rightarrow X$ be the primitive φ_0 -gradient line provided by Proposition 2.16. Then, we have, for every $n \in \mathbb{Z}$, $\varphi_n = \ell(n)^{-1}\varphi_0$.*

PROOF. Let $n \geq 0$. Then, $\varphi_n = \alpha^n(\varphi_0)$. Since $\ell|_{[0, \infty[}$ is the smallest φ -gradient ray starting at the identity, Proposition 6.7 (i) of [CP2] implies that $\varphi_n = \ell(n)^{-1}\varphi_0$. Let us prove now that $\varphi_{-n} = \ell(-n)^{-1}\varphi_0$. We have $\varphi_0 = \alpha^n(\varphi_{-n}) = g_{-n}(n)^{-1}\varphi_{-n}$ where $g_{-n} : [0, \infty[\rightarrow X$ is the smallest φ_{-n} -gradient ray starting at Id (Proposition 6.7 (i) of [CP2]). By the definition of the line ℓ (Proposition 2.16), we have, for every $t \geq -n$, $\ell(t) = g_{-n}(n)^{-1}g_{-n}(t+n)$, which gives, by making $t = n$, $\ell(-n) = g_{-n}(n)^{-1}$. Thus, $\varphi_0 = \ell(-n)\varphi_{-n}$ and therefore $\varphi_{-n} = \ell(-n)^{-1}\varphi_0$. This completes the proof of Lemma 2.17. $\blacksquare$

Let $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$ and let $\bar{\alpha}(\bar{\varphi}) = \bar{\varphi}' = (\varphi'_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$. We note that we have therefore $\varphi'_n = \varphi_{n+1}$ for every $n \in \mathbb{Z}$. Let $\ell = \ell(\bar{\varphi}) : \mathbb{R} \rightarrow X$ (respectively $\ell' = \ell(\bar{\alpha}(\bar{\varphi})) : \mathbb{R} \rightarrow X$) denote the primitive φ_0 -gradient line (respectively the primitive φ_1 -gradient line) provided by Proposition 2.16. Then,

Lemma 2.18.— *We have $\ell' = \ell(1)^{-1}\mathcal{F}_1(\ell)$.*

PROOF. To prove that $\ell' = \ell(1)^{-1}\mathcal{F}_1\ell$, we show that for all $n \geq 0$, ℓ' and $\ell(1)^{-1}\mathcal{F}_1(\ell)$ have the same restriction on $[-n, \infty[$.

Let $n \geq 0$ and let $t \geq -n$. From the definition of ℓ' , we have

$$(2.18.1) \quad \ell'(t) = g'_{-n}(n)^{-1} g'_{-n}(t+n),$$

where $g'_{-n} : [0, \infty[\rightarrow X$ denotes the smallest φ'_n -gradient ray starting at Id .

Since $\varphi'_{-n} = \varphi_{-n+1}$, we have $g'_{-n} = g_{-n+1}$. Therefore, we obtain from (2.18.1),

$$(2.18.2) \quad \ell'(t) = g_{-n+1}(n)^{-1} g_{-n+1}(t+n).$$

By Proposition 6.7 (ii) of [CP2], for all $t \geq 0$, we have

$$g_{-n+1}(t) = g_{-n}(1)^{-1} g_{-n}(t+1),$$

which gives, for $t = n$,

$$g_{-n+1}(n) = g_{-n}(1)^{-1} g_{-n}(n+1).$$

Replacing in (2.18.2), we obtain

$$\ell'(t) = (g_{-n}(1)^{-1} g_{-n}(n+1))^{-1} g_{-n}(1)^{-1} g_{-n}(t+n+1),$$

that is,

$$\ell'(t) = g_{-n}(n+1)^{-1} g_{-n}(1) g_{-n}(1)^{-1} g_{-n}(t+n+1) = g_{-n}(n+1)^{-1} g_{-n}(t+n+1).$$

Since

$$\ell(1) = g_{-n}(n)^{-1} g_{-n}(1+n),$$

we obtain

$$\ell'(t) = \ell(1)^{-1} g_{-n}(n)^{-1} g_{-n}(t+n+1) = \ell(1)^{-1} \mathcal{F}_1 \ell(t),$$

for all $t \geq -n$. Summing up, we have $\ell'(t) = \ell(1)^{-1} -1 \mathcal{F}_1 \ell(t)$ for all $t \in \mathbb{R}$. This proves Lemma 2.18. $\blacksquare$

The following lemma will also be useful.

Lemma 2.19.— *Let $(\varphi, \ell) \in \tilde{\Phi}$ and for every $n \in \mathbb{Z}$, let $\varphi_n = \ell(n)^{-1} \varphi$. Then, for every $n \in \mathbb{Z}$, we have $\alpha(\varphi_n) = \varphi_{n+1}$.*

PROOF. Let us fix an integer $n \in \mathbb{Z}$. We have, by definition, $\alpha(\ell(n)^{-1} \varphi) = a^{-1}(\ell(n)^{-1} \varphi)$ where a is the smallest element in $\mathcal{A}$ satisfying $\ell(n)^{-1} \varphi(Id, a) = 1$. Thus, a is the smallest element in $\mathcal{A}$ satisfying $\varphi(\ell(n), \ell(n)a) = 1$. Since ℓ is a primitive φ -gradient line, we have $\ell(n)a = \ell(n+1)$ (Definition 2.4.1). Therefore, we obtain $\alpha(\varphi_1) = a^{-1}(\ell(n)^{-1} \varphi) = (\ell(n)a)^{-1} \varphi = \ell(n+1)^{-1} \varphi$, that is, $\alpha(\varphi_n) = \varphi_{n+1}$. This proves Lemma 2.19. $\blacksquare$

Definition 2.20 (The map $\tilde{q} : \tilde{\Phi} \rightarrow \overline{\Phi}$).— By Lemma 2.19, we can define the map $\tilde{q} : \tilde{\Phi} \rightarrow \overline{\Phi}$ by setting, for every $(\varphi, \ell) \in \tilde{\Phi}$,

$$\tilde{q}(\varphi, \ell) = (\varphi_n)_{n \in \mathbb{Z}}.$$

with $\varphi_n = \ell(n)^{-1}\varphi$ for every $n \in \mathbb{Z}$. This map $\tilde{q}$ is clearly continuous and is invariant for the action of Γ on $\tilde{\Phi}$. Therefore it induces a map from $\tilde{\Phi}/\Gamma$ to $\bar{\Phi}$.

Definition 2.21 (The map $q : \bar{\Phi} \rightarrow \tilde{\Phi}_0$).— For $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$, we define

$$q(\bar{\varphi}) = (\varphi_0, \ell),$$

where $\ell = \ell(\bar{\varphi}) : \mathbb{R} \rightarrow X$ is the primitive φ_0 -gradient line provided by Proposition 2.16. (We note that the line $\ell = \ell(\bar{\varphi})$ satisfies, by construction, $\ell(0) = Id$, which shows that $q(\bar{\varphi}) \in \tilde{\Phi}_0$, as required.)

This map q is clearly continuous, and in fact we have the following

Proposition 2.22.— *The map $q : \bar{\Phi} \rightarrow \tilde{\Phi}_0$ is a topological conjugacy between the dynamical systems $(\bar{\Phi}, \bar{\alpha})$ and $(\tilde{\Phi}_0, \tilde{\alpha}_0)$. In other words, q is a homeomorphism satisfying $q \circ \bar{\alpha} = \tilde{\alpha}_0 \circ q$. Furthermore, the inverse of q is the restriction to $\tilde{\Phi}_0$ of the map $\tilde{q}$.*

PROOF. Let $\bar{q} : \tilde{\Phi}_0 \rightarrow \bar{\Phi}$ be the restriction to $\tilde{\Phi}_0$ of the map $\tilde{q}$. It is clear that $\bar{q}$ is continuous. By Lemma 2.17, we have $\bar{q} \circ q = Id$.

Let us prove now that $q \circ \bar{q} = Id$.

Let $(\varphi, \ell) \in \tilde{\Phi}_0$. We have, by definition, $\bar{q}(\varphi, \ell) = \bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}}$ where $\varphi_n = \ell(n)^{-1}\varphi$ for every $n \in \mathbb{Z}$. Let us set $q(\bar{\varphi}) = (\varphi_0, \ell')$. We have $\varphi_0 = \ell(0)^{-1}\varphi = \varphi$ and it remains to prove that $\ell' = \ell$.

Let us fix an integer $n \geq 0$. Let us prove that $\ell'(t) = \ell(t)$ for every $t \in [-n, \infty[$. From the definition of the line ℓ' (Proposition 2.16), we have, for $t \in [-n, \infty[$, $\ell'(t) = g_{-n}(n)^{-1}g_{-n}(t+n)$, where $g_{-n} : [0, \infty[\rightarrow X$ is the smallest φ_{-n} -gradient ray starting at the identity. Since $\varphi_{-n} = \ell(-n)^{-1}\varphi$ and since the map $t \mapsto \ell(t-n)$, $t \geq 0$, is the smallest φ -gradient ray starting at $\ell(-n)$, we have $g_{-n}(t) = \ell(-n)^{-1}\ell(t-n)$ for every $t \geq 0$ (Proposition 6.4 of [CP2]), or, equivalently, $g_{-n}(t+n) = \ell(-n)^{-1}\ell(t)$ for every $t \geq -n$.

Thus, for every $t \in [-n, \infty[$, we have $\ell'(t) = g_{-n}(n)^{-1}\ell(-n)^{-1}\ell(t)$.

Now since $g_{-n}(t) = \ell(-n)^{-1}\ell(t-n)$ for all $t \geq 0$, we obtain, by taking $t = n$, $g_{-n}(n) = \ell(-n)^{-1}\ell(0) = \ell(-n)^{-1}$. Thus, we have $\ell'(t) = \ell(t)$ for every $n \geq 0$ and for every $t \geq -n$. Since $n \geq 0$ is arbitrary, we obtain $\ell'(t) = \ell(t)$ for every $t \in \mathbb{R}$. This proves that $q \circ \bar{q} = Id$, which completes the proof of the fact that $\bar{q}$ is the inverse of q .

It remains to prove now that $q \circ \bar{\alpha} = \tilde{\alpha}_0 \circ q$.

Let $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$ and let $\ell = \ell(\bar{\varphi})$ be the primitive φ_0 -gradient ray provided by Proposition 2.16. We have $q(\bar{\varphi}) = (\varphi_0, \ell)$, which implies

$$(2.22.1) \quad \tilde{\alpha}_0 \circ q(\bar{\varphi}) = \ell(1)^{-1}(\varphi_0, \mathcal{F}_1 \ell).$$

By Lemma 2.18, we have $\ell(\bar{\alpha}(\bar{\varphi})) = \ell(1)^{-1}\mathcal{F}_1 \ell(\bar{\varphi})$. Therefore,

$$(2.22.2) \quad q(\bar{\alpha}(\bar{\varphi})) = (\varphi_1, \ell(1)^{-1}\mathcal{F}_1 \ell) = \ell(1)^{-1}(\ell(1)\varphi_1, \mathcal{F}_1 \ell).$$

Since $\ell_{[0,\infty[}$ is the smallest φ_0 -gradient ray starting at the identity, we have, by the definition of the map α (Definition 1.4), $\varphi_1 = \alpha(\varphi_0) = \ell(1)^{-1}\varphi_0$. Thus, replacing in (2.22.2), we obtain

$$(2.22.3) \quad q(\bar{\alpha}(\bar{\varphi})) = \ell(1)^{-1}(\varphi_0, \mathcal{F}_1(\ell)).$$

The relations (2.22.1) and (2.22.3) prove that

$$q \circ \bar{\alpha}(\bar{\varphi}) = \tilde{\alpha}_0 \circ q(\bar{\varphi}).$$

This completes the proof of Proposition 2.22. ■

Thus, the map q allows us to identify $\bar{\Phi}$ with the set of pairs (φ, ℓ) with $\varphi \in \Phi$ and ℓ a primitive φ -gradient line satisfying $\ell(0) = Id$.

Definition 2.23 (The map $h : \bar{\Phi} \rightarrow \tilde{\Phi}/\Gamma$).— We define the map $h : \bar{\Phi} \rightarrow \tilde{\Phi}/\Gamma$ as the composition $h = p \circ q$.

Proposition 2.24.— *The map $h : \bar{\Phi} \rightarrow \tilde{\Phi}/\Gamma$ is a topological conjugacy between the dynamical systems $(\bar{\Phi}, \bar{\alpha})$ and $(\tilde{\Phi}/\Gamma, \alpha^*)$. In other words, h is a homeomorphism satisfying $h \circ \bar{\alpha} = \alpha^* \circ h$. Furthermore, the inverse of h is the map from $\tilde{\Phi}/\Gamma$ to $\bar{\Phi}$ induced by the Γ -invariant map $\tilde{q} : \tilde{\Phi} \rightarrow \bar{\Phi}$.*

PROOF. The fact that h is a conjugacy follows immediately from Propositions 2.15 and 2.21. The last statement of the proposition follows from Proposition 2.22. ■

To help the reader, we have represented some of the maps that we defined in this section in Figure 3.

Figure 3

3.— The dynamical system $(\bar{\Phi}, \bar{\alpha})$ as a two-sided subshift of finite type.

In this section, we prove that the dynamical system $(\bar{\Phi}, \bar{\alpha})$ is conjugate to a two-sided subshift of finite type.

We recall first the definition of a subshift of finite type over $\mathbb{Z}$ or over $\mathbb{N}$. Let S be a finite set, which will be called the alphabet and whose elements will be called the symbols. Let k be a nonnegative integer and let W be a subset of S^{k+1} . We set

$$\Sigma(S, k, W) = \{\sigma : \mathbb{Z} \rightarrow S \text{ such that } \forall n \in \mathbb{Z}, (\sigma(n), \dots, \sigma(n+k)) \in W\}$$

and

$$\Sigma_+(S, k, W) = \{\sigma : \mathbb{N} \rightarrow S \text{ such that } \forall n \in \mathbb{N}. (\sigma(n), \dots, \sigma(n+k)) \in W\}.$$

Definition 3.1 (Subshift of finite type).— A subset $\Sigma \subset S^{\mathbb{Z}}$ (respectively $\Sigma_+ \subset S^{\mathbb{N}}$) is said to be a *one-sided subshift of finite type* (respectively a *two-sided subshift of finite type*) if there exists an integer $k \geq 0$ and a subset $W \subset S^{k+1}$ such that $\Sigma = \Sigma(S, k, W)$ (respectively $\Sigma_+ = \Sigma_+(S, k, W)$).

We note that if Σ (respectively Σ_+) is a one-sided (respectively a two-sided) subshift of finite type, then it is a closed subset of $S^{\mathbb{Z}}$ (respectively $S^{\mathbb{N}}$) which is invariant by the shift map defined by $\sigma \mapsto \sigma'$, where $\sigma'(n) = \sigma(n+1)$ for all $n \in \mathbb{Z}$ (respectively $n \in \mathbb{N}$). The space Σ (respectively Σ_+) is equipped therefore with an induced shift map $T : \Sigma \rightarrow \Sigma$ (respectively $T : \Sigma_+ \rightarrow \Sigma_+$). Clearly, the shift map on Σ is a homeomorphism.

It is well-known that if $\Sigma = \Sigma(S, k, W)$ (respectively if $\Sigma_+ = \Sigma_+(S, k, W)$) is a subshift of finite type, then we can construct a new set of symbols S' and a subset $W' \subset S' \times S'$ such that Σ (respectively Σ_+) is topologically conjugate to $\Sigma' = \Sigma(S', 1, W')$ (respectively $\Sigma'_+ = \Sigma_+(S', 1, W')$), equipped with its shift map.

In [CP2] (Theorem 8.18), we have proved the following

Theorem 3.2.— *The dynamical system (Φ, α) is topologically conjugate to a one-sided subshift of finite type.*

Using this result, we shall deduce that the dynamical system $(\bar{\Phi}, \bar{\alpha})$ is topologically conjugate to a two-sided subshift of finite type. This will be a consequence of a well-known construction which we recall presently.

Let (Σ_+, T) be a one-sided subshift of finite type and let Δ be the set of sequences $(\sigma_n)_{n \in \mathbb{Z}}$ with $\sigma_n \in \Sigma_+$ and $\sigma_{n+1} = T(\sigma_n)$ for all $n \in \mathbb{Z}$.

Let $d : \Delta \rightarrow \Delta$ denote the map defined by $d(\sigma) = \sigma'$, where $\sigma = (\sigma_n)_{n \in \mathbb{Z}} \in \Delta$ and $\sigma' = (\sigma'_n)_{n \in \mathbb{Z}}$ with $\sigma'_n = \sigma_{n+1}$ for all $n \in \mathbb{Z}$.

The next proposition will show that the dynamical system (Δ, d) is topologically conjugate in a natural way to a two-sided subshift of finite type.

By definition, there exists an integer $k \geq 0$ and a subset $W \subset S^{k+1}$ such that $\Sigma_+ = \Sigma_+(S, k, W)$. Let us define the two-sided subshift of finite type $\Sigma \subset S^{\mathbb{Z}}$ by setting $\Sigma = \Sigma(S, k, W)$.

Consider now the map $H : \Delta \rightarrow \Sigma$ defined by

$$(\sigma_n)_{n \in \mathbb{Z}} \mapsto (\sigma_n(0))_{n \in \mathbb{Z}}.$$

Proposition 3.3.— *The map H is a topological conjugacy between the dynamical systems (Δ, d) and (Σ, T) .*

PROOF. Let $\sigma = (\sigma_n)_{n \in \mathbb{Z}} \in \Delta$. For every $n \in \mathbb{Z}$ and for every $m \in \mathbb{N}$, we have $T^m(\sigma_n) = \sigma_{n+m}$. Therefore, we have

$$(3.3.1) \quad \sigma_{n+m}(0) = T^m(\sigma_n)(0) = \sigma_n(m).$$

Let now $\sigma = (\sigma_n)_{n \in \mathbb{Z}}$ and $\tau = (\tau_n)_{n \in \mathbb{Z}}$ be two elements of Δ satisfying $H(\sigma) = H(\tau)$. For every $n \in \mathbb{Z}$, we have $\sigma_n(0) = \tau_n(0)$. Since for all $m \in \mathbb{N}$, we have $\sigma_{n+m}(0) = \tau_{n+m}(0)$, we deduce from (3.3.1) that for all $n \in \mathbb{Z}$ and for all $m \in \mathbb{N}$, we have $\sigma_n(m) = \tau_n(m)$. Therefore, we have $\sigma = \tau$, which shows that the map H is injective.

For the surjectivity, let $(s_n)_{n \in \mathbb{Z}} \in \Sigma$. For all $n \in \mathbb{Z}$, the sequence $s_n, s_{n+1}, \dots, s_{n+k}$ is in U . Therefore, for all $n \geq 0$, the infinite sequence $s_n, s_{n+1}, \dots$ represents an element $\sigma_n \in \Sigma_+$. From the definition of the shift on Σ , we have $\sigma_{n+1} = T(\sigma_n)$ for all $n \in \mathbb{Z}$. Thus, the sequence $(\sigma_n)_{n \in \mathbb{Z}}$ is in Δ . This shows the surjectivity of H .

The bi-continuity of H follows easily from the definition of the topologies involved (pointwise convergence). It remains to prove now that $H \circ d = T \circ H$.

For $\sigma = (\sigma_n)_{n \in \mathbb{Z}} \in \Delta$, let $\sigma' = (\sigma'_n)_{n \in \mathbb{Z}}$ with $\sigma'_n = \sigma_{n+1}$ for every $n \in \mathbb{Z}$. Then, we have

$$T(H(\sigma)) = T((\sigma_n(0))_{n \in \mathbb{Z}}) = (\sigma'_n(0))_{n \in \mathbb{Z}} = H(\sigma') = H(d(\sigma)).$$

This completes the proof of Proposition 3.3 ■

From Theorem 3.2 and Proposition 3.3, we deduce

Theorem 3.4.—*The dynamical system $(\bar{\Phi}, \bar{\alpha})$ is topologically conjugate to a two-sided subshift of finite type.* ■

Theorem 3.4 implies that the dynamical systems $(\tilde{\Phi}/\Gamma, \alpha^*)$ and $(\tilde{\Phi}_0, \tilde{\alpha}_0)$ are also topologically conjugate to a two-sided subshift of finite type.

Remark. The coding that we obtain for $(\bar{\Phi}, \bar{\alpha})$ can be made geometrically explicit, using the explicit coding for (Φ, α) that we gave in [CP2]. What we have to do is to associate to each $\bar{\varphi} = (\varphi_n)_{n \in \mathbb{Z}} \in \bar{\Phi}$ the sequence $(\sigma_n)_{n \in \mathbb{Z}}$, where $\sigma_n = \rho(\varphi_n)$ is the restriction of the primitive h_n of φ_n satisfying $h_n(Id) = 0$ to the “initial fat segment” $V(\varphi_n)$ of φ (using the terminology and the notations of [CP2]).

Let us note finally that Theorem 3.4 gives us another way to see that the space $\bar{\Phi}$ is compact and metrizable (and therefore that $\tilde{\Phi}_0$ and $\tilde{\Phi}/\Gamma$ are likewise compact and metrizable).

4.— The suspension of $(\bar{\Phi}, \bar{\alpha})$.

We recall first the notion of *suspension* (or *mapping torus*) of a homeomorphism.

Let E be a topological space and $f : E \rightarrow E$ a homeomorphism. We define the *suspension of f* , denoted by $\text{Susp}(E, f)$, to be the quotient space $(E \times \mathbb{R})/\mathbb{Z}$, where the $\mathbb{Z}$ -action is generated by the homeomorphism of $E \times \mathbb{R}$ which sends (x, u) to $(f(x), u - 1)$.

Equivalently, the space $\text{Susp}(E, f)$ can be obtained from $E \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for every $x \in E$.

The flow $\tilde{\phi}$ on $E \times \mathbb{R}$ defined by $\tilde{\phi}_t(x, u) = (x, t + u)$ commutes with the $\mathbb{Z}$ -action and gives rise to a canonical flow $\phi_t : \text{Susp}(E, f) \rightarrow \text{Susp}(E, f)$, which we shall call the *suspension flow* on $\text{Susp}(E, f)$.

In this section, we introduce a space $\tilde{\Phi}_{\mathbb{R}}$ which is equipped with a Γ -action and with a flow. We prove that there exists a natural topological conjugacy between the quotient space $\bar{\Phi}_{\mathbb{R}} = \tilde{\Phi}_{\mathbb{R}}/\Gamma$ equipped with its quotient flow $\mathcal{F}^*$ and the spaces $\text{Susp}(\bar{\Phi}, \bar{\alpha}) \simeq \text{Susp}(\tilde{\Phi}/\Gamma, \alpha^*) \simeq \text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0)$ equipped with their respective suspension flows.

Definition 4.1 (The space $\tilde{\Phi}_{\mathbb{R}}$ and the flow $\tilde{\mathcal{F}}$ on $\tilde{\Phi}_{\mathbb{R}}$).— We let $\tilde{\Phi}_{\mathbb{R}}$ be the set of pairs (φ, ℓ) where $\varphi \in \Phi$ and where $\ell \in GX$ is a geodesic line which is in the $\mathcal{F}$ -orbit of a primitive φ -gradient line. (Recall that $\mathcal{F}$ is the geodesic flow on the space of geodesics GX : see Definition 2.1.) We have a natural inclusion $\tilde{\Phi}_{\mathbb{R}} \subset \Phi \times GX$, and we equip $\tilde{\Phi}_{\mathbb{R}}$ with the induced topology. There is a natural flow $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)_{t \in \mathbb{R}}$ on $\tilde{\Phi}_{\mathbb{R}}$ defined by the rule

$$\tilde{\mathcal{F}}_t(\varphi, \ell) = (\varphi, \mathcal{F}_t \ell)$$

for $(\varphi, \ell) \in \tilde{\Phi}_{\mathbb{R}}$ and $t \in \mathbb{R}$. We note that $\tilde{\mathcal{F}}_1$, the time-1 map of the flow $\tilde{\mathcal{F}}$, restricted to the space $\tilde{\Phi} \subset \tilde{\Phi}_{\mathbb{R}}$, is the homeomorphism $\tilde{\alpha}$ defined in 2.6.

Definition 4.2 (The action of Γ on $\tilde{\Phi}_{\mathbb{R}}$).— We consider the left action of the group Γ on $\tilde{\Phi}_{\mathbb{R}}$ defined by the rule $\gamma(\varphi, \ell) = (\gamma\varphi, \gamma\ell)$. (We note that by Lemma 2.5, if $\mathcal{F}_t \ell$ is a primitive φ -gradient line, then $\mathcal{F}_t(\gamma\ell)$ is a primitive $\gamma\varphi$ -gradient line.) It is easy to see (as we saw in Proposition 2.8 for the action of Γ on $\tilde{\Phi}$) that the action of Γ on $\tilde{\Phi}_{\mathbb{R}}$ is free and properly discontinuous. We equip $\tilde{\Phi}_{\mathbb{R}}/\Gamma$ with the quotient topology. The action of Γ on $\tilde{\Phi}_{\mathbb{R}}$ commutes with the flow $\tilde{\mathcal{F}}$ and we have therefore an induced flow on $\tilde{\Phi}_{\mathbb{R}}/\Gamma$, which we denote by $\mathcal{F}^* = (\mathcal{F}_t^*)_{t \in \mathbb{R}}$.

Definition 4.3 (The map $p' : \tilde{\Phi}_0 \times \mathbb{R} \rightarrow \tilde{\Phi}_{\mathbb{R}}$).— We define the map $p' : \tilde{\Phi}_0 \times \mathbb{R} \rightarrow \tilde{\Phi}_{\mathbb{R}}$ by the formula

$$p'((\varphi, \ell), u) = \tilde{\mathcal{F}}_u(\varphi, \ell)$$

for $(\varphi, \ell) \in \tilde{\Phi}_0$ and $u \in \mathbb{R}$. (Recall that we have $\tilde{\mathcal{F}}_u(\varphi, \ell) = (\varphi, \mathcal{F}_u \ell)$, by definition.) It is clear that p' is continuous and injective.

Lemma 4.4.— *Let $((\varphi, \ell), u)$ and $((\varphi', \ell'), u') \in \tilde{\Phi}_0 \times \mathbb{R}$. Assume that $p'((\varphi', \ell'), u') = \gamma p'((\varphi, \ell), u)$ for some $\gamma \in \Gamma$ and let $n = u - u'$. Then, we have $n \in \mathbb{Z}$, $\gamma = \ell(n)^{-1}$ and $(\varphi', \ell') = \tilde{\alpha}_0^n(\varphi, \ell)$.*

PROOF. By definition, we have $\varphi' = \gamma\varphi$ and $\mathcal{F}_{u'} \ell' = \gamma \mathcal{F}_u \ell$. Therefore,

$$\gamma \ell(n) = \gamma \ell(u - u') = \gamma \mathcal{F}_u \ell(-u') = \mathcal{F}_{u'} \ell'(-u') = \ell'(0) = Id.$$

Thus, $\gamma = \ell(n)^{-1}$, which shows in particular that $n \in \mathbb{Z}$. We can write then $\varphi' = \ell(n)^{-1}\varphi$ and $\mathcal{F}_u\ell' = \ell(n)^{-1}\mathcal{F}_u\ell$. The last inequality gives

$$\ell' = \ell(n)^{-1}\mathcal{F}_{u-u'}\ell = \ell(n)^{-1}\mathcal{F}_n\ell.$$

Thus, we have $(\varphi', \ell') = \ell(n)^{-1}(\varphi, \mathcal{F}_n\ell)$ which by Lemma 2.14 implies $(\varphi', \ell') = \tilde{\alpha}_0^n(\varphi, \ell)$. This proves Lemma 4.4. $\blacksquare$

Lemma 4.5.— *Let $\bar{p}' : \tilde{\Phi}_0 \times \mathbb{R} \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$ be the map obtained by composing $p' : \tilde{\Phi}_0 \times \mathbb{R} \rightarrow \tilde{\Phi}_{\mathbb{R}}$ with the canonical projection $\tilde{\Phi}_{\mathbb{R}} \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$. Then, $\bar{p}'$ is surjective.*

PROOF. Let $(\varphi, \ell) \in \tilde{\Phi}_{\mathbb{R}}$ and consider an arbitrary vertex $\gamma \in \Gamma$ belonging to the image of ℓ , that is, $\gamma = \ell(u)$ for some $u \in \mathbb{R}$. Then, we have, $\gamma^{-1}\ell(u) = \gamma^{-1}\gamma = Id$. Thus, $\gamma^{-1}(\varphi, \mathcal{F}_u\ell) \in \tilde{\Phi}_0$ and therefore $\gamma^{-1}(\varphi, \ell) = (\gamma^{-1}\varphi, \gamma^{-1}\ell)$ is in the image of p' . This proves that $\bar{p}'$ is surjective. $\blacksquare$

Lemma 4.6.— *For $(\varphi, \ell) \in \tilde{\Phi}_0$ and $u \in \mathbb{R}$, we have*

$$p'(\tilde{\alpha}_0(\varphi, \ell), u - 1) = \ell(1)^{-1}p'((\varphi, \ell), u).$$

PROOF. We have

$$\begin{aligned} p'(\tilde{\alpha}_0(\varphi, \ell), u - 1) &= p'(\ell(1)^{-1}(\varphi, \mathcal{F}_1\ell), u - 1) \\ &= p'((\ell(1)^{-1}\varphi, \ell(1)^{-1}\mathcal{F}_1\ell), u - 1) \\ &= (\ell(1)^{-1}\varphi, \ell'), \end{aligned}$$

where $\ell' = \mathcal{F}_{u-1}(\ell(1)^{-1}\mathcal{F}_1\ell) = \ell(1)^{-1}\mathcal{F}_u\ell$. Thus, we have

$$p'(\tilde{\alpha}_0(\varphi, \ell), u - 1) = \ell(1)^{-1}(\varphi, \mathcal{F}_u\ell) = \ell(1)^{-1}p'((\varphi, \ell), u).$$

This completes the proof of Lemma 4.6. $\blacksquare$

Definition 4.7 (The map $p_{\mathbb{R}} : \text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0) \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$).— Lemma 4.6 implies that the map $\bar{p}' : \tilde{\Phi}_0 \times \mathbb{R} \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$ is invariant by the action of $\mathbb{Z}$ on $\tilde{\Phi}_0 \times \mathbb{R}$ that is generated by the homeomorphism sending an element $((\varphi, \ell), u) \in \tilde{\Phi}_0 \times \mathbb{R}$ to the element $(\tilde{\alpha}_0(\varphi, \ell), u - 1)$. Therefore, $\bar{p}'$ induces a map $p_{\mathbb{R}} : \text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0) \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$. It is clear that $p_{\mathbb{R}}$ is continuous.

Let us recall now that if E and E' are topological spaces and if $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ and $\mathcal{F}' = (\mathcal{F}'_t)_{t \in \mathbb{R}}$ are flows on E and E' respectively, then the dynamical systems $(E, \mathcal{F})$ and $(E', \mathcal{F}')$ are said to be *topologically conjugate* if there exists a homeomorphism $h : E \rightarrow E'$ satisfying $h \circ \mathcal{F}_t = \mathcal{F}'_t \circ h$ for every $t \in \mathbb{R}$. Such a homeomorphism h is called a *topological flow conjugacy* between $(E, \mathcal{F})$ and $(E', \mathcal{F}')$ or (more concisely) a *conjugacy* between the flows $\mathcal{F}$ and $\mathcal{F}'$.

Proposition 4.8.— *The map $p_{\mathbb{R}} : \text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0) \rightarrow \tilde{\Phi}_{\mathbb{R}}/\Gamma$ is a topological conjugacy between the suspension flow on $\text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0)$ and the flow $\mathcal{F}^*$ on $\tilde{\Phi}_{\mathbb{R}}/\Gamma$.*

PROOF. Lemma 4.5 implies that $p_{\mathbb{R}}$ is surjective, and Lemma 4.4 shows that $p_{\mathbb{R}}$ is injective. Since $p_{\mathbb{R}}$ is continuous and since the space $\text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0)$ is compact, $p_{\mathbb{R}}$ is a homeomorphism.

Finally, we see from the definitions that p' sends the flow on $\tilde{\Phi}_0 \times \mathbb{R}$ which acts as the Identity $\times$ the translations on the factor $\mathbb{R}$, to the flow $\tilde{\mathcal{F}}$ on $\tilde{\Phi}_{\mathbb{R}}$, implying that $p_{\mathbb{R}}$ conjugates the suspension flow on $\text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0)$ with the flow $\mathcal{F}^*$ on $\tilde{\Phi}_{\mathbb{R}}/\Gamma$. This proves Proposition 4.8. $\blacksquare$

We note now that if $f : E \rightarrow E$ and $f' : E' \rightarrow E'$ are homeomorphisms, and if $h : E \rightarrow E'$ is a topological conjugacy between the dynamical systems (E, f) and (E', f') , then there is a natural topological conjugacy between the spaces $\text{Susp}(E, f)$ and $\text{Susp}(E', f')$ equipped with their suspension flows. Thus, from the topological conjugacy $p : \tilde{\Phi}_0 \rightarrow \tilde{\Phi}/\Gamma$ of Proposition 2.15 (respectively the topological conjugacy $q : \bar{\Phi} \rightarrow \tilde{\Phi}_0$ of Proposition 2.22), we deduce a topological conjugacy between the spaces $\text{Susp}(\tilde{\Phi}_0, \tilde{\alpha}_0)$ (respectively $\text{Susp}(\bar{\Phi}, \bar{\alpha}_0)$) and $\text{Susp}(\tilde{\Phi}, \bar{\alpha})$, equipped with their suspension flows. Thus, we obtain in particular, from Theorem 3.4 and Proposition 4.8 the following

Corollary 4.9.— *There exists a two-sided subshift of finite type (Σ, T) and a topological conjugacy between the space $\text{Susp}(\Sigma, T)$ equipped with its suspension flow and the space $\tilde{\Phi}_{\mathbb{R}}/\Gamma$ equipped with the flow $\mathcal{F}^*$.* $\blacksquare$

5.— The geodesic flow associated to Γ .

In this section, we consider the geodesic flow $(\hat{G}, \hat{\mathcal{F}})$ associated to Γ , and the quotient space $\bar{G} = \hat{G}/\Gamma$ equipped with the quotient flow $\bar{\mathcal{F}}$. We prove that there exists a two-sided subshift of finite type (Σ, T) and a map $\text{Susp}(\Sigma, T) \rightarrow \bar{G} = \hat{G}/\Gamma$ which is continuous, surjective and uniformly finite-to-one and which sends each orbit of the suspension flow on $\text{Susp}(\Sigma, T)$ onto an $\bar{\mathcal{F}}$ -orbit of $\bar{G}$.

We start by recalling the following result of Gromov ([Gro2], Theorem 8.3.C, see [Cha] for details on the proof).

Theorem 5.1.— *For each hyperbolic group Γ , there exists a metric space $\hat{G} = \hat{G}(\Gamma)$ which satisfies the following properties :*

(5.1.1) *The space $\hat{G}$ is geodesic and proper. It is equipped with an isometric action of Γ , an isometric involution $i : \hat{G} \rightarrow \hat{G}$ and a flow $\hat{\mathcal{F}} = (\hat{\mathcal{F}}_t)_{t \in \mathbb{R}}$ such that these three actions are compatible in the following sense : For every $\gamma \in \Gamma$, for every $\hat{g} \in \hat{G}$ and for every $t \in \mathbb{R}$, we have $\gamma \hat{\mathcal{F}}_t(\hat{g}) = \hat{\mathcal{F}}_t(\gamma \hat{g})$, $i(\hat{\mathcal{F}}_t \hat{g}) = \hat{\mathcal{F}}_{-t}(i\hat{g})$ and $\gamma i(\hat{g}) = i(\gamma \hat{g})$.*

(5.1.2) *The action of Γ on $\hat{G}$ is properly discontinuous and cocompact. (Note that this implies that the space $\hat{G}$ is quasi-isometric to Γ , that $\hat{G}$ is hyperbolic as a metric space and that we have a canonical homeomorphism $\partial\Gamma \rightarrow \partial\hat{G}$ which is Γ -equivariant.)*

(5.1.3) *For every $\hat{g} \in \hat{G}$, the map $t \mapsto \hat{\mathcal{F}}_t \hat{g}$ is a quasi-isometric embedding of $\mathbb{R}$ into $\hat{G}$. In particular, we can associate to each $\hat{g} \in \hat{G}$ two distinct points in $\partial\hat{G} = \partial\Gamma$, defined by $\hat{g}(-\infty) = \lim_{t \rightarrow -\infty} \hat{\mathcal{F}}_t(\hat{g})$ and $\hat{g}(\infty) = \lim_{t \rightarrow \infty} \hat{\mathcal{F}}_t(\hat{g})$.*

(5.1.4) The map from $\hat{G}$ to $\partial^2 \hat{G} = \partial^2 \Gamma$ defined by $\hat{g} \mapsto (\hat{g}(-\infty), \hat{g}(\infty))$ induces a canonical homeomorphism $\hat{G}/\mathbb{R} \rightarrow \partial^2 \Gamma$. (Here, $\partial^2 \hat{G} = \partial^2 \Gamma$ denotes the space consisting of distinct pairs (ξ, η) with ξ and $\eta \in \partial \hat{G} = \partial \Gamma$ and $\xi \neq \eta$.) This implies that $\hat{G}$ is homeomorphic to the product space $\partial^2 \Gamma \times \mathbb{R}$.

(5.1.5) Let $\hat{X}$ be a geodesic proper hyperbolic metric space equipped with a properly discontinuous cocompact isometric action of Γ and consider the space $G\hat{X}$ of geodesic lines $(: \mathbb{R} \rightarrow \hat{X}$ equipped with the metric and with the three actions described in Section 2 above (Definition 2.1). Then, there exists a (non-canonical) continuous Γ -equivariant quasi-isometric map $\mathcal{Q} : G\hat{X} \rightarrow \hat{G}$ which satisfies $\mathcal{Q} \circ i_X = i \circ \mathcal{Q}$ and which maps homeomorphically every orbit of the geodesic flow on $G\hat{X}$ onto an $\hat{\mathcal{F}}$ -orbit in $\hat{G}$.

(5.1.6) (Uniqueness.) If $\hat{G}_1$ and $\hat{G}_2$ are spaces which satisfy the four properties 5.1.1 to 5.1.4 verified by $\hat{G}$, then there exists a (non-canonical) Γ -equivariant homeomorphism $\hat{G}_1 \rightarrow \hat{G}_2$ which conjugates the involutions on $\hat{G}_1$ and $\hat{G}_2$ and which maps homeomorphically each $\mathbb{R}$ -orbit in $\hat{G}_1$ onto an $\mathbb{R}$ -orbit in $\hat{G}_2$. ■

Let us note the following useful facts:

1. — Property 5.1.3 implies that the orbits of $\hat{\mathcal{F}}$ are all homeomorphic to the real line (there are no closed orbits).
2. — Let $\hat{X}$ be a space satisfying the hypothesis of Property 5.1.5, let $\hat{g} \in \hat{G}$ and consider the two distinct points $\hat{g}(-\infty)$ and $\hat{g}(\infty) \in \partial \hat{G}$. Since $\partial \hat{X} = \partial \hat{G}$, there exists a geodesic in $\hat{X}$ having $\hat{g}(-\infty)$ and $\hat{g}(\infty)$ as endpoints. Thus, the map $\mathcal{Q} : G\hat{X} \rightarrow \hat{G}$ of Property 5.1.5 is surjective.
3. — The hypothesis in Property 5.1.5 is satisfied in the particular case where $\hat{X}$ is X , the Cayley graph of our group Γ .
4. — Suppose that the space $\hat{X}$ in Property 5.1.5 satisfies furthermore the following property: a geodesic line joining any pair of distinct points in $\partial \hat{X}$ is unique up to parametrization. Then, the map $\mathcal{Q} : G\hat{X} \rightarrow \hat{G}$ is a homeomorphism. Indeed, we have, by Property 5.1.2, a canonical homeomorphism $\partial^2 \Gamma \rightarrow \partial^2 \hat{X}$, and the existence of the canonical homeomorphism $\hat{G}/\mathbb{R} \rightarrow \partial^2 \Gamma = \partial^2 \hat{X}$ (Property 5.1.4) implies that the map $\mathcal{Q} : G\hat{X} \rightarrow \hat{G}$ is bijective. Since this map is furthermore continuous and proper, it is a homeomorphism. Thus, in the case considered, the space of geodesic lines $G\hat{X}$, equipped with its geodesic flow, is a model for the geodesic flow space $\hat{G}$ associated to Γ .
5. — Let Y be a proper geodesic metric space which satisfies the global Alexandroff $\text{CAT}(\chi)$ inequality for some $\chi < 0$ and let Γ be a group of isometries of Y acting on Y in a convex-cocompact manner (see [Boi] and [Coo]). Then, Γ is hyperbolic and $\partial \Gamma$ can be identified with the limit set $\Lambda \subset \partial Y$ of this action of Γ . Let GY be the space of geodesics of Y , equipped with its metric defined in Section 2 above (Definition 2.1), and let $\Omega \subset GY$ be the space of geodesic lines $g : \mathbb{R} \rightarrow Y$ satisfying $g(-\infty)$ and $g(\infty) \in \Lambda$. We equip Ω with the induced metric. Since Y satisfies the $\text{CAT}(\chi)$ inequality with $\chi < 0$, there is a unique geodesic line (up to parametrization) joining two distinct given points in ∂Y . It follows that Ω is a model for $\hat{G}(\Gamma)$. We note that in the special case where Γ acts cocompactly

on Y , we have $\Lambda = \partial Y$ and therefore $\hat{G} = \Omega = GY$. For example, if Γ is the fundamental group of a closed manifold M equipped with a Riemannian metric of negative sectional curvature, then the universal cover $\tilde{M}$ satisfies a global CAT(λ) inequality for some $\lambda < 0$ and we obtain $\hat{G}(\Gamma) = G\tilde{M} = T^1(\tilde{M})$, the unit tangent bundle of $\tilde{M}$ equipped with the Riemannian geodesic flow, and $\hat{G}(\Gamma)/\Gamma = T^1(M)$, the unit tangent bundle of M equipped with its geodesic flow.

Proposition 5.2.— *Let $\varphi \in \Phi$, let $\xi \in \partial\Gamma$ be the point at infinity of φ and let $\eta \in \partial\Gamma$ with $\eta \neq \xi$. Then, there exists a primitive φ -gradient line $\ell : \mathbb{R} \rightarrow X$ satisfying $\ell(-\infty) = \eta$. (Of course, we have $\ell(\infty) = \xi$.) Furthermore, the number of images of such lines ℓ is bounded above by a constant N_2 which depends only on X (that is, on the group Γ with its given set of generators).*

PROOF. We start with an arbitrary geodesic line $L : \mathbb{R} \rightarrow X$ satisfying $L(-\infty) = \eta$ and $L(\infty) = \xi$, with $L(0) \in \Gamma$ (which implies $\ell(\mathbb{Z}) \subset \Gamma$). For every $n \in \mathbb{N}$, let $g_n : [0, \infty[\rightarrow X$ be the smallest φ -gradient ray starting at $L(-n)$ and let $r_n : [-n, \infty[\rightarrow X$ be the map defined by $r_n(t) = g_n(t+n)$ for all $t \in [-n, \infty[$. Thus, in particular, $r_n(-n) = g_n(0) = L(-n)$. For every $n \in \mathbb{N}$, the point $r_n(t)$ converges to ξ as $t \rightarrow \infty$. By the δ -hyperbolicity of X , we have, for all $n \in \mathbb{N}$, $|r_n(0) - L(0)| \leq 8\delta$ (see for example Corollary 3.5 of Chapter 1 of [CP1]). Since the maps $r_n : [-n, \infty[\rightarrow X$ are geodesics and since the set $\{r_n(0)\}$ is contained in the compact set $B(L(0), 8\delta)$, Ascoli's theorem shows that there is a subsequence (r_{n_k}) of (r_n) and a geodesic line $\ell : \mathbb{R} \rightarrow X$ such that $r_{n_k} \rightarrow \ell$ uniformly on compact sets as $k \rightarrow \infty$. As the space X is a graph, the convergence of (r_{n_k}) to ℓ as $k \rightarrow \infty$ implies that given $i \in \mathbb{Z}$, we can find $K = K(i)$ such that $r_{n_k}(i) = \ell(i)$ for every $k \geq K$. Now since two smallest φ -gradient rays starting at the same vertex coincide, the equality $r_{n_k}(i) = \ell(i)$ implies that r_{n_k} and ℓ coincide on the ray $[i, \infty[$ for every $k \geq K$. This shows in particular that ℓ is a primitive φ -gradient line.

By the δ -hyperbolicity of X , we have $|r_{n_k}(t) - L(t)| \leq 8\delta$ for every k and for every $t \geq -n_k$ (see for example Corollary 3.4 of Chapter 1 of [CP1]). Letting $k \rightarrow \infty$, we obtain $|\ell(t) - L(t)| \leq 8\delta$ for every $t \in \mathbb{R}$. Thus, we have $\ell(-\infty) = L(-\infty) = \eta$. This proves the existence of a primitive φ -gradient line ℓ with the required properties.

We prove now that there are only finitely many images of such lines ℓ (associated to a given $\varphi \in \Phi$ and $\eta \in \partial\Gamma$). Consider again an arbitrary geodesic line $L : \mathbb{R} \rightarrow X$ satisfying $L(-\infty) = \eta$, $L(\infty) = \xi$ and $L(0) \in \Gamma$. Let $(x_n = L(t_n))_{n \in \mathbb{Z}}$ be a sequence of vertices on the image set $L(\mathbb{R})$ satisfying $t_{n+1} > t_n$ and $|x_n - x_{n+1}| > 16\delta$ for every $n \in \mathbb{Z}$. Finally, for every $n \in \mathbb{Z}$, let $B_n = B(x_n, 8\delta)$. If $\ell : \mathbb{R} \rightarrow X$ is any primitive φ -gradient line satisfying $\ell(-\infty) = \eta$ and $\ell(\infty) = \xi$, then, by the δ -hyperbolicity of X we have $\ell(\mathbb{R}) \cap B_n \neq \emptyset$ for every $n \in \mathbb{Z}$ (see for example Corollary 3.3 of Chapter 1 of [CP1]).

For the rest of the proof of Proposition 5.2, we shall use the two following lemmas :

Lemma 5.3.— *Let $n \in \mathbb{Z}$, let $x \in B_n$ and let $r : [0, \infty[\rightarrow X$ be a geodesic ray starting at x and satisfying $r(\infty) = \xi$. Then, the image of r has nonempty intersection with the ball B_{n+1} .*

PROOF. We have $r(\infty) = \xi = L(\infty)$. Consider the infinite geodesic triangle consisting of the union of $r([0, \infty[)$ with the positive subray of L starting at x_n and with a geodesic

segment $[x, x_n]$. By the δ -hyperbolicity of X , every point on one side of this triangle is at distance $\leq 8\delta$ from the union of the two other sides (see for example Proposition 3.2 of Chapter 1 of [CP1]). We have $\text{dist}(x_{n+1}, [x, x_n]) > 8\delta$. Therefore, there exists a point y on $r([0, \infty[)$ with $|x_{n+1} - y| \leq 8\delta$. Thus, $y \in B_{n+1}$ and the image of r has therefore nonempty intersection with B_{n+1} . ■

Lemma 5.4.— *Let $\varphi \in \Phi$. If two primitive φ -gradient lines coincide at some point, then they coincide everywhere after that point. In other words, if $\ell_1 : \mathbb{R} \rightarrow X$ and $\ell_2 : \mathbb{R} \rightarrow X$ are two primitive φ -gradient lines and if t_1 and $t_2 \in \mathbb{R}$ satisfy $\ell_1(t_1) = \ell_2(t_2)$, then we have $\ell_1(t_1 + t) = \ell_2(t_2 + t)$ for every $t \geq 0$.*

PROOF. Consider first the particular case where $\ell_1(t_1) = \ell_2(t_2)$ is a vertex of X . Then, the statement follows from the uniqueness of the smallest φ -gradient ray starting at any given vertex (see [CP2], the proof of Proposition 6.2.). Therefore, we need only consider the case where the point $\ell_1(t_1) = \ell_2(t_2)$ is at the interior of some edge e of X . Since ℓ_1 and ℓ_2 are gradient lines, they run through the edge e in the same direction. Thus, the problem is reduced to the particular case considered at the beginning of the proof. This completes the proof of Lemma 5.4. ■

To continue the proof of Proposition 5.2, let $\ell_1 : \mathbb{R} \rightarrow X$ and $\ell_2 : \mathbb{R} \rightarrow X$ be two primitive φ -gradient lines satisfying $\ell_i(-\infty) = \eta$ and $\ell_i(\infty) = \xi$ for $i = 1$ and 2 and such that the images $\ell_1(\mathbb{R})$ and $\ell_2(\mathbb{R})$ are distinct. Using the fact that if the images of ℓ_1 and ℓ_2 intersect at some point of X , then they coincide after this point (Lemma 5.4), we see that if $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_n \neq \emptyset$ for all $n \in \mathbb{Z}$, then we have $\ell_1(\mathbb{R}) = \ell_2(\mathbb{R})$. Since the images of ℓ_1 and ℓ_2 are distinct, we can find an integer n_0 such that $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_{n_0} = \emptyset$. Now if $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_{n_0} = \emptyset$, then Lemmas 5.3 and 5.4 imply that $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_{n_0-1} = \emptyset$. Thus, we see by induction that if $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_{n_0} = \emptyset$, then we have $\ell_1(\mathbb{R}) \cap \ell_2(\mathbb{R}) \cap B_n = \emptyset$ for every $n \leq n_0$.

Suppose now that there exist $M \geq 1$ primitive φ -gradient lines $\ell_1, \dots, \ell_M$ whose images are all distinct. For every pair of integers i and j satisfying $1 \leq i < j \leq M$, we can find an integer $n_{i,j}$ such that for all $n \leq n_{i,j}$, we have $\ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}) \cap B_n = \emptyset$. Taking $n \leq n_{i,j}$ for every i and j , we obtain a ball B_n such that $\ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}) \cap B_n = \emptyset$ for all $1 \leq i < j \leq M$. Since each line ℓ_i has nonempty intersection with B_n , it contains a vertex of B_n . Thus, we can find M distinct vertices of X in the ball B_n . This shows that M is bounded above by the number of vertices of B_n , that is, we have $M \leq N_2 = C(C-1)^c$ where C is the cardinality of the generating set A and $c = [8\delta]$. This proves Proposition 5.2. ■

Proposition 5.5.— *The map $\Phi \rightarrow \partial\Gamma$ which associates to each cocycle $\varphi \in \Phi$ its point at infinity $\varphi(\infty)$ is Γ -equivariant, surjective and at most N_1 -to-one for some $N_1 > 0$ depending only on the group Γ with its given set of generators.*

PROOF. See [CP1], Propositions 7.4 and 7.9, Chapter 3, where N_1 is given explicitly as $N_1 = (2N_0 + 1)^{k_0}$, where $k_0 = [8\delta + 1]$ and N_0 is the number of vertices of X which are at distance $\leq k_0$ from the identity. ■

Corollary 5.6.— *The map $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ defined by $(\varphi, \ell) \mapsto \ell$ is continuous and at most N_1 -to-one, with N_1 being the constant of Proposition 5.5.*

PROOF. Let $\ell \in GX$. By Proposition 5.5, the number of cocycles $\varphi \in \Phi$ satisfying $\varphi(\infty) = \ell(\infty)$ is bounded above by N_1 . The same constant bounds therefore the cardinality of the set of $(\varphi, \ell) \in \tilde{\Phi}_{\mathbb{R}}$ mapping to ℓ . ■

Let $\tilde{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ be the map obtained by composing the map $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ of Corollary 5.6 with the map $Q : GX \rightarrow \hat{G}$ defined in Property 5.1.5 of Theorem 5.1, with $\hat{X} = X$, the Cayley graph of Γ . We can now prove the following

Proposition 5.7.— *The map $\tilde{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ is continuous, surjective, Γ -equivariant and at most N_3 -to-one for some constant N_3 depending only on Γ equipped with its given set of generators. Furthermore, $\tilde{\pi}_{\mathbb{R}}$ sends bijectively every $\tilde{\mathcal{F}}$ -orbit in $\tilde{\Phi}_{\mathbb{R}}$ onto an $\tilde{\mathcal{F}}$ -orbit in $\hat{G}$.*

PROOF. By Property 5.1.5 of Theorem 5.1, the map $Q : GX \rightarrow \hat{G}$ is continuous and Γ -equivariant. It is clear that the map $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ is also continuous and Γ -equivariant. Thus, $\tilde{\pi}_{\mathbb{R}}$ is continuous and Γ -equivariant. Let us prove the surjectivity. Let $\hat{g} \in \hat{G}$ and let η and $\xi \in \partial\hat{G} = \partial\Gamma$ denote the points $\eta = \hat{g}(-\infty)$ and $\xi = \hat{g}(\infty)$. By Proposition 5.5, we can find a cocycle $\varphi \in \Phi$ satisfying $\varphi(\infty) = \xi$, and the number of such cocycles is bounded by a constant N_1 depending only on the group Γ together with its fixed generating set. Since $\eta \neq \xi$, we can find now, by Proposition 5.2, a primitive φ -gradient line $\ell : \mathbb{R} \rightarrow X$ satisfying $\ell(-\infty) = \eta$. Now, by Property 5.1.5 of Theorem 5.1, the map $Q : GX \rightarrow \hat{G}$ maps bijectively every orbit of the geodesic flow $\mathcal{F}$ on GX onto an $\tilde{\mathcal{F}}$ -orbit in $\hat{G}$. Thus, there exists a unique geodesic line $\ell' : \mathbb{R} \rightarrow X$ in the $\mathcal{F}$ -orbit of ℓ which is mapped to $\hat{g}$ by Q . This proves in particular the surjectivity of $\tilde{\pi}_{\mathbb{R}}$.

Now since the number of images of primitive φ -gradient lines ℓ satisfying $\ell(-\infty) = \eta$ and $\ell(\infty) = \xi$ is bounded above by N_2 , a constant that depends only on Γ with its given set of generators (Proposition 5.2), we conclude that the number of inverse images of $\hat{g} \in \hat{G}$ by the map $\tilde{\pi}_{\mathbb{R}}$ is bounded above by $N_3 = N_1 \times N_2$, a constant that depends only on the group Γ together with its given set of generators.

Finally, It is clear also that the restriction of the map $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ to an arbitrary $\tilde{\mathcal{F}}$ -orbit is bijective. Thus, $\tilde{\pi}_{\mathbb{R}}$ sends bijectively every $\tilde{\mathcal{F}}$ -orbit in $\tilde{\Phi}_{\mathbb{R}}$ onto an $\tilde{\mathcal{F}}$ -orbit in $\hat{G}$. This completes the proof of Proposition 5.7. ■

We consider now the quotient spaces $\hat{G}/\Gamma = \overline{G}$ and $\tilde{\Phi}_{\mathbb{R}}/\Gamma$, equipped with the quotient topologies. As the action of Γ on $\hat{G}$ is properly discontinuous and cocompact (Property 5.1.2 of Theorem 5.1), the space $\overline{G}$ is Hausdorff and compact. We already know that the action of Γ on $\tilde{\Phi}_{\mathbb{R}}$ is free and properly discontinuous (see the remark contained in Definition 4.2). We know also that the action of Γ on $\tilde{\Phi}_{\mathbb{R}}$ preserves the flow $\tilde{\mathcal{F}}$ and that there is a natural induced flow on the quotient space $\tilde{\Phi}_{\mathbb{R}}/\Gamma$, which we denoted by $\mathcal{F}^*$. In the same way, since the action of Γ on $\hat{G}$ preserves the flow $\tilde{\mathcal{F}}$ (Property 5.1.1 of Theorem 5.1), there is a natural induced flow on $\overline{G} = \hat{G}/\Gamma$, which we denote by $\overline{\mathcal{F}} = (\overline{\mathcal{F}}_t)_{t \in \mathbb{R}}$. Finally, since the map $\tilde{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ is Γ -equivariant, it induces a natural quotient map $\overline{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \overline{G}$. We have the following

Proposition 5.8.— *The map $\overline{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \overline{G}$ is continuous, surjective and at most*

N_3 -to-one, with N_3 being the constant of Proposition 5.7. Furthermore, $\bar{\pi}_{\mathbb{R}}$ sends each $\mathcal{F}^*$ -orbit of $\tilde{\Phi}_{\mathbb{R}}/\Gamma$ onto an $\bar{\mathcal{F}}$ -orbit of $\bar{G}$.

PROOF. The surjectivity and the continuity of $\bar{\pi}_{\mathbb{R}}$ follow from the surjectivity and the continuity of $\tilde{\pi}_{\mathbb{R}}$. In the same manner, the fact that $\bar{\pi}_{\mathbb{R}}$ sends $\mathcal{F}^*$ -orbits in $\tilde{\Phi}_{\mathbb{R}}/\Gamma$ onto $\bar{\mathcal{F}}$ -orbits of $\bar{G}$ follows from the fact that $\tilde{\pi}_{\mathbb{R}}$ sends $\hat{\mathcal{F}}$ -orbits of $\tilde{\Phi}_{\mathbb{R}}$ onto $\hat{\mathcal{F}}$ -orbits of $\hat{G}$. It remains to see that $\bar{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ is at most N_3 -to-one. Let $\bar{g} \in \bar{G}$. Then, $\bar{g}$ is the Γ -orbit of an element $\hat{g} \in \hat{G}$. The set $\bar{\pi}_{\mathbb{R}}^{-1}(\bar{g})$ is the set of Γ -orbits of $\tilde{\pi}_{\mathbb{R}}^{-1}(\hat{g})$. As the cardinality of $\tilde{\pi}_{\mathbb{R}}^{-1}(\hat{g})$ is bounded above by N_3 , the cardinality of $\bar{\pi}_{\mathbb{R}}^{-1}(\bar{g})$ is bounded above by this same constant. This completes the proof of Proposition 5.8. ■

Combining now Proposition 1.21, Proposition 5.8 and Theorem 2.5, we obtain the following

Corollary 5.9.— *There exists a two-sided subshift of finite type (Σ, T) and a map from $\text{Susp}(\Sigma, T)$ to $\bar{G} = \hat{G}/\Gamma$ which is continuous, surjective and at most N_3 -to-one, with N_3 being a constant which depends only on the group Γ together with its given set of generators. Furthermore, the map $\text{Susp}(\Sigma, T) \rightarrow \bar{G}$ sends each orbit of the suspension flow onto an $\bar{\mathcal{F}}$ -orbit of $\bar{G}$.*

The special case where Γ is a free group, which we describe now in detail, is an instance where the map $\text{Susp}(\Sigma, T) \rightarrow \bar{G}$ can be made to be a topological flow conjugacy. We shall describe also the case where $\Gamma = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the map $\bar{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \bar{G}$ is not injective.

The case where Γ is the free group.

Let us consider the particular case where Γ is a free group on n generators $a_1, \dots, a_n$. Let A denote the generating set $\{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$. We equip A with an arbitrary total order relation. The Cayley graph X associated to (Γ, A) is then the homogeneous tree of degree $2n$. Several of the maps which we have considered above (the map $\pi : \Phi \rightarrow \partial\Gamma$ of Proposition 5.3, the map $\tilde{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ of Proposition 5.7, the map $\tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \bar{G}$ of Proposition 5.8 and the map $\text{Susp}(\Sigma, T) \rightarrow \bar{G}$ of Corollary 5.9), instead of being only finite-to-one, are homeomorphisms in this special case. Furthermore, we can give explicit descriptions of the spaces Φ , $\tilde{\Phi}$, $\tilde{\Phi}_{\mathbb{R}}$, $\bar{\Phi}$, $\tilde{\Phi}/\Gamma$, $G\tilde{X}/\Gamma$ and $\bar{G}$, associated to (Γ, A) .

We start with the space Φ of cocycles on X . There is an explicit conjugacy between Φ , equipped with the action of the map α , and a one-sided subshift of finite type. To describe this conjugacy, we recall first that every cocycle on X is a Busemann cocycles (see [CP2], Corollary 5.7). Thus, there is a natural one-to-one correspondence between Φ and the set of geodesic rays starting at the identity. Now since every geodesic ray in X is entirely determined by the sequence of labels of its consecutive edges, the space Φ , which we regard now as the space of geodesic rays in X starting at the identity, can be naturally identified with the one-sided subshift of finite type $\Sigma_+ = \Sigma_+(A, 1, W)$, where $W \subset A \times A$ is the set of pairs (a, b) with $ab \neq 1$. With this identification, the map $\alpha : \Phi \rightarrow \Phi$ is just the shift map $(x_n)_{n \geq 0} \mapsto (x_{n+1})_{n \geq 0}$ on Σ_+ (independently of the choice of the order relation on A).

Consider now the space $\tilde{\Phi}$ of pairs (φ, ℓ) , with $\varphi \in \Phi$ and $\ell : \mathbb{R} \rightarrow X$ a primitive φ -gradient line. The point $\ell(\infty)$ determines in a unique way the cocycle φ . On the other hand, given $\xi \in \partial\Gamma$, every geodesic line $\ell : \mathbb{R} \rightarrow X$ with $\ell(0) \in \Gamma$ and $\ell(\infty) = \xi$ is the primitive φ -gradient line for the unique cocycle $\varphi \in \Phi$ satisfying $\varphi(\infty) = \xi$. Thus, there is a natural one-to-one correspondence between the space $\tilde{\Phi}$ and the space of geodesic lines $\ell : \mathbb{R} \rightarrow X$ satisfying $\ell(0) \in \Gamma$.

Consider now the space $\tilde{\Phi}_{\mathbb{R}}$. The map $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ sending each element (φ, ℓ) to the line ℓ is a homeomorphism which is Γ -equivariant and which conjugates the action of $\tilde{\mathcal{F}}$ on $\tilde{\Phi}_{\mathbb{R}}$ with the action of $\mathcal{F}$ on GX . The space GX of geodesics, equipped with its natural flow $\mathcal{F}$, is a model for the geodesic flow space $\hat{G}$. (This is the content of Remark 4 after Theorem 5.1.). With these identifications, the homeomorphism $\tilde{\Phi}_{\mathbb{R}} \rightarrow GX$ becomes the map $\tilde{\pi}_{\mathbb{R}} : \tilde{\Phi}_{\mathbb{R}} \rightarrow \hat{G}$ of Proposition 5.7. Thus, in the case considered and with the natural identifications that we have made, the maps $\tilde{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \overline{G}$ and $\text{Susp}(\Sigma, T) \rightarrow \overline{G}$ of Proposition 5.8 and Corollary 5.9 are topological flow conjugacies.

We note now that the subspace $\tilde{\Phi}_0 \subset \tilde{\Phi}_{\mathbb{R}}$ (and therefore, the spaces $\overline{\Phi}$ and $\tilde{\Phi}/\Gamma$) can be naturally identified with the subspace GX_0 of GX consisting of the geodesic lines $\ell : \mathbb{R} \rightarrow X$ satisfying $\ell(0) = Id$.

The space GX_0 can be identified with the two-sided subshift of finite type $\Sigma = \Sigma(A, 1, W)$, where A and W are defined as above, that is, $A = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\} \subset \Gamma$ and $W = \{(a, b) ; ab \neq Id\} \subset A \times A$. This identification is done by representing each $\ell \in GX_0$ by the sequence $(x_n)_{n \in \mathbb{Z}}$ where x_n denotes the label of the oriented edge going from $\ell(n)$ to $\ell(n+1)$. With the identifications $\overline{\Phi} \simeq GX_0 \simeq \Sigma$, the map $\bar{\alpha} : \overline{\Phi} \rightarrow \overline{\Phi}$ is just the shift map $T : \Sigma \rightarrow \Sigma$, and we have $\overline{G} \simeq \tilde{\Phi}_{\mathbb{R}}/\Gamma \simeq GX/\Gamma \simeq \text{Susp}(\Sigma, T)$, these identifications preserving the flows.

Finally let us give another description of $\overline{G}$.

We start with an alternative description of the space GX/Γ (which we identify as above with the space $\overline{G}$). The group Γ acts freely and properly discontinuously on X and the quotient space $\overline{X} = X/\Gamma$ is a graph with one vertex and $2n$ edges, each edge being attached by its two endpoints to that unique vertex. The canonical map $X \rightarrow \overline{X}$ is a universal covering map. Let us say that a map $g : \mathbb{R} \rightarrow \overline{X}$ is a *local geodesic line* if for every $t \in \mathbb{R}$, there exists a neighborhood of t such that the restriction of g to this neighborhood is geodesic. It is easy to see that $g : \mathbb{R} \rightarrow \overline{X}$ is a local geodesic line if and only if a lift (or, equivalently, any lift) of g to the universal cover X is a geodesic line. This leads to a canonical homeomorphism between the space $\overline{G} = GX/\Gamma$ and the space $G^{loc} \overline{X}$ of local geodesic lines in $\overline{X}$, equipped with the topology of uniform convergence on compact sets. This homeomorphism conjugates the flows on the two spaces.

Let v denote the (unique) vertex of $\overline{X}$ and let $G_0^{loc} \overline{X} \subset G^{loc} \overline{X}$ be the set of local geodesic lines $g : \mathbb{R} \rightarrow \overline{X}$ satisfying $g(0) = v$. The space GX_0 of geodesic lines $\ell : \mathbb{R} \rightarrow X$ satisfying $\ell(0) = Id$ is mapped homeomorphically onto $G_0^{loc} \overline{X}$ by the projection map. We have therefore the following natural identifications

$$G_0^{loc} \overline{X} \simeq \Sigma \subset \text{Susp}(\Sigma, T) \simeq G^{loc} \overline{X} \simeq \overline{G}.$$

The case where $\Gamma = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Let us consider now the case where Γ is the group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, equipped with the symmetric system of generators $A = \{(1,0), (-1,0), (0,\bar{1})\}$. We endow A with an arbitrary order relation. The Cayley graph X associated to (Γ, A) is a simplicial graph homeomorphic to a bi-infinite ladder (see Figure 4). The space ∂X consists in two points, $-\infty$ and ∞ , and the set of flow orbits in GX is infinite countable. The space $\hat{G}$ is homeomorphic to $\mathbb{R} \times \{-\infty, \infty\}$, that is, to the union of two disjoint copies of $\mathbb{R}$.

Figure 4

Since the action of Γ on $\hat{G}$ is cocompact (Property 5.1.2 of Theorem 5.1), the element $(0,1)$ acts non-trivially, as a translation, on each copy of $\mathbb{R}$. However, the involution $(0,\bar{1})$ acts trivially on $\hat{G}$. Indeed, if this action were non-trivial, then it would interchange the two points at infinity of some $\mathbb{R}$ -orbit in $\hat{G}$. Since there is a Γ -equivariant map $GX \rightarrow \hat{G}$ which sends homeomorphically $\mathbb{R}$ -orbits in GX to $\mathbb{R}$ -orbits in $\hat{G}$ (Property 5.1.5 of Theorem 5.1), the action of $(0,\bar{1})$ on some orbit in GX would interchange these two points at infinity, which is a contradiction. Thus, $(0,\bar{1})$ acts trivially on $\hat{G}$. We deduce therefore that the quotient space $\overline{G} = \hat{G}/\Gamma$ is homeomorphic to the disjoint union of two circles.

Let us describe now the space Φ of cocycles on X . Consider first the Busemann cocycles. We recall again that a Busemann cocycle is associated to some geodesic ray in X starting at a vertex and that two such geodesic rays whose images coincide up to a compact initial segment define the same Busemann cocycle. There are exactly four equivalence classes of geodesic rays in X , which we describe now. If we represent this Cayley graph as a horizontal ladder (as in Figure 4), then these four classes are represented by horizontal rays converging respectively to the left and to the right, on the top and on the bottom part of the ladder. This gives four Busemann cocycles in Φ , each one being associated to one of these classes of geodesic rays in X . In Figure 4 (a) and (b), we have represented the restriction on the set of vertices Γ of X of two horofunctions $h : X \times X \rightarrow \mathbb{R}$ whose cocycles are the Busemann cocycles associated to the two classes of geodesic rays converging to the right. The two other Busemann cocycles are obtained by taking the symmetric images of Figure 4 (a), and (b), with respect to a vertical axis passing through a vertical edge.

In Figure 4 (c), we have represented the values on Γ of a horofunction $h : X \times X \rightarrow \mathbb{R}$ whose associated cocycle is *not* a Busemann cocycle. In this example, the value of h on the midpoint of a vertical edge is $\frac{1}{2}$ + the common value taken by h on its endpoints. It is easy to see that the three cocycles defined by Figure 4 (a), (b) and (c), together with the three other cocycles obtained by taking symmetric images with respect to a vertical axis (and these are the ones whose point at infinity is to the left of the figure), constitute all the elements in Φ . Thus, the cardinality of Φ is 6. Now for each of the three cocycles φ represented in Figure 4, there are two distinct images of primitive φ -gradient lines, one being the upper side of the ladder, and the other one being the lower side (for any choice of total order relation on A). Thus, the number of $\hat{\mathcal{F}}$ -orbits in $\hat{\Phi}_{\mathbb{R}}$ is 12. The involution $(0, \bar{1}) \in \Gamma$ interchanges the upper and the lower part of the ladder. Thus, the space $\hat{\Phi}_{\mathbb{R}}/\Gamma$ contains exactly 6 orbits, and these orbits are closed. We can conclude now that the map $\hat{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \overline{G}$ is three-to-one and sends homeomorphically each $\hat{\mathcal{F}}$ -orbit to an $\overline{\mathcal{F}}$ -orbit.

Index of maps

- $\alpha : \Phi \rightarrow \Phi$ (Definition 1.4)
- $\mathcal{F}_t : GX \rightarrow GX$ (Definition 2.1)
- $\bar{\alpha} : \bar{\Phi} \rightarrow \bar{\Phi}$ (Definition 2.3)
- $\hat{\alpha} : \check{\Phi} \rightarrow \check{\Phi}$ (Definition 2.6)
- $\alpha^* : \check{\Phi}/\Gamma \rightarrow \check{\Phi}/\Gamma$ (Definition 2.11)
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- $\bar{\pi}_{\mathbb{R}} : \hat{\Phi}_{\mathbb{R}}/\Gamma \rightarrow \bar{G}$ (Proposition 5.8)

References

- [Ano] D. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, Proc. Steklov Inst. Math. 90 (1967).
- [Bot] R. Bott, Marston Morse and his mathematical works, Bulletin (new Series) Amer. Math. Soc., Volume 3, Number 3, 907-944 (1980).
- [Bou] M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace, *L'enseignement Mathématique*, **41**, 63-102 (1995).
- [Bow] R. Bowen, Symbolic dynamics for hyperbolic flows, Amer. J. Math. 95, 429-460 (1973).
- [Cha] C. Champetier, Petite simplification dans les groupes hyperboliques, Ann. Fac. Sci. Toulouse Math. (6) 3 (1994), no. 2, 161-221.
- [CDP] M. Coornaert, T. Delzant and A. Papadopoulos, Géométrie et Théorie des Groupes, Lecture Notes in Mathematics, Vol. 1441, Springer Verlag (1990).
- [Coo] M. Coornaert, Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov, Pacific J. Math. 159 (1993), no. 2, 241-270.
- [CP1] M. Coornaert and A. Papadopoulos, Symbolic Dynamics and Hyperbolic Groups, Lecture Notes in Mathematics, Vol. 1539, Springer Verlag (1993).
- [CP2] M. Coornaert and A. Papadopoulos, Horofunctions and symbolic dynamics on Gromov hyperbolic groups, preprint, IRMA Strasbourg No. 1998/21 (1998).
- [GH] E. Ghys and P. de la Harpe (Ed.) Sur les groupes hyperboliques d'après Mikhael Gromov, Progress in Mathematics, 83, Birkhäuser Boston, Inc., Boston, MA, 1990.
- [Gro1] M. Gromov, Three remarks on geodesic dynamics and fundamental groups, State University of NY, Stony Brook.
- [Gro2] M. Gromov, Hyperbolic Groups, in Essays in group Theory (S. M. Gersten, Ed.), MSRI Publications, Vol. 8, Springer verlag (1987).
- [Had] J. Hadamard, Les surfaces à courbures opposées et leurs lignes géodésiques. J. Math. Pures Appl. (5) t. 4 27-74 (1898).
- [Mor1] M. Morse, A one-to-one representation of geodesics on a surface of negative curvature, Am. J. Math. 43, 33-51 (1921).
- [Mor2] M. Morse, Recurrent geodesics on a surface of negative curvature, Trans. Amer. Math. Soc. 22, 84-110 (1921).
- [Ser] C. Series, Symbolic dynamics for geodesic flows, Proceedings of the International Congress of Mathematicians, Berkeley 1986, p. 1210-1215.

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