Quantum searching with continuous variables

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A fast quantum search algorithm for continuous variables is presented. The result is the quantum continuous variable analog of Grover’s algorithm originally proposed for qubits. Continuous variable analogous of Hadamard and Fourier transform operations are used in conjunction with inversion about average of quantum states to allow the approximate identification of an unknown quantum state in a way that gives a square root speed-up over search algorithms using classical continuous variables. Also, we show that quantum search algorithm is robust for generalised Fourier transformation on continuous variables.

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Quantum systems can register and process information in ways that classical systems cannot. As a result, it is possible for quantum computers to perform certain computational tasks faster than any classical computer [1–9]. It is progressively becoming clear that at the heart of quantum computation lies two basic quantum phenomena, one is the quantum interference and the other is quantum entanglement. The fact that these phenomena could be used to provide speedups over classical computation was first suggested by Feynman [2] (and confirmed by Lloyd [9]) in the context of quantum simulation. Deutsch and Jozsa [4] and subsequently Simon [6] showed that certain purely computational problems could be sped up by quantum computers. The real upsurge of interest in this field came after Shor’s [10] remarkable discovery of an algorithm for factoring large numbers [11]. Subsequently, a fast quantum search algorithm has been discovered by Grover [12], which takes \( O(\sqrt{N}) \) steps to search an unmarked item in a unsorted list of \( N \) entries. The search process in a classical computer takes \( O(N) \) number of steps. Using properties of unitary transformation, Bennett et al [13] have shown that search cannot be accomplished in less than \( O(\sqrt{N}) \) steps. Zalka [15] has shown that the search algorithm is optimal. Farhi and Gutmann [14] have shown that the time taken to reach a target state is \( O(\sqrt{N}) \), within a Hamiltonian description of search algorithm. Grover’s algorithm has already been implemented by Chuang et al [16] and Jones et al [17] using nuclear magnetic resonance quantum computers. Further, Grover [18] has generalised his algorithm which uses almost any unitary transformation instead of W-H transformation and selective inversion of phase of the qubit basis states. Lloyd [19] has argued that quantum search would work even without entanglement. Recently, the time-dependent generalisation and unitary perturbation of Grover’s algorithm has been studied and various bounds have been obtained [20].

These algorithms are usually implemented on quantum systems with discrete spectra, such as a collection of two-level atoms, ions, or spin-1/2 particles (called qubits). However, there are other class of quantum systems whose observable form continuous spectra. So, it is a curious question to ask how these algorithms can be implemented with quantum systems having continuous variables? With the recent advancement in our understanding of manipulation of continuous quantum information in teleportation [21], error-correction codes [22,23] and its feasibility of implementing with linear devices [24] it is natural to ask whether one can provide the quantum algorithms over continuous variables. In fact, the usefulness of quantum computation over continuous variables has been emphasised by Lloyd and Braunstein [25]. They have shown that universal quantum computation over continuous variable is not only possible, but could be effected using simple non-linear operations combined with linear operations. These operations form a universal set of quantum gates for continuous variables that allow the performance of ‘quantum floating point’ arithmetic. While discrete quantum computation can be thought of as the coherent manipulation of qubits, continuous quantum computation can be thought of as the manipulation of ‘qunats’, where the qunat (pronounced as ‘Q nat’) is the unit of continuous quantum information.

In this letter, we propose a fast quantum search algorithm with continuous variables. Here a continuous
Consider a compact region of the state space divided for each member of the set \( K \). Let \( x_f \) be the centre of the subvolume corresponding to \( k_f \). In the context of this continuous variable embedding, calling the function \( f \) corresponds to adjoining an extra state to the system, originally in the state \(|0\rangle\), and applying an operator \( U_f : |x\rangle|0\rangle \rightarrow |x\rangle|1\rangle \) if \( x \) belongs to the region corresponding to \( k_f \), \(|x\rangle|0\rangle \rightarrow |x\rangle|0\rangle \) otherwise. Clearly, if one samples the region at random by applying the operator to a series of random points, it will take \( O(N) \) calls of the operator to find \( k_f \).

If one exploits the power of quantum superposition and entanglement, however, fewer function calls are required. Let us pick an initial state \(|x_f\rangle = |x_1, x_2, ..., x_n\rangle_i\) of the quantum computer with continuous spectrum at random. The final (target) state is \(|x_f\rangle = |x_1, x_2, ..., x_n\rangle_f\). We need a suitable unitary operator, which can take the initial state to the final state. Like the Hadamard transformation in discrete computation, one of the basic operation with continuous variable is a Fourier transform between position and momentum variable in phase space.

By defining the Fourier transformation as an active operation on \( n \) “qunats” state \(|x\rangle\), we can write

\[
\mathcal{F}|x\rangle = \frac{1}{\sqrt{N}} \int dxe^{2\pi i xy}|y\rangle, \tag{1}
\]

where \( xy = x_1y_1 + ... + x_ny_n \), \(|y\rangle = |y_1, y_2, ..., y_n\rangle\) and \( x \) and \( y \) are both in position basis. This has been used by one of the present authors \([22,24]\) in developing an error correction code for continuous variables. This Fourier transformation can be straightforwardly applied in physical situations. For example, when \(|x\rangle\) represents quadrature states of a set of modes of the electromagnetic field, \(|y\rangle\) is simply the conjugate quadrature.

Suppose, we apply the unitary operator \( \mathcal{F} \) to a basis state \(|x_i\rangle\), then the amplitude of finding the system in the target basis \(|x_f\rangle\) is \((x_f|\mathcal{F}|x_i\rangle| = \mathcal{F}_{ji} = \frac{1}{\sqrt{N}} e^{2\pi i x_i x_f}\). Therefore, the probability of finding the system in the final “qunats” state will be given by \(|\mathcal{F}_{ji}|^2 = \frac{1}{N}\). Hence, we have to repeat the experiment at least \( 1/|\mathcal{F}_{ji}|^2 = \pi^2 \) times to get successfully the state \(|x_f\rangle\). Here, we prove that search algorithm based on continuous variable can take \( \sqrt{\pi} \) steps to reach the final state starting from an initial state. (Here, we may identify the number of entries \( N \) with \( \pi^2 \).

The next operator we need is the unitary operator, which can invert the sign of a basis state \(|x\rangle\). We can define the selective inversion operator for a continuous basis \(|x\rangle\) as

\[
I_x = 1 - 2P_{\Delta x}, \tag{2}
\]

where \( P_{\Delta x} \) is the projection operator for continuous variables. Unlike the discrete case we cannot define the projection operator for the basis \(|x\rangle\) as \( P_x = |x\rangle\langle x|\), because the operator \( P_x \) is an ill defined and it will not satisfy \( P_x^2 = P_x \). The correct projection operator is defined \([26]\) as

\[
P_{\Delta x} = \int_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}} dx'|x\rangle\langle x'|. \tag{3}
\]

The reason for this definition is that we cannot project an arbitrary state represented in terms of continuous basis state onto a point to get the exact eigenvalue. There will be always a spread within an interval. So we can only project a state around \( x_0 \) having a selectivity of measuring apparatus \( \Delta x \). It is not possible to design a device which make a perfectly selective measurement of a
First, we show the action of $C$. We calculate the distance between the resulting state $|x_f\rangle$ and $|x\rangle$ as expected. With the help of inversion operator we can construct a compound operator $\mathcal{C}$ defined as

$$\mathcal{C} = -I_x, \mathcal{F}^\dagger I_{x_f} \mathcal{F}. \quad (4)$$

It may be remarked that the selective inversion of the target state $|x_f\rangle$ can be achieved by attaching an ancilla qubit and considering the quantum XOR circuit for continuous variables [22]. Let us define a basis $\mathcal{F}^\dagger |x_f\rangle = |\tilde{x}_f\rangle$. We can show that the operator $\mathcal{C}$ can preserve the subspace spanned by the basis $|x_i\rangle$ and $|\tilde{x}_f\rangle$. First, we show the action of $\mathcal{C}$ on $|x_i\rangle$. This can be expressed as

$$\mathcal{C}|x_i\rangle = |x_i\rangle - 4P_{\Delta x_i} \mathcal{F}^\dagger P_{\Delta x_f} \mathcal{F}|x_i\rangle + 2\mathcal{F}^\dagger P_{\Delta x_f} \mathcal{F}|x_i\rangle \quad (5)$$

where $P_{\Delta x_i} = \int x_{i1} \rho(x') |x'\rangle |x'\rangle \langle x'| \langle x'|$, $x_{i1} = x_0 + \frac{\Delta x_i}{2}$, $x_{i2} = x_0 - \frac{\Delta x_i}{2}$ and similarly for $P_{\Delta x_f}$. Using these facts, we simplify the above equation

$$\mathcal{C}|x_i\rangle = (1 - \frac{4}{\pi^2}) |x_i\rangle + \frac{2}{\pi^{n/2}} \int_{x_{i1}}^{x_{i2}} d x' e^{2 i x_i x_f} \mathcal{F}^\dagger |x_f\rangle. \quad (6)$$

Similarly, we can evaluate the action of $\mathcal{C}$ on $|\tilde{x}_f\rangle$. It is given by

$$\mathcal{C}|\tilde{x}_f\rangle = |\tilde{x}_f\rangle - \frac{2}{\pi^{n/2}} \int_{x_{i1}}^{x_{i2}} d x' e^{2 i x_i x_f} |x_f\rangle. \quad (7)$$

Thus, the operator $\mathcal{C}$ creates superposition two “qubits” as Grover’s operator creates linear superposition of two qubits. Once this is recognised, we can easily obtain the total number of steps required in reaching the target basis. Here, we use some geometric structures from the projective Hilbert space of a quantum system, to obtain the number of steps. The projective Hilbert space admits a natural measure of distance called Fubini-Study distance [27]. The distance between any two states (need not be normalised) $|\psi_1\rangle$ and $|\psi_2\rangle$ whose projections on $\mathcal{P}$ are $\Pi(\psi_1)$ and $\Pi(\psi_2)$, respectively can be defined as

$$d^2(\psi_1, \psi_2) = 4 \left(1 - |\frac{\psi_1}{\|\psi_1\|} |\frac{\psi_2}{\|\psi_2\|}|^2\right). \quad (8)$$

Here, the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ can be quantum state over continuous variable or discrete variable. During quantum search over continuous variables, we want to reach a state $|\tilde{x}_f\rangle$ from an initial state $|x_i\rangle$. This means we have to travel the Fubini-Study distance between these states which is given by $d^2(|x_i\rangle, |\tilde{x}_f\rangle) = 4(1 - \frac{1}{\pi^2})$.

Application of the operator $\mathcal{C}$ creates a new basis state. We calculate the distance between the resulting state $\mathcal{C}|x_i\rangle = |x_i^{(1)}\rangle$ and the initial state $|x_i\rangle$. We note that the overlap of these states is given by

$$\langle x_i | \mathcal{C}|x_i\rangle = (1 - \frac{4}{\pi^2}) \langle x_i | x_i\rangle + \frac{2}{\pi^2} \Delta x_f \quad (9)$$

For large database search $N = \pi^n$ is very large and if we assume that the measuring device has a narrow selectivity, then $\Delta x_f$ is also small. Hence, we can neglect the second term in (9) (as it is a product of two small terms). With, this idea in mind we can evaluate the distance between these states given by

$$d^2(|x_i\rangle, |x_i^{(1)}\rangle) = \frac{32}{\pi^n} \quad (10)$$

This shows that in one application of the search operator $\mathcal{C}$ we can move the initial basis a distance $O(\sqrt{\pi^n})$. Therefore, to travel the full distance on the quantum state space we need $N_s$ number of steps, where $N_s$ will be given by

$$N_s = \frac{d(|x_i\rangle, |x_i^{(1)}\rangle)}{d(|x_i\rangle, |\tilde{x}_f\rangle)} \approx O(\sqrt{\pi^n}). \quad (11)$$

This shows that a quantum computer based on quants can take $O(\sqrt{\pi^n})$ applications of $\mathcal{C}$ to reach the target state, which otherwise would have taken $O(n\pi^n)$ number of steps by the application of $\mathcal{F}$ on $|x_i\rangle$. This is the quantum search algorithm with continuous variables.

Now we show that the quantum search algorithm over continuous variables is robust to some extent. Instead of the Fourier transform $\mathcal{F}$ if we replace a generalised Fourier transform (GFT) in the search operator $\mathcal{C}$ still the algorithm works, i.e., we do get a square root reduction in the number of steps. We define a generalised Fourier transform as an active operation in the position basis $|x\rangle$ as

$$\mathcal{F}^{(\theta)}|x\rangle = \frac{i}{\pi \sin \theta} \int dy \exp \left[-\frac{i}{\sin \theta} \left((x^2 + y^2) \cos \theta - 2xy\right)\right] |y\rangle. \quad (12)$$

The GFT with a flexible angle $\theta$ gives a physical change of the basis $|x\rangle$ by any desired amount. The GFT for $\theta = 2\pi m$, $m$ being an integer corresponds to no change of basis. The GFT for $\theta = \frac{\pi}{2}$ corresponds to the Fourier transform defined in (1) (up to a constant phase shift equal to $\frac{\pi}{\sqrt{2}}$, $n$ being the number of “qubits”). If we apply GFT to an initial basis $|x_i\rangle$ then by probability rules of quantum theory we have to do at least $O((\pi \sin \theta)^n)$ number of trials to reach a target state $|x_f\rangle$. We will prove that the generalised search operator acting on continuous variables will take $O((\sqrt{\pi \sin \theta})^n)$ steps to find the unknown item.

Now, the search operator with this GFT takes the form

$$\mathcal{C}^{(\theta)} = -I_x, \mathcal{F}^{(\theta)}^\dagger I_{x_f} \mathcal{F}^{(\theta)} \quad (13)$$
We can see the action of the generalised search operator on the basis $|x_i\rangle$, given by

$$
C^{(\theta)}|x_i\rangle = (1 - \frac{4}{\pi \sin \theta})|x_i\rangle + 2\left(\frac{i}{\pi \sin \theta}\right)^{n/2} \int_{x_i}^{x_f} dx' \exp\left[-\frac{i}{\sin \theta}(x_i^2 + x_f^2) \cos \theta - 2x_i x_f\right] \mathcal{F}|x'\rangle
$$

Similarly, the action of the generalised search operator $C^{(\theta)}$ on $|\tilde{x}\rangle$ can be calculated. It is given by

$$
C^{(\theta)}|\tilde{x}\rangle = |\tilde{x}\rangle - 2\left(\frac{-i}{\pi \sin \theta}\right)^{n/2} \int_{x_i}^{x_f} dx' \exp\left[\frac{i}{\sin \theta}(x_i^2 + x_f^2) \cos \theta - 2x_i x_f\right] |x'\rangle
$$

Again, the generalised search operator creates linear superposition of "qunats" in the search process. Now, we can calculate the Fubini-Study distances to know how many steps are needed to reach the target state. The Fubini-Study distance between the states $|x_i\rangle$ and $|\tilde{x}_f\rangle$ is $d^2(|x_i\rangle, |\tilde{x}_f\rangle) = 4(1 - (\frac{\pi \sin \theta}{\pi \sin \theta})^n)$. Notice that single application of the search operator $C^{(\theta)}$ moves the initial state by a distance $d(|x_i\rangle, |x_i^{(1)}\rangle)$, given by

$$
d^2(|x_i\rangle, |x_i^{(1)}\rangle) = \left(\frac{32}{\pi \sin \theta}\right)^n.
$$

Using same argument as before, we can show that to travel a distance $d(|x_i\rangle, |\tilde{x}_f\rangle)$ we need $N_s$ number of steps, given by

$$
N_s = \frac{d(|x_i\rangle, |\tilde{x}_f\rangle)}{d(|x_i\rangle, |x_i^{(1)}\rangle)} \approx O((\pi \sin \theta)^n).
$$

Thus, using a generalised Fourier transform we have proved that there is a square root reduction in the number of steps working with continuous variables. As expected for an angle $\theta = \frac{\pi}{2}$ we get back the original result with the search operator $C$. This result is similar to the recent result of Grover [18], where search algorithm for qubits has been generalised for arbitrary unitary transformations.

In conclusion, we have provided for the first time an efficient, known algorithm (in discrete case) such as quantum searching to be implemented on a quantum computer with continuous variables. The key elements in this generalisation are the continuous analogue of Hadamard transformation and inversion operator which constitute the search operator for qunats in an infinite dimensional Hilbert space. We find that a square root speed up is possible with quantum computers based on qunats. Also, the continuous search is possible with almost any Fourier transformations. This may be practically implemented for any number of large data base search using linear and non-linear optical devices with the qunats being played by the electromagnetic fields. It may be advantageous also for large data base search to use continuous quantum search algorithm, rather than its discrete counter part.

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