Inflation and quintessence with nonminimal coupling

Valerio Faraoni

Research Group in General Relativity (RggR)
Université Libre de Bruxelles, Campus Plaine CP 231
Bvd. du Triomphe, 1050 Bruxelles, Belgium

Abstract

The nonminimal coupling (NMC) of the scalar field to the Ricci curvature is unavoidable in many cosmological scenarios. Inflation and quintessence models based on nonminimally coupled scalar fields are studied, with particular attention to the balance between the scalar potential and the NMC term $\xi R \phi^2 / 2$ in the action. NMC makes acceleration of the universe harder to achieve for the usual potentials, but it is beneficial in obtaining cosmic acceleration with unusual potentials. The slow-roll approximation with NMC, conformal transformation techniques, and other aspects of the physics of NMC are clarified.

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The idea of cosmological inflation is legitimately regarded as a breakthrough of modern cosmology: it solves the horizon, flatness and monopole problem, and it provides a mechanism for the generation of density perturbations needed to seed the formation of structures in the universe [1, 2]. The essential qualitative feature of inflation, the acceleration of the universe, is also required (albeit at a different rate) at the present epoch of the universe in order to explain the data from high redshift supernovae [3]. If confirmed, the latter imply that a form of matter with negative pressure ("quintessence") is beginning to dominate the dynamics of the universe. Scalar fields have been proposed as natural models of quintessence [4, 5, 6, 7, 8].

Inflation is believed to be driven by a scalar field, apart possibly from the $R^2$ inflationary scenario in higher derivative theories of gravity [11], or in supergravity (e.g. [9, 10]).

The inflaton field $\phi$ obeys the Klein-Gordon equation [15]

$$\Box \phi - \xi R \phi - \frac{dV}{d\phi} = 0 ,$$

where $V(\phi)$ is the scalar field potential, $R$ denotes the Ricci curvature of spacetime, and the term $-\xi R \phi$ in Eq. (1.1) describes the explicit nonminimal coupling (NMC) of the field $\phi$ to the Ricci curvature [13, 14]. A possible mass term $m^2 \phi^2/2$ for the field $\phi$, and the cosmological constant $\Lambda$ are embodied in the expression of $V(\phi)$.

Eq. (1.1) is derived from the Lagrangian density

$$\mathcal{L}\sqrt{-g} = \left[ \frac{R}{16\pi G} - \frac{1}{2} \nabla^c \phi \nabla_c \phi - V(\phi) - \frac{\xi}{2} R \phi^2 \right] \sqrt{-g} ,$$

where $g$ is the determinant of the metric tensor $g_{ab}$, and $\nabla_c$ is the covariant derivative operator. In inflationary theories it is assumed that the scalar field dominates the evolution of the universe and that no forms of matter other than $\phi$ are included in the Lagrangian density (1.2).

Two values of the coupling constant $\xi$ are most often encountered in the literature: $\xi = 0$ (minimal coupling) and $\xi = 1/6$ (conformal coupling) [17], while the possibility $|\xi| \gg 1$ (strong coupling) has also been considered many times, for both signs of $\xi$ [18, 19, 20, 21, 22, 24, 6].

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1 Introduction

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Contrarily to common belief, the introduction of NMC is not a matter of taste; NMC is instead forced upon us in many situations of physical and cosmological interest. There
are many compelling reasons to include an explicit nonminimal (i.e. $\xi \neq 0$) coupling in the action: NMC arises at the quantum level when quantum corrections to the scalar field theory are considered, even if $\xi = 0$ for the classical, unperturbed, theory [16]; NMC is necessary for the renormalizability of the scalar field theory in curved space [14, 25, 26]. But what is the value of $\xi$? This problem has been addressed at both the classical and the quantum level ([27, 28, 29], and references therein). The answer depends on the theory of gravity and of the scalar field adopted; in most theories used to describe inflationary scenarios, it turns out that a value of the coupling constant $\xi \neq 0$ cannot be avoided.

In general relativity, and in all other metric theories of gravity in which the scalar field $\phi$ is not part of the gravitational sector, the coupling constant necessarily assumes the value $\xi = 1/6$ [27, 31, 28]. The study of asymptotically free theories in an external gravitational field, described by the Lagrangian density

$$\mathcal{L}_{AF}\sqrt{-g} = \sqrt{-g} \left( aR^2 + bG_{GB} + cC_{abcd}C^{abcd} - \xi R\phi^2 + \mathcal{L}_{matter} \right)$$

(1.3)

(where $G_{GB}$ is the Gauss-Bonnet invariant and $C_{abcd}$ is the Weyl tensor) shows a scale-dependent coupling parameter $\xi(\tau)$. In Refs. [29, 30] it was shown that asymptotically free GUTs have a $\xi$ depending on a renormalization group parameter $\tau$, and that $\xi(\tau)$ converges to $1/6$, $\infty$, or to any initial condition $\xi_0$ as $\tau \to \infty$ (this limit corresponds to strong curvature conditions and to the early universe), depending on the gauge group and on the matter content of the theory. In Ref. [32] it was also obtained that $|\xi(\tau)| \to +\infty$ in $SU(5)$ GUTs. Similar results were derived in finite GUTs without running of the gauge coupling, with the convergence of $\xi$ to its asymptotic value being much faster [29, 30]. An exact renormalization group study of the $\lambda\phi^4$ theory shows that $\xi = 1/6$ is a stable infrared fixed point [33].

In the large $N$ limit of the Nambu-Jona-Lasinio model, $\xi = 1/6$ [34]; in the $O(N)$-symmetric model with $V = \lambda\phi^4$, $\xi$ is generally nonzero and depends on the coupling constants $\xi_i$ of the individual bosonic components [35]. Higgs fields in the standard model have $\xi \leq 0$ or $\xi \geq 1/6$ [36]. Only a few investigations produce $\xi = 0$: the minimal coupling is predicted if $\phi$ is a Goldstone boson with a spontaneously broken global symmetry [37], for a semiclassical scalar field with backreaction and cubic self-interaction [36], and for theories formulated in the Einstein conformal frame [28, 38]. In view of the above results, it is wise to incorporate an explicit NMC between $\phi$ and $R$ in the inflationary paradigm and in quintessence models.

A conservative approach to inflation and quintessence employs general relativity as the underlying gravity theory (exceptions are $R^2$, extended, hyperextended and stringy
inflation and the extended quintessence model of Ref. [8]), and conformal coupling is unavoidable in general relativity, as well as in any metric theory of gravity in which the scalar field is part of the non-gravitational sector (e.g. when $\phi$ is a Higgs field) [27, 31, 28].

The viability of an inflationary scenario and the constraints on the inflationary model are profoundly affected by the presence of NMC and by the value of the coupling constant $\xi$ ([28] and references therein; [39, 40, 41]). The analysis of the various inflationary scenarios considered in the literature usually leads to the result that NMC makes it harder to achieve inflation with a given potential that is known to be inflationary for $\xi = 0$ [43, 41, 44, 38]. There are two main reasons for this difficulty:

1) The common attitude in the literature on nonminimally coupled scalar fields in inflation is that the coupling constant $\xi$ is a free parameter to fine-tune at one’s own will in order to solve problems of the inflationary scenario under consideration. The fine-tuning of certain parameters of inflation is reduced by fine-tuning the extra parameter $\xi$ instead. For example, the self-coupling constant $\lambda$ of the scalar field in the chaotic inflation potential $V = \lambda \phi^4$ is subject to the constraint $\lambda < 10^{-12}$ coming from the observational limits on the amplitude of fluctuations in the cosmic microwave background. This constraint makes the scenario uninteresting because the energy scale predicted by particle physics is much higher. The constraint on $\lambda$ is reduced by fine-tuning $\xi$ instead [45, 39, 40, 46, 18]; while the fine-tuning of $\xi$ is less drastic than that of the self-coupling constant $\lambda$ by several orders of magnitude [39], one cannot be satisfied with the fact that NMC is introduced ad hoc to improve the fine tuning problems (and still does not completely cure them). A more rigorous approach consists in studying the prescriptions for the value of $\xi$ given in the literature (which are summarized in Ref. [28]) and the consequences of NMC for the known inflationary scenarios. The philosophy of this approach is that NMC is often unavoidable and the value of $\xi$ is not arbitrary but is determined by the underlying physics. Once the value of the coupling constant $\xi$ is predicted, one does not have anymore the freedom to adjust its value and the fine-tuning problems that may plague the inflationary scenario reappear. Several inflationary scenarios turn out to be theoretically inconsistent when one takes into account the appropriate values of the coupling constant [28, 38].

2) Most of the inflationary scenarios are built upon the slow-roll approximation [1, 42], in which the Einstein-Friedmann dynamical equations are solved. It is more difficult to achieve the slow rolling of the scalar field when $\xi \neq 0$. In fact, an almost flat section of the potential $V(\phi)$ gives slow rollover of $\phi$ when $\xi = 0$, but its shape is distorted by
the NMC term $\xi R\phi^2/2$ in the Lagrangian density (1.2). The extra term plays the role of an effective mass term for the inflaton. The phenomenon was described by Abbott [43] in the new inflationary scenario with the Ginzburg-Landau potential, by Futamase and Maeda [41] in chaotic inflation, and by Fakir and Unruh [19]; and the generalization to any slow roll inflationary potential is straightforward [28]. This mechanism is quantitatively discussed in Sec. 6.

How general are the previous conclusions? They hold for particular inflationary scenarios, and the conclusion that it is always more difficult to achieve a sufficient amount of inflation in the presence of NMC is premature. In principle, it is possible that a suitable scalar field potential $V(\phi)$ be balanced by the NMC term $\xi R\phi^2/2$ in the Lagrangian density (1.2), thus producing an “effective potential” [23] which is inflationary and even gives a slow-roll regime. In this situation, NMC would make it easier to achieve inflation, thus opening the possibility for a wider class of scalar field potentials to be considered. This possibility is studied in this paper; the discussion is kept as general as possible, without specifying a particular inflationary scenario until it is necessary.

In a previous paper [28], the theoretical consistency of the known inflationary scenarios was studied from the point of view of the theoretical prescriptions for the value of $\xi$ and of the fine-tuning of the parameters. Calculations of density perturbations with nonminimally coupled scalar fields have been performed in Refs. [48, 49, 50, 51, 24, 52, 53], while observational constraints on $\xi$ were derived in Refs. [28, 52, 24, 53]. Here instead we study the effect of NMC by analyzing the dynamical equations for the scale factor of the universe and the scalar field, without specifying the value of the coupling constant $\xi$. Aspects of the physics of NMC which give rise to ambiguities in the literature are also clarified.

Throughout this paper it is assumed that gravity is described by Einstein’s theory with a scalar field as the only source of matter, as described by the Lagrangian density (1.2). Only in Secs. 2 and 4.3 the presence of a different kind of matter in addition to the scalar field is allowed.

The plan of the paper is as follows: in Sec. 2 the possible ways of writing the Einstein equations in the presence of NMC are discussed and compared, together with the corresponding conservation laws and with the issue of the effective gravitational constant. In Sec. 3 the positivity of the energy density of a nonminimally coupled scalar field is discussed in the context of cosmology. In Sec. 4 a necessary condition for the acceleration of a universe driven by a nonminimally coupled scalar field is derived; this is relevant for both inflation and quintessence models based on scalar fields. The question of whether the acceleration can occur due to pure NMC without a potential $V(\phi)$ is answered. In Sec. 5, scalar field potentials that are known to be inflationary for $\xi = 0$ are studied.
and it is shown that NMC spoils inflation rather than helping it. In Sec. 6 the slow-roll approximation to inflation with NMC and the attractor behavior of de Sitter solutions are studied. This is relevant for the calculation of density and gravitational wave perturbations and, ultimately, for the comparison with observations of the cosmic microwave background. Sec. 7 presents a discussion of conformal transformation techniques used in cosmology with NMC, while Sec. 8 contains a discussion and the conclusions.

2 Field equations and conservation laws

When discussing nonminimally coupled scalar fields, many authors choose to reason in terms of an effective gravitational constant instead of keeping a $\phi$-dependent term in the left hand side of the Einstein equations. In this section this approach is discussed and compared with the more conservative approaches using a $\phi$-independent gravitational constant, and the corresponding conservation equations are studied. The following discussion has early parallels, for special cases in Refs. [54, 55], and a recent but incomplete one in Ref. [56].

One begins from the action

$$S = S_g[g_{cd}] + S_{int}[g_{cd}, \phi] + S_\phi[g_{cd}, \phi] + S_m[g_{cd}, \psi_m] =$$

$$= \int d^4x \sqrt{-g} \left[ \left( \frac{1}{2\kappa} - \frac{\xi \phi^2}{2} \right) R - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right] + S_m[g_{cd}, \psi_m] , \quad (2.1)$$

where $\kappa \equiv 8\pi G$, $S_g = (2\kappa)^{-1} \int d^4x \sqrt{-g} R$ is the purely gravitational part of the action, $S_{int} = -\xi/2 \int d^4x \sqrt{-g} R \phi^2$ is an explicit interaction term between the gravitational and the $\phi$ fields, $S_\phi$ describes the purely material part of the action associated with the scalar field, and the remainder $S_m$ describes matter fields other than $\phi$, collectively denoted by $\psi_m$.

The variation of the action (2.1) with respect to $\phi$ leads to the Klein-Gordon equation (1.1). By varying (2.1) with respect to $g_{ab}$ and using the well known formulas [12]

$$\delta \left( \sqrt{-g} \right) = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} , \quad (2.2)$$

$$\delta \left( \sqrt{-g} R \right) = \sqrt{-g} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) \delta g^{ab} \equiv \sqrt{-g} G_{ab} \delta g^{ab} \quad (2.3)$$

(where $G_{ab}$ is the Einstein tensor), one obtains the Einstein equations in the form

$$(1 - \kappa \xi \phi^2) G_{ab} = \kappa \left( \bar{T}_{ab}[\phi] + \bar{T}_{ab}[\psi_m] \right) \equiv \kappa \bar{T}^{(total)}_{ab} , \quad (2.4)$$
where
\[
\tilde{T}_{ab}[\phi] = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi - V_{ab}.
\] (2.5)

and
\[
\tilde{T}_{ab}[\psi_m] = -\frac{2}{\sqrt{-g}} \frac{\delta S_m[\psi_m, g_{cd}]}{\delta g^{ab}}.
\] (2.6)

One can also rewrite Eq. (2.4) by taking the factor \(\kappa \xi \phi^2 G_{ab}\) to the right hand side,
\[
G_{ab} = \kappa \tilde{T}_{ab},
\] (2.7)

where
\[
\tilde{T}_{ab} = \tilde{T}^{(total)}_{ab} + \xi \phi^2 G_{ab}.
\] (2.8)

By taking a different approach, the coefficient of the Ricci scalar in the action (2.1) can be written as \((16\pi G_{eff})^{-1}\), where
\[
G_{eff} \equiv \frac{G}{1 - 8\pi G\xi \phi^2}
\] (2.9)
is an effective, \(\phi\)-dependent, gravitational coupling. This way of proceeding is analogous to the familiar identification of the Brans-Dicke scalar field \(\phi_{BD}\) with the inverse of an effective gravitational constant \((G_\phi = \phi_{BD}^{-1})\) in the gravitational sector of the Brans-Dicke action
\[
S_{BD} = \int d^4 x \sqrt{-g} \left( \phi_{BD} R - \frac{\omega}{\phi_{BD}} \nabla^a \phi_{BD} \nabla_a \phi_{BD} \right).
\] (2.10)

By adopting this point of view in the case of a nonminimally coupled scalar field, one divides Eq. (2.4) by the factor \(1 - \kappa \xi \phi^2\) to obtain the Einstein equations in the form
\[
G_{ab} = \kappa_{eff} \left( \tilde{T}_{ab}[\phi] + \tilde{T}_{ab}[\psi_m] \right) = \kappa_{eff} \tilde{T}^{(total)}_{ab}
\] (2.11)
(where \(\kappa_{eff} \equiv 8\pi G_{eff}\)), which looks more familiar to the relativist’s eye.

The approach using the effective gravitational coupling (2.9) has been used to investigate the situation in which \(G_{eff} = 0\), and the “antigravity” regime corresponding to \(G_{eff} < 0\) [57].

A third possibility is to use the form of the Einstein equations
\[
G_{ab} = \kappa \left( T_{ab}[\phi] + T_{ab}[\phi, \psi_m] \right) \equiv \kappa T^{(total)}_{ab},
\] (2.12)
where

$$T_{ab}[\phi] \equiv \frac{1}{1 - \kappa \xi \phi^2} \tilde{T}_{ab}[\phi], \quad (2.13)$$

$$T_{ab}[\phi, \psi_m] \equiv \frac{1}{1 - \kappa \xi \phi^2} \tilde{T}_{ab}[\psi_m], \quad (2.14)$$

and the gravitational coupling is given by the true constant $G$.

If $\xi \leq 0$ the forms (2.4), (2.7), (2.11) and (2.12) of the Einstein equations are all equivalent (apart from the conservation of the corresponding stress-energy tensors, which is discussed later). If instead $\xi > 0$, caution must be exercised to ensure that the factor $1 - \kappa \xi \phi^2$ by which Eq. (2.4) is divided does not vanish. The division by $1 - \kappa \xi \phi^2$ used to write Eqs. (2.11) and (2.12) unavoidably introduces the two critical values of the scalar field

$$\pm \phi_c = \pm \frac{m_{pl}}{\sqrt{8\pi \xi}} \quad (\xi > 0), \quad (2.15)$$

which are barriers that the scalar field cannot cross. At $\phi = \pm \phi_c$ the effective gravitational coupling (2.9), its gradient, and the stress-energy tensor $\tilde{T}_{ab}^{(total)}$ in Eq. (2.12) diverge. Therefore, solutions of the field equations can only be obtained for which $|\phi| < \phi_c$ or $|\phi| > \phi_c$ at all times. For $\xi > 0$, one has obtained a restricted form of the field equations and a restricted class of solutions; any solution $\phi$ of the original theory described by Eq. (2.4) which crosses the barriers $\pm \phi_c$, is lost in passing to the picture of Eq. (2.11) or of Eq. (2.12).

Although the caveat on the division by the factor $(1 - \kappa \xi \phi^2)$ looks trivial, surprisingly it is missed in the literature on scalar field cosmology with NMC, and the restricted range of validity of the solutions goes unnoticed. In particular, investigations of the coupled Einstein-Klein-Gordon equations using dynamical systems methods and aiming at determining generic solutions and attractors, are put in jeopardy by the previous considerations if they employ the forms (2.11) or (2.12) of the Einstein equations. For example, the approach of Eq. (2.4) is used in Ref. [44], which makes correct statements on the general class of solutions of the field equations with NMC, while the parallel treatment of Ref. [58]) using Eq. (2.11) cannot claim to study general solutions.

We proceed by discussing the conservation equations for $T_{ab}$ and $\tilde{T}_{ab}$. The approach of Eq. (2.4) for $\xi > 0$ uses a truly constant gravitational coupling $G$, but the field equations (2.4) do not guarantee covariant conservation of $\tilde{T}_{ab}^{(total)}$: in fact the contracted Bianchi identities $\nabla^b G_{ab} = 0$ yield

$$\nabla^b \tilde{T}_{ab}^{(total)} = \frac{-2}\kappa \xi \phi^2 \tilde{T}_{ab}^{(total)} \nabla^b \phi \quad (2.16)$$
when the denominator is nonvanishing. The covariant divergence $\nabla^b T^{(total)}_{ab}$ vanishes only for the trivial case $\phi = \text{const.}$ and approximately vanishes in regions of spacetime where $\phi$ is nearly constant. When the scalar $\phi$ is the only source of gravity, $\tilde{T}_{ab}^{(total)} = \tilde{T}_{ab}[\phi]$, the constancy of $\phi$ corresponds to de Sitter solutions (if $\xi \leq 0$), with the energy-momentum tensor of quantum vacuum $\tilde{T}_{ab} = -V(\phi)g_{ab}$ and equation of state $P = -\rho$.

On the contrary, in the approach based on Eq. (2.12) the relevant stress-energy tensor $T^{(total)}_{ab}$ is covariantly conserved,

$$\nabla^b T^{(total)}_{ab} = 0 ,$$ (2.17)

as a consequence of the contracted Bianchi identities. This is probably the reason why the approach based on Eqs. (2.12) has been preferred over the alternative formulations. However, the loss of generality in the solutions for $\xi > 0$ must be kept in mind.

To give an idea of how the conservation equation for ordinary matter is modified by NMC, we consider the case of a dust fluid acting as the source of gravity together with the nonminimally coupled scalar field. When the scalar identically vanishes, the equation $\nabla^b T_{ab} = 0$ for the stress-energy tensor $T_{ab}[\psi] = \rho u_a u_b$ (where $u^a$ is the dust four-velocity) implies the geodesic equation for fluid particles

$$u^b \nabla_b u^a = 0 ,$$ (2.18)

and the conservation equation for the energy density $\rho$

$$\frac{d\rho}{d\lambda} + \rho \nabla_b u_b = 0 ,$$ (2.19)

where $\lambda$ is an affine parameter along the geodesics. When the nonminimally coupled scalar appears together with the dust, Eq. (2.16) yields

$$\left( \frac{d\rho}{d\lambda} + \rho \nabla^b u_b + \frac{2\kappa \xi \rho \phi}{1 - \kappa \xi \phi^2} \frac{d\phi}{d\lambda} \right) u_a + \rho \frac{D u_a}{D \lambda} = 0 ,$$ (2.20)

from which one derives again the geodesic equation $D u^a / D\lambda \equiv u^b \nabla_b u^a = 0$ and the modified conservation equation

$$\frac{d\rho}{d\lambda} + \rho \nabla_b u_b + \frac{2\kappa \xi \phi \rho}{1 - \kappa \phi^2} \frac{d\phi}{d\lambda} = 0 .$$ (2.21)

The geodesic hypothesis \cite{12} is satisfied since test particles move on geodesics. In the weak field limit, the modified conservation equation (2.21) reduces to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) + \frac{2\kappa \xi \phi}{1 - \kappa \phi^2} \left( \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \vec{v} \right) \rho = 0 .$$ (2.22)
Finally, we consider the approach using Eq. (2.7); it employs the truly constant gravitational coupling $G$ and it guarantees that the stress-energy tensor $\tilde{T}_{ab}$ is covariantly conserved,

$$\nabla^b \tilde{T}_{ab} = 0,$$

(2.23)
as can be deduced by using the contracted Bianchi identities and Eq. (2.7).

### 3 Positivity of the energy density

It is acknowledged [59, 56] that the energy density of a nonminimally coupled scalar field has a sign that depends on the particular solution $\phi$ and on the spacetime metric $g_{ab}$. This statement is easy to understand upon inspection of the rather complicated expression (4.4) for the energy density of a nonminimally coupled scalar in a Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. Since it is difficult or impossible to establish \textit{a priori} the sign of $\rho$, the minimal physical requirement $\rho \geq 0$ has to be checked \textit{a posteriori} for the known solutions of the field equations.

In this section we limit the discussion to homogeneous and isotropic cosmologies; scalar fields are extremely important in this context, due to their role as inflaton, dark matter, and quintessence. In this context, it is possible to improve substantially on the subject of the positivity of the energy density.

In a spatially flat or closed (curvature index $K \geq 0$) FLRW universe dominated by a scalar field, the solution $(a(t), \phi(t))$ of the field equations satisfies the Hamiltonian constraint

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3} \dot{\phi}^2 - \frac{K}{a^2},$$

(3.1)

which follows from the field equations in the form (2.7) and (2.8), and from which it is immediate to deduce that the energy density $\rho$ is always non-negative for a solution of the Einstein equations, in spite of the complication of the expression of $\rho$ in terms of $\phi$, $\dot{\phi}$, $a$ and $H = \dot{a}/a$ (see for example Eq. (6.4)).

Note that the different forms of the field equations considered in the previous section lead to different stress-energy tensors, and therefore to different definitions of energy density of the scalar field. Hence, by using field equations different from (2.7) and (2.8) one is not able to draw conclusions on the sign of $\rho$. 

9
4 Necessary conditions for the acceleration of the universe

In the rest of this paper we specialize our considerations to cosmology. In this section, we study a necessary condition for the universe to accelerate when a nonminimally coupled scalar field is the dominant source of gravity. This is relevant for both inflation and quintessence models based on scalar fields with NMC. While such inflationary models are well known \cite{39, 58, 52, 53, 24, 21, 22, 60}, the use of scalar fields with NMC as dark matter \cite{20, 61} and, more recently, as quintessence models \cite{6, 7, 8} is less known. The necessary condition for cosmic acceleration derived here is applied in the following sections.

The study of necessary conditions for the acceleration of the universe enables one to determine whether NMC helps, or makes it difficult to achieve inflation for a given scalar field potential, in comparison with the corresponding situation for minimal coupling.

We begin by considering the spatially flat Einstein-de Sitter universe with line element
\[
d s^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \tag{4.1}
\]
in comoving coordinates \((t, x, y, z)\). In this section we adopt the form (2.12) of the Einstein field equations keeping in mind the caveat of Sec. 2; the Einstein-Friedmann equations are
\[
\ddot{a} = -\frac{\kappa}{6}(\rho + 3P) , \tag{4.2}
\]
\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \rho , \tag{4.3}
\]
where an overdot denotes differentiation with respect to the comoving time \(t\). The energy density \(\rho\) and pressure \(P\) of the scalar field are given by the diagonal components of the stress-energy tensor \(T_{ab}[\phi]\) of Eq. (2.13):
\[
\rho = \left( 1 - \kappa \xi \phi^2 \right)^{-1} \left[ \frac{(\dot{\phi})^2}{2} + V(\phi) + 6\xi H\dot{\phi} \phi \right] , \tag{4.4}
\]
\[
P = \left( 1 - \kappa \xi \phi^2 \right)^{-1} \left[ \left( \frac{1}{2} - 2\xi \right) \dot{\phi}^2 - V(\phi) - 2\xi \phi \dot{\phi} - 4\xi H\phi \dot{\phi} \right] . \tag{4.5}
\]
Equations (4.1), (4.4) and (4.5) yield, in the case of minimal coupling \((\xi = 0)\),
\[
\ddot{a} = -\frac{\kappa}{3} \left( \dot{\phi}^2 - V \right) . \tag{4.6}
\]
An inflationary era in the evolution of the universe includes as an essential feature an accelerated expansion, $\ddot{a} > 0$ (of course, other ingredients are required for a successful inflationary scenario: a natural mechanism of entry into inflation, a sufficient amount of expansion, a graceful exit mechanism, acceptable scalar and tensor perturbations, etc). It is clear from Eq. (4.6) that when $\xi = 0$ a necessary (but not sufficient) condition for acceleration is given by $V > 0$. It is useful to keep in mind that the slow-roll approximation used to solve the equations of inflation, corresponds to the dominance of the scalar field potential energy density $V(\phi)$ over its kinetic energy density, $V(\phi) >> \dot{\phi}^2/2$. In the slow-roll approximation for $\xi = 0$, $\rho \approx V$ and hence the necessary condition for acceleration $V > 0$ reduces to the minimal requirement of the positivity of the scalar field energy density, and of the existence of real solutions of Eq. (4.2).

What is the necessary condition for acceleration analog to $V > 0$ when $\xi \neq 0$? The first part of this section is devoted to answer this question.

By definition, acceleration corresponds to the condition $\rho + 3P < 0$; upon use of Eqs. (4.4) and (4.5), this inequality is equivalent to

$$\left(1 - 3\xi\right)\dot{\phi}^2 - V - 3\xi\phi \left(\ddot{\phi} + H\dot{\phi}\right) < 0.$$  \hspace{1cm} (4.7)

The Klein-Gordon equation (1.1), which takes the form

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \frac{dV}{d\phi} = 0,$$  \hspace{1cm} (4.8)

is then used to substitute for $\ddot{\phi}$, obtaining

$$\left(1 - 3\xi\right)\dot{\phi}^2 - V + 3\xi^2 R\phi^2 + 6\xi H\phi \dot{\phi} + 3\xi \phi \frac{dV}{d\phi} < 0,$$  \hspace{1cm} (4.9)

and Eq. (4.4) can be used to rewrite the definition of cosmic acceleration (4.9) as

$$x \equiv \left(1 - \kappa \xi \phi^2\right) \rho - 2V + \dot{\phi}^2 \left(\frac{1}{2} - 3\xi\right) + 3\xi^2 R\phi^2 + 3\xi \phi \frac{dV}{d\phi} < 0.$$  \hspace{1cm} (4.10)

To proceed, one assumes the weak energy condition $\rho \geq 0$. Due to the difficulty of handling the dynamical equations analytically when $\xi \neq 0$, in the rest of this section we restrict ourselves to values of the coupling constant $\xi \leq 1/6$. Albeit limited, this semi-infinite range covers many of the prescriptions for the value of $\xi$ given in the literature [28]. One then has $-2V + 3\xi \phi dV/d\phi \leq x < 0$ and a necessary condition for cosmic acceleration to occur when $\xi \leq 1/6$ is

$$V - \frac{3\xi}{2} \phi \frac{dV}{d\phi} > 0.$$  \hspace{1cm} (4.11)
Eq. (4.11) reduces to the well known necessary condition for acceleration $V > 0$ for minimal coupling.

Unfortunately, the necessary and sufficient condition for acceleration (4.9) is not very useful in general because different terms, which depend on the solution $(a(t), \phi(t))$ and have opposite signs can balance one another and hamper a general analysis of the problem. In practice, one is compelled to adopt one of the specific forms of $V(\phi)$ considered in the literature and solve the equations for $a(t)$ and $\phi(t)$ for specific examples. However, a few considerations of general character can still be given.

First, we take the point of view that a potential $V(\phi)$ is given, for example by a particle physics theory, and we study the effect of introducing NMC in the field equations. The discussion is kept as general as possible, without specifying the value of $\xi$.

Keeping in mind the necessary condition (4.11) for acceleration of the universe in the $\xi = 0$ case, consider an even potential $V(\phi) = V(-\phi)$ which is increasing for $\phi > 0$. This is the case, e.g., of a pure mass term $m^2\dot{\phi}^2/2$, or of the quartic potential $V = \lambda \phi^4$, or of their combination $V(\phi) = m^2\phi^2/2 + \lambda \phi^4 + V_0$, where $V_0$ is constant. For $0 < \xi < 1/6$, one has $\xi \phi dV/d\phi > 0$ and it is harder to satisfy the necessary condition (4.11) for acceleration than in the minimal coupling case. Hence one can say that, for this class of potentials, it is harder to achieve acceleration of the universe, and hence inflation. If instead $\xi < 0$, the necessary condition for cosmic acceleration is more easily satisfied than in the $\xi = 0$ case, but one is not entitled to say that with NMC it is easier to achieve inflation (because a necessary, and not a sufficient condition for acceleration is considered).

Let us consider now an even potential $V(\phi) = V(-\phi)$ such that $dV/d\phi < 0$ for $\phi > 0$. This is the case, e.g., of the Ginzburg-Landau potential $V(\phi) = \lambda (\phi^2 - v^2)^2$ for $0 < \phi < v$, or of an inverted harmonic oscillator potential [62], which approximates the potential for natural inflation

$$V_{ni}(\phi) = \lambda^4 \left[ 1 + \cos \left( \frac{\phi}{f} \right) \right]$$  \hspace{1cm} (4.12)

around its maximum at $\phi = 0$. For $0 < \xi \leq 1/6$, it is easier to satisfy the necessary condition (4.11) for acceleration when $\xi \neq 0$ than when $\xi = 0$ but, again, this does not allow one to conclude that the universe actually accelerates its expansion. If $\xi < 0$ instead, it is harder to achieve acceleration than in the $\xi = 0$ case.

The inequality (4.11) can be read in a different way: assume, for simplicity, that $V > 0$ and $\phi > 0$ (it is straightforward to generalize to the case in which $V$ or $\phi$, or
both, are negative). Then, if \( 0 \leq \xi \leq 1/6 \), (4.11) is equivalent to
\[
\frac{d}{d\phi} \left\{ \ln \left[ \frac{V}{V_0} \left( \frac{\phi}{\phi_0} \right)^{\frac{2}{\pi}} \right] \right\} < 0 ,
\]
(4.13)
where \( V_0 \) and \( \phi_0 \) are arbitrary (positive) constants. Due to the fact that the logarithm is a monotonically increasing function of its argument, the necessary condition for cosmic acceleration (4.11) amounts to require that the potential \( V(\phi) \) grows with \( \phi \) slower than the power-law potential \( V_{\text{crit}}(\phi) \equiv V_0 (\phi/\phi_0)^{\frac{2}{\pi}} \). If instead \( \xi < 0 \), the necessary condition for cosmic acceleration amounts to require that \( V \) grows faster than \( V_{\text{crit}}(\phi) \) as \( \phi \) increases. This criterion is further developed in Sec. 4.3.

4.1 No acceleration without scalar field potential

Taking to the extreme the possibility of a balance between the potential \( V(\phi) \) and the term \( \xi R \dot{\phi}^2/2 \) in (1.2), the question arises of whether it is possible to obtain acceleration of the universe with a free, massless scalar field with no cosmological constant (i.e. \( V = 0 \)) for suitable values of the coupling constant \( \xi \). In particular, we are interested to the case of strong coupling \( |\xi| >> 1 \), which has been considered many times in the literature \([45, 39, 61, 20, 22, 24, 6]\).

Inflation driven by a pure NMC term turns out to be impossible for negative values of \( \xi \). In fact, by assuming that the expansion of the universe is accelerated and that \( V = 0 \), Eq. (4.9) and the expression of the Ricci curvature in an Einstein-de Sitter space
\[
R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) > 6H^2
\]
(4.14)
yield, for \( \xi < 0 \),
\[
(1 - 3\xi)\dot{\phi}^2 + 3\xi^2 R \dot{\phi}^2 + 6\xi H \dot{\phi} \phi \geq (\dot{\phi} + 3\xi H \phi)^2 \geq 0 ,
\]
(4.15)
thus contradicting (4.9). Therefore, the combined assumptions \( \ddot{a} > 0 \) and \( V = 0 \) lead to an absurdity. The previous analysis fails to yield conclusions when \( \xi > 0 \) because terms of different signs can balance in the left hand side of (4.9). The previous reasoning does not make use of the weak energy condition.

The discussion can easily be extended to values of \( \xi \) in the range \( 0 < \xi \leq 1/6 \) by using an independent argument: by rewriting (4.9) as
\[
\dot{\phi}^2 \left( \frac{1}{2} - 3\xi \right) < - \left[ 3\xi^2 R \phi^2 + (1 - \kappa \xi \phi^2) \rho \right] ,
\]
(4.16)
and using $\rho \geq 0$, one concludes that the right hand side is negative when the cosmic expansion accelerates and that (4.16) can be satisfied only if the term on the left hand side is negative, i.e. if $\xi > 1/6$, which contradicts the assumptions. Therefore,

for $\xi \leq 1/6$, the NMC term alone cannot act as an effective potential to provide acceleration of the universe.

A further argument (again for $\xi \leq 1/6$) consists in noting that the necessary condition for acceleration (4.11) is not satisfied if $V(\phi)$ vanishes identically. Unfortunately no conclusion can be obtained analytically when $\xi > 1/6$.

4.2 Negative potentials

In the usual studies of inflation and quintessence with $\xi = 0$, only positive scalar field potentials are considered. The reason is easy to understand: in the slow-rollover approximation to inflation $V >> \dot{\phi}^2/2$, $\rho \approx V$, and $V > 0$ corresponds to $\rho > 0$, a minimal requirement. However, this is no longer true when $\xi \neq 0$ and $\rho$ is given by the more complicated expression (4.4). Indeed, negative scalar field potentials have been considered in the literature on NMC, in inflation [59] or in other contexts [63, 64, 36].

The question of whether the positive term $\xi R\phi^2/2$ can balance a negative $V(\phi)$ arises. In the toy model of Ref. [65] a negative potential $V$ is balanced by the coupling term $\xi R\phi^2/2$ in such a way that inflation is achieved: there, a closed FLRW universe dominated by a conformally coupled scalar field is investigated in Einstein’s gravity. By assuming the equation of state $P = (\gamma - 1)\rho$, the potential deemed necessary for inflation is derived numerically for small values of the constant $\gamma$; the resulting $V(\phi)$ is significantly different from the corresponding potential derived analytically in Ref. [66] for the case $\xi = 0$ and for the same values of $\gamma$.

Mathematically, the possibility of a negative $V(\phi)$ which is inflationary in the presence of NMC extends the range of potentials explored so far, but the meaning of a negative scalar field potential $V(\phi)$ remains unclear and the latter is probably unpalatable to most particle physicists.

4.3 Quintessence models

In quintessence models based on a nonminimally coupled scalar field, the energy density of the latter is beginning to dominate the dynamics of the universe, but there is also ordinary matter with energy density $\rho_m \propto a^{-3}$ and vanishing pressure $P_m = 0$. Eq. (4.4)
is modified according to

\[ \rho = \left(1 - \kappa \xi \phi^2 \right)^{-1} \left[ \rho_m + \frac{(\dot{\phi})^2}{2} + V(\phi) + 6\xi H\dot{\phi} \right] = \frac{\rho_m}{1 - \kappa \xi \phi^2} + \rho_\phi, \tag{4.17} \]

where \( V(\phi) \) is an appropriate quintessential potential. The necessary and sufficient condition for the acceleration of the universe \( \rho + 3P < 0 \) is written as

\[ y \equiv \frac{\rho_m}{2} + (1 - 3\xi) \phi^2 - V + 6\xi H\dot{\phi} + 3\xi^2 R\phi^2 + 3\xi \phi \frac{dV}{d\phi} < 0, \tag{4.18} \]

where the Klein-Gordon equation (4.8) has been used to substitute for \( \ddot{\phi} \). As before, one obtains a necessary condition for the accelerated expansion of the universe by rewriting the quantity \( y \) as

\[ y = \frac{\rho_m}{2} + (1 - \kappa \xi \phi^2) \rho_\phi - 2V + \dot{\phi}^2 \left( \frac{1}{2} - 3\xi \right) + 3\xi^2 R\phi^2 + 3\xi \phi \frac{dV}{d\phi} < 0, \tag{4.19} \]

and by assuming that \( \rho_m \) and \( \rho_\phi \) be non-negative; one obtains again Eq. (4.11) as a necessary condition for the acceleration of a universe in which quintessence is modelled by a nonminimally coupled scalar field, in the additional presence of ordinary matter.

The analysis can be refined by noting that, when quintessence dominates, \( \rho_\phi \approx V \). By introducing the matter and scalar field energy densities measured in units of the critical density \( \rho_c \) (respectively, \( \Omega_m = \rho_m/\rho_c \) and \( \Omega_\phi = \rho_\phi/\rho_c \)), one has \( \rho_m \approx V \Omega_m/\Omega_\phi \) and

\[ y = \left( -1 + \frac{\Omega_m}{2\Omega_\phi} - \kappa \xi \phi^2 \right) V + \left( \frac{1}{2} - 3\xi \right) \phi^2 + 3\xi^2 R\phi^2 + 3\xi \phi \frac{dV}{d\phi} < 0. \tag{4.20} \]

For \( \xi \leq 1/6 \) one has

\[ \left( -1 + \frac{\Omega_m}{2\Omega_\phi} \right) |_{0} - \kappa \xi \phi^2 \right) V + 3\xi \phi \frac{dV}{d\phi} \leq y < 0, \tag{4.21} \]

where the ratio \( \Omega_m/\Omega_\phi \) has been approximated by its present value (which is correct at least around the present epoch). By assuming again, for simplicity, that \( V \) and \( \phi \) be positive, the necessary condition for quintessential inflation for \( \xi \leq 1/6 \) is

\[ \frac{d}{d\phi} \left\{ \ln \left[ \frac{V}{V_0} \left( \frac{\phi_0}{\phi} \right)^{\alpha} \exp \left( -\frac{\kappa}{6} \phi^2 \right) \right] \right\} < 0, \tag{4.22} \]
where $V_0$ and $\phi_0$ are constants and

$$\alpha = \left(1 - \frac{\Omega_m}{2\Omega_\phi}\right) \frac{1}{3\xi}.$$  

(4.23)

Then, to have quintessential expansion with nonminimal coupling and $0 < \xi \leq 1/6$, one needs a potential $V(\phi)$ that does not grow with $\phi$ faster than the function

$$C(\phi) = V_0 \left(\frac{\phi}{\phi_0}\right)^{\alpha} \exp\left(\frac{\kappa}{6} \phi^2\right).$$

(4.24)

If instead $\xi < 0$, $V(\phi)$ must grow faster than $C(\phi)$. The necessary conditions for cosmic acceleration in a quintessence-dominated universe are useful for future reference in studies of quintessence models with NMC.

5 Fixing the scalar field potential

Due to the complication of the coupled Einstein-Klein-Gordon equations when $\xi \neq 0$, general analytical considerations on the occurrence of inflation with nonminimally coupled scalar fields are necessarily quite limited, as seen in Sec. 2. However, one can (at least partially) answer the following meaningful question:

*is it harder or easier to achieve acceleration of the universe with NMC for the potentials that are known to be inflationary in the minimal coupling case?*

Since in many situations these potentials are motivated by a high energy physics theory, they are of special interest. In order to appreciate the effect of the inclusion of a NMC term in a given inflationary scenario, we study some exact solutions for popular inflationary potentials, and the necessary condition (4.11) for the occurrence of inflation.

5.1 $V = 0$

In order to illustrate the qualitative difference between minimal and nonminimal coupling, it is sufficient to compare the solution for $V = 0$, $\xi = 0$ with the corresponding solution for the special value of the NMC coupling constant $\xi = 1/6$. For minimal coupling, one has the stiff equation of state $P = \rho$, and the scale factor $a(t) = a_0 t^{1/3}$, as can be deduced by the inspection of Eqs. (4.4) and (4.5) (we exclude the trivial case of Minkowski space). In the $V = 0$, $\xi = 1/6$ case, the Klein-Gordon equation is conformally invariant, corresponding to the vanishing of the trace of $T_{ab}[\phi]$, to the radiation
equation of state \( P = \rho/3 \), and to the scale factor \( a(t) = a_0 t^{1/2} \). This is in agreement with the fact that there are no accelerated universe solutions for \( V = 0 \) and any value of \( \xi \), because the necessary condition (4.11) cannot be satisfied in this case.

### 5.2 \( V = V_0 = \text{constant} \)

For \( \xi = 0 \) a constant potential can be regarded as a cosmological constant in the Einstein equations. Viceversa, a \( \Lambda \)-term in the Einstein equations,

\[
G_{ab} = \Lambda g_{ab} + \kappa T_{ab} ,
\]

(5.1)
can be incorporated into the scalar field potential by means of the substitution

\[
V(\phi) \rightarrow V(\phi) + \frac{\Lambda}{\kappa} ;
\]

(5.2)
this is true subject to the condition that the scalar field is constant.

The equivalence between cosmological constant and constant scalar field potential does no longer hold when \( \xi \neq 0 \) and the form (2.12) of the Einstein equations is used. In fact, in this case the addition of a cosmological constant term to the left hand side of the Einstein equations (2.12),

\[
\kappa T_{ab} \rightarrow \kappa T_{ab} + \Lambda g_{ab}
\]

(5.3)
is equivalent to the substitution

\[
V(\phi) \rightarrow V(\phi) + \frac{\Lambda}{\kappa} \left( 1 - \kappa \xi \phi^2 \right) = V(\phi) + V_1(\phi) .
\]

(5.4)
The extra piece \( V_1(\phi) = \frac{\Lambda}{\kappa} (1 - \kappa \xi \phi^2) \) in the potential does not correspond to a mere shift in the potential energy density (usually identified with the vacuum energy), but it also adds a self-interaction with the shape of an inverted harmonic oscillator. This is an example of how different things are when \( \xi \) is allowed to be different from zero, and testifies of the difference between the physical interpretations associated to the different ways of writing the field equations discussed in Sec. 2. When \( \xi \neq 0 \), a constant potential cannot be interpreted as the vacuum energy density coming from the left hand side of the Einstein equations.

In the case \( V = V_0 \), the necessary condition (4.11) for cosmic acceleration when \( \xi \leq 1/6 \) coincides with the corresponding condition for minimal coupling, \( V = \Lambda/\kappa > 0 \).
A negative $V_0$ does not give rise to acceleration of the universe, and hence to inflation. While, for $\xi = 0$, a negative $\Lambda$ violates the weak energy condition, this may no longer be true for $\xi \neq 0$. The necessary and sufficient condition for acceleration when $\xi = 0$ is $\Lambda > \kappa \dot{\phi}^2/2$, which reminds of the slow-roll condition.

The $\xi = 0$, $V = \text{const.}$ case itself deserves a comment. In this case, the Einstein-Friedmann equations with $V = \Lambda/\kappa$ have the familiar de Sitter solution (historically, the prototype of inflation)

$$a(t) = a_0 \exp(\dot{H} t), \quad \dot{H} = 0, \quad \dot{\phi} = 0,$$

(5.5)
corresponding to the vacuum equation of state $P = -\rho$. In addition one has, for $\Lambda \neq 0$, the exact solution

$$a(t) = a_1 \left[ \sinh \left( \sqrt{3\Lambda} t \right) \right]^{1/3},$$

(5.6)
corresponding to the non-constant scalar field

$$\phi(t) = \pm \sqrt{\frac{2}{3\kappa}} \ln \left[ \tanh \left( \frac{\sqrt{3\Lambda}}{2} t \right) \right] + \phi_0,$$

(5.7)

where $a_1$ and $\phi_0$ are integration constants. The latter solution is asymptotic to (5.5) at late times $t \to +\infty$, in agreement with the cosmic no-hair theorems [1], but it exhibits a big-bang singularity at $t = 0$ (where $a(t) \approx t^{1/3}$), while the FLRW universe described by the de Sitter solution (5.5) has been expanding forever. In addition, the solution (5.6) and (5.7) corresponds to an effective equation of state that changes with time, and interpolates between the two extremes $P = \rho$ ("stiff" equation of state) at early times $t \to 0$ and the vacuum equation of state $P = -\rho$ at late times. The exact solution (5.6) and (5.7) tells us two things (note that we are not even talking about the more complicated NMC in this example):

1) contrarily to naive statements found in the literature, fixing the scalar field potential $V(\phi)$ does not fix the equation of state, and therefore the scale factor $a(t)$;

2) a given potential $V(\phi)$ may correspond to very different equations of state, depending on the solution $(g_{ab}, \phi)$ of the field equations.

To conclude this subsection, we note that one can impose that the solution (5.5) holds (in the spirit, e.g., of Ref. [20]); in this case, if $\xi = 0$, the Klein-Gordon [68] equation implies that $dV/d\phi|_{\phi_0} = 0$. If instead $\xi \neq 0$, by imposing that the solution
(5.5) holds, the Klein-Gordon equation implies that \( dV/d\phi|_{\phi_0} = -12\xi H^2 \phi_0 \). A positive linear potential \( V = \lambda \phi \) achieves de Sitter expansion with constant \( \phi \) if \( \xi < 0 \) and \( H^2 = \lambda/(12|\xi|) \).

5.3 \( V = m^2 \phi^2/2 \)

A pure mass term is perhaps the most natural “potential” for a scalar field, and an example of the class of even potentials for which \( \phi dV/d\phi > 0 \) considered in Sec. 4. For \( \xi = 0 \), it corresponds to chaotic inflation [69], while for \( \xi < 0 \), it can still generate inflation. For example, the exponentially expanding solution

\[
H = H_* = \frac{m}{(12|\xi|)^{1/2}}, \quad \phi = \phi_* = \frac{1}{(\kappa|\xi|)^{1/2}},
\]

(5.8)

has been studied, not in relation to the early universe, but as the description of short periods of unstable exponential expansion of the universe which occur after the star formation epoch, well into the matter dominated era [20]. The fact that the Ricci curvature \( R \) is constant for this particular solution makes this case particularly suitable for the interpretation of the \( \xi R \phi^2/2 \) term in the Lagrangian density as a negative mass term, which balances the intrinsic mass term \( m^2 \phi^2/2 \) in the potential, thus conspiring to give a vanishing effective mass \( m_{\text{eff}} = \sqrt{m^2 - |\xi| R} \). However, the so called late time mild inflationary scenario [20] based on the relation \( m_{\text{eff}} = 0 \) is unphysical, as is best seen by studying the scalar wave tails in the corresponding spacetime [70]. In fact, the scenario corresponds to a spacetime in which a massive scalar field propagates sharply on the light cone at every point; this occurs because the usual tail due to the intrinsic mass \( m \) is cancelled by a second tail term describing the backscattering of the \( \phi \)-waves off the background curvature of spacetime [70].

For \( \xi > 0 \), the nonminimally coupled scalar field has been studied by Morikawa [61], who found no inflation. The \( \xi > 0 \) case was studied in order to explain the reported periodicity in the redshift of galaxies [71]. If the model was correct, the parameter \( \xi \) could be determined directly from astronomical observations, and it would provide information on whether general relativity is the correct theory of gravity [72]. However, the prevailing opinion among astronomers is that the reported periodicity in galactic redshifts is not genuine, but is an artifact of the statistics used to analyze the astronomical data. The nonminimally coupled, massive, scalar field model may however be resurrected in the future in conjunction with the more recent claims of redshift periodicities for large scale structures [73].
From the point of view of this paper, the introduction of NMC destroys the acceleration of the cosmic expansion for large positive values of $\xi$ when $V(\phi) = m^2 \phi^2 / 2$. This is relevant since we were not able to draw conclusions for $\xi > 1/6$ in Sec. 4.

### 5.4 Quartic potential

The potential $V = \lambda \phi^4$ corresponds to chaotic inflation for $\xi = 0$. When $\xi \neq 0$ we limit ourselves to consider the case of conformal coupling. For $\xi = 1/6$, the Klein-Gordon equation (4.8) is conformally invariant, corresponding to the vanishing of the trace $T = \rho - 3P$ of the scalar field stress-energy tensor, to the radiation equation of state $P = \rho / 3$ and to the non-inflationary expansion law $a(t) \propto t^{1/2}$. The introduction of conformal coupling destroys the acceleration occurring in the minimal coupling case for the same potential; however, accelerated solutions can be recovered by breaking the conformal symmetry with the introduction of a mass for the scalar, or of a cosmological constant (which, in this respect, behaves in the same manner [74]). Exact accelerating and non-accelerating solutions corresponding to integrability of the dynamical system for the potential $V = \Lambda + m^2 \phi^2 / 2 + \lambda \phi^4$ are presented in Ref. [75] for special sets of the parameters ($\Lambda, m, \lambda$).

### 5.5 $V = \lambda \phi^n$

In general, the necessary condition for cosmic acceleration (4.11) depends from the particular solution of the Klein-Gordon equation, which is not known a priori. However, this dependence disappears for power-law potentials. This case contains those of the previous subsections and also the potential $V \propto \phi^{-|\beta|}$, which approximates the potential for intermediate inflation [76] and has been used in quintessence models [5, 4, 64, 6, 7]. The previous examples can be extended to the case of a potential proportional to an even power of the scalar field, which is associated to chaotic inflation for $\xi = 0$. The necessary condition (4.11) for the occurrence of accelerated cosmic expansion then becomes

$$\lambda \left(1 - \frac{3n \xi}{2}\right) > 0$$

(5.9)

when $\xi \leq 1/6$. Under the assumption $\lambda > 0$ corresponding to a positive scalar field potential, the necessary condition (4.11) for acceleration of the cosmic expansion fails to be satisfied when $\xi \geq 2/3n$, independently of the solution $\phi$ and of the initial conditions $(\phi_0, \dot{\phi}_0)$. This is interesting for $n \geq 4$. Hence, also in this case, NMC destroys acceleration in the range of values of $\xi$ ($2/3n, 1/6$).
The potential \( V = \lambda \phi^n \) with \( n > 6 \) gives rise to power-law inflation \( a = a_0 t^p \), where

\[
p = \frac{1 + (n - 10)\xi}{(n - 4)(n - 6)|\xi|}
\]  \hspace{1cm} (5.10)

([41]; see also Ref. [59] for the case \( \xi = 1/6 \)). The universe is accelerating if \( p > 1 \). The range of values \( 6 < n \leq 10 \) is interesting for superstring theories [77, 62]; however the scenario is fine-tuned for \( p > 0 \). For \( \xi < 0 \) the solution is accelerating only if \( 6 < n < 4 + 2\sqrt{3} \approx 7.464 \).

### 5.6 Exponential potential

The potential \( V = V_0 \exp\left(-\sqrt{2\kappa/p} \phi \right) \) is associated to power-law inflation \( a = a_0 t^p \) when \( \xi = 0 \) and \( \phi > 0 \). An exponential potential is the fingerprint of theories which are reformulated in the Einstein conformal frame by means of a suitable conformal transformation of a theory previously set in the Jordan conformal frame (see Ref. [78] for an explanation of this terminology and for the relevant formulas). This class of theories include Kaluza-Klein and higher-dimensional theories with compactified extra dimensions, scalar-tensor theories of gravity, higher derivative and string theories [78]. In this case, the low energy prediction for the coupling constant yields the value \( \xi = 0 \) [28]. Nevertheless, one can consider a positive exponential potential also for \( \xi \neq 0 \), and in this case the necessary condition for cosmic acceleration is

\[
\frac{\phi}{m_{pl}} > \frac{1}{6\xi} \sqrt{\frac{p}{\pi}} \quad \left( 0 < \xi \leq \frac{1}{6} \right),
\]  \hspace{1cm} (5.11)

\[
\frac{\phi}{m_{pl}} < \frac{1}{6|\xi|} \sqrt{\frac{p}{\pi}} \quad (\xi < 0).
\]  \hspace{1cm} (5.12)

### 6 NMC and the slow-roll approximation to inflation

For minimal coupling the equations of inflation are solved in the slow-roll approximation [1, 42], which amounts to assume that the solution is approximately the de Sitter space

\[
(H, \phi) = \left( \sqrt{\frac{\Lambda}{3}}, 0 \right);
\]  \hspace{1cm} (6.1)
the slow-roll approximation works because the solution (6.1) is an attractor of the dynamical equations for $\xi = 0$, and slow-roll inflation is a quasi-de Sitter expansion [1, 79, 62, 42].

The slow-roll approximation is often used to solve also the equations for $\xi \neq 0$; however it is unknown whether the de Sitter solution (6.1) is still an attractor in this case, and the use of the slow-roll approximation is unjustified unless this question is affirmatively answered.

We adopt the form (2.7) and (2.8) of the field equations, which guarantees the generality of the solution, the covariant conservation of the stress-energy tensor, the weak energy condition, and the constancy of the gravitational coupling. The equations of motion can be written as the Klein-Gordon equation (1.1), the trace of the Einstein equations, and the Hamiltonian constraint, respectively:

\begin{equation}
R = 6 \left( \dot{H} + 2H^2 \right) = \kappa \left( \rho - 3P \right), \quad (6.2)
\end{equation}

\begin{equation}
3H^2 = \kappa \rho, \quad (6.3)
\end{equation}

where the energy density and pressure are given by

\begin{equation}
\rho = \frac{\dot{\phi}^2}{2} + 3\xi H^2 \phi^2 + 6\xi H \phi \dot{\phi} + V \quad (6.4)
\end{equation}

\begin{equation}
P = \frac{\dot{\phi}^2}{2} - V - \xi \left( 4H \phi \dot{\phi} + 2\phi^2 + 2\phi \ddot{\phi} \right) - \xi \left( 2\dot{H} + 3H^2 \right) \phi^2. \quad (6.5)
\end{equation}

The equations of motion can be reduced to a two-dimensional system of first order equations for the variables $H$ and $\phi$ [47, 80],

\begin{equation}
-6H \left[ 1 + \xi \left( 6\xi - 1 \right) \kappa \phi^2 \right] + \kappa \left( 6\xi - 1 \right) \phi^2 - 12H^2 + 12\xi (1 - 6\xi) \kappa H^2 \phi^2 + 4\kappa V - 6\kappa \phi \frac{dV}{d\phi} = 0, \quad (6.6)
\end{equation}

\begin{equation}
-\frac{\kappa}{2} \dot{\phi}^2 - 6\xi \kappa H \phi \dot{\phi} + 3H^2 - 3\kappa \xi H^2 \phi^2 - \kappa V = 0; \quad (6.7)
\end{equation}

then it is clearly convenient to formulate the problem in terms of the variables $H$ and $\phi$. One can rewrite the system (6.6) and (6.7) as two equations that explicitly give the vector field $(\dot{H}, \dot{\phi})$ of the system. In the language of dynamical systems, the fixed points of this system are the de Sitter solutions corresponding to constant Hubble function and scalar field. It is straightforward to check that, for $V = \Lambda/\kappa \geq 0$, the solutions

\begin{equation}
(H, \phi) = \left( \pm \sqrt{\frac{\Lambda}{3}}, 0 \right) \quad (6.8)
\end{equation}

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satisfy Eqs. (1.1), (6.2), and (6.3) for arbitrary values of $\xi$. The slow-roll formalism is only meaningful when applied around a stable de Sitter solution (6.8), otherwise small perturbations of the background run away from it and from inflation. Hence one asks whether the solutions (6.8) are stable or unstable fixed points; the answer is given by a local stability analysis. When the potential $V = \Lambda/\kappa$ is left unchanged the equations for the perturbations $\delta H$ and $\delta \phi$ defined by

$$H = H_0 + \delta H, \quad \phi = \delta \phi,$$

yield perturbations that decrease exponentially with time for the expanding solution (6.8) and therefore stability for any $\xi \geq 0$; there is instability for $\xi < 0$. The contracting solution (6.8) is unstable for any value of $\xi$. It is significant that the sign of the coupling constant $\xi$ affects the stability of the solution.

However, it is more interesting to consider perturbations of the equations of motion corresponding to a perturbed potential

$$V(\phi) = \frac{\Lambda}{\kappa} + V_0' \delta \phi + \frac{V_0''}{2} \delta \phi^2 + \frac{V_0'''}{6} \delta \phi^3 + \frac{V_0^{(IV)}}{24} \delta \phi^4 + \ldots.$$

The cosmological constant is then seen as the zeroth order approximation of the potential. The density and pressure perturbations are given by

$$\delta \rho = \frac{\delta \phi^2}{2} + 3\xi H_0^2 \delta \phi^2 + 6\xi H_0 \delta \phi \delta \dot{\phi} + \frac{V_0''}{2} \delta \phi^2 + \ldots,$$

$$\delta P = \frac{\delta \phi^2}{2} - 4\xi H_0 \delta \phi \delta \dot{\phi} - 2\xi \delta \phi^3 - 2\xi \delta \phi \delta \ddot{\phi} - 3\xi H_0^2 \delta \phi^2 - \frac{V_0''}{2} \delta \phi^2 + \ldots,$$

where ellipsis denote higher order contributions and the Klein-Gordon equation implies $V_0' \equiv dV/d\phi|_0 = 0$. The perturbations satisfy the equations of motion

$$\ddot{\delta \phi} + 3H_0 \delta \dot{\phi} + \left(12\xi H_0^2 + V_0''\right) \delta \phi + \ldots = 0,$$

$$\delta H = \frac{\kappa}{6H_0} \left(\frac{\delta \phi^2}{2} + 3\xi H_0^2 \delta \phi^2 + 6\xi H_0 \delta \phi \delta \dot{\phi} + \frac{V_0''}{2} \delta \phi^2\right) + \ldots.$$

By assuming fundamental solutions of the form

$$\delta \phi = \epsilon e^{\alpha t}$$

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one finds the algebraic equation for $\alpha$

$$\alpha^2 + 3H_0 \alpha + 12\xi H_0^2 + V''_0 = 0 . \quad (6.16)$$

Let us first analyze the stability of the expanding de Sitter solution (6.8); the fundamental solutions $\delta \phi_{1,2}$ corresponding to

$$\alpha_{1,2} = \frac{3H_0}{2} \left( -1 \pm \sqrt{1 - \frac{16\xi}{3} - \frac{4V''_0}{3\Lambda}} \right) \quad (6.17)$$

are exponentially decreasing (or constant) when $1 - 16\xi/3 - 4V''_0/3\Lambda$ is not greater than unity, which corresponds to stability and is achieved for $\xi \geq -V''_0/(4\Lambda)$. There is instability when $\xi < -V''_0/(4\Lambda)$.

Note that, for $\xi = 0$, there is stability for $V''_0 > 0$ which happens, e.g., when the potential has a minimum $\Lambda/\kappa$ at $\phi = 0$; a solution starting at any value of $\phi$ is attracted towards the minimum (in slow-roll if the potential is sufficiently flat). If instead $V''_0 < 0$ and the potential has a maximum, the solution starting at $\phi = 0$ runs away from it.

When $\xi \neq 0$, the potential and the NMC term $\xi R\phi^2/2$ balance; if $V(\phi)$ has a minimum $V_0 = \Lambda/\kappa$ at $\phi = 0$ the solution is unstable for large negative values of $\xi$. If instead the potential has a maximum $\Lambda/\kappa$ at $\phi = 0$, then the expanding de Sitter solution is unstable for any negative $\xi$, and stable only for $-4\Lambda\xi \leq V''_0 < 0$. Again, the stability character of the de Sitter solution is determined not only by the shape of the potential, but also by the value of the coupling constant $\xi$. This analysis makes exact the previous qualitative considerations of Refs. [43, 19, 41, 28] on the balance between $\xi R\phi^2/2$ and $V(\phi)$, and is not limited to the case in which $V(\phi)$ has an extremum at $\phi = 0$.

The contracting de Sitter solution (6.8) is unstable for any value of $\xi$, as is deduced by repeating the analysis above.

### 7 Conformal transformation techniques

Conformal transformation techniques are often used to reduce the study of a cosmological scenario with a nonminimally coupled scalar field to the problem of a minimally coupled field, with considerable mathematical simplification (see Ref. [78] for a review). The "Jordan conformal frame" in which the scalar field couples nonminimally to the Ricci curvature is mapped into the "Einstein frame" in which the (transformed) scalar is minimally coupled. The two frames are not physically equivalent, and care must be taken in applying conformal techniques [81, 78, 82].
The conformal transformation is given by

\[ g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad (7.1) \]

where

\[ \Omega = \sqrt{1 - \kappa \xi \phi^2}, \quad (7.2) \]

and the scalar field is redefined according to

\[ d\tilde{\phi} = \frac{\sqrt{1 - \kappa \xi (1 - 6\xi)\phi^2}}{1 - \kappa \xi \phi^2} d\phi. \quad (7.3) \]

The “new” scalar \( \tilde{\phi} \) in the Einstein frame \((\tilde{g}_{ab}, \tilde{\phi})\) is minimally coupled,

\[ \Box \tilde{\phi} - \frac{d\tilde{V}}{d\tilde{\phi}} = 0, \quad (7.4) \]

where

\[ \tilde{V} \left( \tilde{\phi} \right) = \frac{V \left[ \phi \left( \tilde{\phi} \right) \right]}{(1 - \kappa \xi \phi^2)^2} \quad (7.5) \]

and \( \phi = \phi \left( \tilde{\phi} \right) \) is obtained by integrating and inverting Eq. (7.3). The conformal transformation technique is useful to solve the equations of cosmology in the Einstein frame and then map the solutions \((\tilde{g}_{ab}, \tilde{\phi})\) back into the physical solutions \((g_{ab}, \phi)\) of the Jordan frame with NMC. Although from the mathematical point of view it is convenient to obtain exact solutions with NMC in this way (see e.g. Ref. [59]), in general the procedure is not very interesting from the physical point of view. In fact, one starts from a known solution for a potential \( \tilde{V} \left( \tilde{\phi} \right) \) motivated by particle physics in the unphysical Einstein frame, and one obtains a solution in the physical Jordan frame which corresponds to a potential \( V(\phi) \) with no physical justification, and therefore not very interesting. Furthermore, if a solution is inflationary in one frame, its conformally transformed counterpart in the other frame is not necessarily inflationary. To give an example, we consider a conformally coupled scalar field. Starting with the potential \( \tilde{V} \left( \tilde{\phi} \right) = \lambda \tilde{\phi}^4 \) in the Einstein frame, one integrates Eq. (7.3) and uses Eq. (7.5) to obtain

\[ V(\phi) = \left( \frac{3}{2\kappa} \right)^2 \lambda \left( 1 - \frac{\kappa}{6\phi^2} \right)^2 \ln^4 \left[ \frac{\sqrt{\kappa/6} \phi + 1}{\sqrt{\kappa/6} \phi - 1} \right]. \quad (7.6) \]
While the quartic potential in the unphysical Einstein frame is everyday’s routine, one would be hard put to justify the potential (7.6).

There is, however, a meaningful situation in which an inflationary solution is mapped into another inflationary solution by the conformal transformation: the slow-roll approximation described in the previous section. To prove this statement, one begins by noting that an exact de Sitter solution (6.8) is invariant under the conformal transformation (7.1), (7.2), and (7.3). In fact, when $\phi$ is constant, Eq. (7.1) reduces to a rescaling of the metric by a constant factor (which can be absorbed into a coordinate rescaling), and the scalar $\phi$ is mapped into another constant $\tilde{\phi}$. Moreover, it is proven in Sec. 6 that a de Sitter solution is an attractor point in the phase space for suitable values of the coupling constant $\xi$, with nonminimal coupling as well as with minimal coupling; hence, for these suitable values of $\xi$, the conformal transformation maps an attractor of the Jordan frame into an attractor of the Einstein frame. It is therefore meaningful to consider the slow-roll approximation to inflation in both frames.

In the Jordan frame the Hubble parameter is given by

$$a = a_0 \exp[H(t)t], \quad(7.7)$$

$$H(t) = H_0 + \delta H(t), \quad(7.8)$$

where $H_0$ is constant and $|\delta H| << |H_0|$. In the Einstein frame one has the line element

$$ds^2 = \Omega^2 dt^2 = -d\tilde{t}^2 + \bar{a}^2 \left(dx^2 + dy^2 + dz^2\right), \quad(7.9)$$

where $d\tilde{t} = \Omega dt$ and $\bar{a} = \Omega a$. The Hubble parameter in the Einstein frame is

$$\ddot{H} \equiv \frac{1}{\bar{a}} \frac{d\bar{a}}{d\tilde{t}} = \frac{1}{\Omega} \left(H + \frac{\dot{\Omega}}{\Omega}\right), \quad(7.10)$$

where a dot denotes differentiation with respect to the Jordan frame comoving time $t$. For an exact de Sitter solution $H = \text{const.}$ implies $\ddot{H} = \text{const.}$ and vice versa. A slow-roll inflationary solution in the Jordan frame satisfies Eq. (7.8) and

$$\phi(t) = \phi_0 + \delta \phi(t), \quad(7.11)$$

where $\phi_0$ is constant and $|\delta H| << |H_0|, |\delta \phi| << |\phi_0|$; the corresponding Einstein frame quantities are

$$\ddot{H} = \frac{1}{\sqrt{1 - \kappa \xi \phi_0^2}} \left( H_0 + \delta H + \frac{\kappa \xi \phi_0 H_0}{1 - \kappa \xi \phi_0^2} \delta \phi - \frac{\kappa \xi \phi_0}{1 - \kappa \xi \phi_0^2} \delta \dot{\phi} \right) = \bar{H}_0 + \delta \bar{H} \quad(7.12)$$
and

$$\tilde{\phi} = \phi_0 + \delta\phi$$  \hspace{1cm} (7.13)

where, to first order,

$$\tilde{H}_0 = \frac{H_0}{\sqrt{1 - \kappa \xi \phi_0^2}},$$  \hspace{1cm} (7.14)

$$\frac{\delta \tilde{H}}{\tilde{H}} = \frac{\delta H}{H_0} + \frac{\kappa \xi \phi_0^2}{1 - \kappa \xi \phi_0^2} \left( \frac{\delta \phi}{\phi_0} - \frac{\delta \tilde{\phi}}{H_0 \phi_0} \right),$$  \hspace{1cm} (7.15)

$$\delta \tilde{\phi} = \frac{\sqrt{1 - \kappa \xi (1 - 6\xi) \phi_0^2}}{1 - \kappa \xi \phi_0^2} \delta \phi.$$

The smallness of the Jordan frame quantities in Eq. (7.15) guarantees the smallness of the deviation from a de Sitter solution $\delta \tilde{H}/\tilde{H}$ in the Einstein frame; slow-roll inflation in the Jordan frame implies slow-roll inflation in the Einstein frame. The converse is not true, as shown in Ref. [59] in the special case $\xi = 1/6$, and therefore some caution must be taken when mapping back solutions from the Einstein to the Jordan frame. These considerations are relevant for the calculation of density and gravitational wave perturbations with NMC aimed at testing NMC inflation with present and future satellite observations [49, 50, 51, 48, 24]. One must take special care when computing quantum fluctuations and applying the conformal transformation (7.1), (7.2) and (7.3) since, in general, the vacuum state of one conformal frame is changed into a different state in the other frame [83, 18, 49, 84].

The conformal transformation is only defined for $\xi < 0$ and, if $\xi > 0$, for values of $\phi$ such that $\phi \neq \pm \phi_c = \pm (\kappa \xi)^{-1/2}$. For large values of $\xi$, this is a serious limitation on the usefulness of conformal transformation techniques. When $\xi > 0$ and the nonminimally coupled scalar field approaches the critical values $\pm \phi_c$, $g_{ab}$ degenerates, $\tilde{\phi}$ diverges and the conformal transformation technique cannot be applied. This happens when $\phi \simeq 0.199 \xi^{-1/2} m_{pl}$, which induces the unreasonable constraint $|\phi| < 0.2 m_{pl}$ if $\xi$ is of order unity (for example, chaotic inflation requires $\phi$ larger than about $5 m_{pl}$ [41]). In particular, for strong positive coupling $\xi >> 1$, the critical value $|\phi_c|$ corresponds to very low energies.

The conformal transformation technique cannot provide solutions with $\phi$ crossing the barriers $\pm \phi_c$, even when such solutions are physically admissible. In this sense, the conformal technique has the same limitations of the form of the field equations (2.11) and (2.12) discussed in Sec. 2. The transformed scalar $\tilde{\phi}$ in the Einstein frame can be
explicitly expressed in terms of \( \phi \) by integrating Eq. (7.3),

\[
\tilde{\phi} = \sqrt{\frac{3}{2\kappa}} \ln \left[ \frac{\xi \sqrt{6\kappa \phi^2} + \sqrt{1 - \xi (1 - 6\xi) \kappa \phi^2}}{\xi \sqrt{6\kappa \phi^2} - \sqrt{1 - \xi (1 - 6\xi) \kappa \phi^2}} \right] + f(\phi), \tag{7.17}
\]

where

\[
f(\phi) = \left( \frac{1 - 6\xi}{\kappa \xi} \right)^{1/2} \arcsin \left( \sqrt{\xi (1 - 6\xi) \kappa \phi^2} \right) \tag{7.18}
\]

for \( 0 < \xi \leq 1/6 \) and

\[
f(\phi) = \left( \frac{6\xi - 1}{\kappa \xi} \right)^{1/2} \arcsinh \left( \sqrt{\xi (6\xi - 1) \kappa \phi^2} \right) \tag{7.19}
\]

for \( \xi \geq 1/6 \). Eqs. (7.17)-(7.19) show that \( \tilde{\phi} \rightarrow \pm \infty \) in the Einstein frame as \( \phi \rightarrow \pm \phi_c \) in the Jordan frame. Any nonminimally coupled solution \( \phi \) crossing the barriers \( \pm \phi_c \) cannot be found by applying the conformal transformation technique.

An explicit example of such a solution is the one corresponding to a nonminimally coupled scalar field which is constant and equal to one of the critical values. In this case the field equations (1.1), (6.2), and (6.3) yield \( R = 6 \left( \dot{H} + 2H^2 \right) = 0 \) and \( V = 0 \). The solution

\[
a = a_0 \sqrt{t - t_0}, \quad \phi = \pm \frac{1}{\sqrt{\kappa \xi}} \quad (\xi > 0), \tag{7.20}
\]

corresponds to the vanishing of the trace of the energy-momentum tensor \( \tilde{T}_{ab} [\phi] = \tilde{T}_{ab}^{(\text{total})} \) in Eqs. (2.7) and (2.8), and to the radiation equation of state \( P = \rho/3 \).

### 8 Discussion and conclusions

Scalar fields are a basic ingredient of particle physics and cosmology and many arguments strongly suggest that a scalar field must couple nonminimally to the Ricci curvature of spacetime in the theories of gravity and of the scalar field used to build most scenarios of inflation and quintessence. Therefore, one cannot ignore NMC in these models. In this paper we approach several topics in the physics of NMC, from a general (i.e. not limited to a specific potential \( V(\phi) \)) point of view.

First, it is shown that the possible forms of writing the field equations are not equivalent, and it is pointed out that some of them lead to loss of generality and to a restricted
class of solutions. This is not a problem when one focuses on a specific solution \((g_{ab}, \phi)\), but it compromises studies that aim at generality like, e.g., the dynamical system analysis of the equations of cosmology in the phase space. Further, a shadow is cast on the reality of the time-variability of the effective gravitational constant \(G_{\text{eff}}(t)\) in many cosmological scenarios using NMC. In fact, the time-variability may be removed by passing to a different form of the field equations, and this interpretation problem deserves attention in the future.

The conservation equations for the different forms of the field equations are discussed, and it emerges that different formulations lead to different definitions of the energy density and pressure of the scalar field. Due to this fact, the problem of whether a nonminimally coupled scalar field satisfies the weak energy condition becomes fuzzy. In this paper, the form of the field equations (2.7) and (2.8) is preferred because \(i\) it does not lead to loss of generality, \(ii\) the stress-energy tensor (2.8) is covariantly conserved and satisfies the weak energy condition in spatially flat and closed FLRW universes, and \(iii\) the gravitational coupling is constant, and there are no interpretation problems with a time-varying \(G_{\text{eff}}(t)\).

The crucial feature of inflationary and quintessence models, i.e. the acceleration of the universe, is studied when the universe is dominated by a nonminimally coupled scalar field. The inclusion of the NMC term \(\xi R \phi^2 / 2\) in the Lagrangian density seems to make it harder to achieve cosmic acceleration for most potentials that are known to be inflationary when \(\xi = 0\). This conclusion derives from the dynamical equations for the scalar field and the scale factor and does not rely upon the slow-roll approximation, nor does it arise from independent consistency requirements of the kind discussed in Ref. [28]. In addition to the dynamical arguments, one must keep in mind that a given inflationary scenario must be consistent with the theoretical prescriptions for the value of \(\xi\), which further constrain the known scenarios [28, 38]. Fine-tuning arguments or, in other words, the genericity of inflation, are also an issue [41, 44].

The NMC term \(\xi R \phi^2 / 2\) can balance a suitable scalar field potential \(V(\phi)\) and induce cosmic acceleration with a wider class of potentials than it is normally considered. However, the NMC term in the Lagrangian cannot completely substitute for a potential and induce an acceleration epoch when \(V = 0\). We have proven this statement in Sec. 4 for values of the coupling constant \(\xi \leq 1/6\), but we could not reach a conclusion for \(\xi > 1/6\). A new solution for \(\xi = 0\) expanding from a big-bang singularity and quickly approaching a de Sitter space is also presented.

Since almost all the inflationary scenarios proposed to date are based on the slow-roll approximation, the role of de Sitter solutions (which are fixed points) as attractor points in the phase space is crucial. We study this issue in the presence of NMC and we find
that the stability of the expanding de Sitter solution (6.8) is determined not only by the shape of the potential $V(\phi)$ (as is the case of minimal coupling), but also by the value of the coupling constant $\xi$. A more general analysis including perturbations which are space-dependent or anisotropic is needed to confirm the stability; however, our local perturbation analysis is sufficient to establish instability for

$$\xi < \xi_0 \equiv -\frac{V''}{4\Lambda},$$

where $\Lambda = \kappa V_0$. Note that $\xi_0 = -\eta_0/4$, where $\eta = V''/(\kappa V)$ is one of the slow-roll parameters used in the slow-roll approximation for minimal coupling [62].

Contracting de Sitter solutions are always unstable, as in the $\xi = 0$ case. These considerations set precise limits for the domain in which the slow-roll approximation is meaningful in the presence of NMC, and is fundamental for the computation of scalar and tensor perturbations. Ultimately, the amplitudes and spectral indices of these perturbations are the predictions of the theory to be compared with observations of the cosmic microwave background.

Conformal transformation techniques are widely used in scalar field cosmology [78] and it is useful to clarify their link with the slow-roll approximation. We prove that slow-roll inflation in the physical Jordan frame (in which the scalar field is nonminimally coupled) implies slow-roll inflation in the unphysical Einstein frame (but not vice versa), and make explicit the limitations intrinsic to the use of conformal transformations. Analytic examples are given in which the conformal transformation method cannot be applied.

Recently, there has been a great deal of work on NMC in both inflationary and quintessence models; this paper justifies certain assumptions and methods used, solves some of the problems posed, and provide caveats on difficulties that were overlooked. Our considerations will be applied in the future to specific models; other areas in which NMC is relevant include quantum cosmology, classical and quantum wormholes, and the stability of boson stars.

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References


[11] Nevertheless, even in $R^2$-inflation, a scalar field is sometimes added to the scenario in order to increase the amount of inflation [10].


[15] The metric signature is $-+++$ and $G$ denotes Newton’s constant. The speed of light and Planck’s constant assume the value unity and $m_{pl} = G^{-1/2}$ is the Planck mass. The components of the Ricci tensor are given in terms of the Christoffel symbols $\Gamma^a_{\alpha\beta}$ by $R_{\mu\nu} = \Gamma^a_{\mu\rho,\nu} - \Gamma^a_{\nu\rho,\mu} + \Gamma^a_{\mu\rho} \Gamma^\rho_{\alpha\nu} - \Gamma^a_{\nu\rho} \Gamma^\rho_{\alpha\mu}$, $\Box \equiv g^{ab} \nabla_a \nabla_b$, and the abstract index notation [12] is adopted.

[17] The value 1/6 of the coupling constant $\xi$ makes the Klein-Gordon equation (1.1) in four spacetime dimensions conformally invariant if $V = 0$ or $V = \lambda \phi^4$ [12, 16].


[23] See the cautionary notes in Refs. [28, 47] on the use of an effective potential description of the effects of NMC.


[68] For de Sitter solutions, the Klein-Gordon equation is not independent of the Einstein field equations [67, 47].


