Some Remarks on the Bel–Robinson Tensor*

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Abstract

In this paper we present our point of view on correct physical interpretation of the Bel–Robinson tensor within the framework of the standard General Relativity (GR), i.e., within the framework of the GR without supplementary elements like arbitrary vector field, distinguished tetrads field or second metric. We show that this tensor arises as a consequence of the Bianchi identities and, in a natural manner, it is linked to the differences of the canonical gravitational energy–momentum calculated in normal coordinates NC(P).
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I. INTRODUCTION

In analogy to the symmetric energy–momentum tensor of the electromagnetic field, the Bel–Robinson superenergy tensor is defined as follows (see eg. [1,2])

\[ T^{iklm} = R_{iabl} R_{ab}^{k m} + *R_{iabl} R_{ab}^{k m} \]

\[ = R_{iabl} R_{ab}^{k m} + R_{iabm} R_{ab}^{l k} - \frac{1}{2} g^{ik} R^{abcd} R_{abc}^{m} , \]  

where \( R_{iabl} \) is the Riemann tensor and \(*\) indicates the usual dual operation

\[ *R_{iabl} := \frac{1}{2} \eta_{iadc} R_{bldc}^{dc} . \]  

The unusual properties (see eg. [2]) of the Bel–Robinson tensor intrigued physicists and a large number of papers were devoted to the understanding of its physical sense within the framework of the General Relativity (GR). Recently, in the Ref. [3] (see also [4]) the authors try to connect this tensor with square of the energy–momentum and derive from it (by taking a suitable defined "square root") an energy–momentum tensor of the gravitational field. In some special cases such square root really exists. However, it seems that this is an incorrect physical idea.

In order to understand the physical content of the Bel–Robinson tensor correctly in GR one should take into account the following, fundamental facts:

1. The Bel–Robinson tensor can be obtained as a consequence of the Bianchi identities independently of the Einstein equations.

2. In the framework of the standard General Relativity (GR) \(^1\) the gravitational field has non–tensorial strenghts \( \Gamma_{kl}^{i} \) and has no (and cannot have any) energy–momentum tensor but only the so–called pseudotensors. It is a consequence of the Einstein Equivalence Principle (EEP).

\(^1\)The standard General Relativity has a very good experimental confirmation; especially its main postulate — Einstein Equivalence Principle (see eg. [5,6,7]).
3. The Bel–Robinson tensor appears explicitly in the expansion of the differences of the gravitational energy–momentum calculated in normal coordinates NC(P) by use of the canonical energy–momentum pseudotensor \( Et^k_i \). Namely, it is a part of the differences \( Et^k_i(y) - Et^k_i(P) \). Here \( y \) means normal coordinates NC(P) which have point \( P \) as origin.

In this paper, in Sec. II and in Sec. III, we will consider the above three facts more intensively and show that they uniquely indicate the link between Bel–Robinson tensor and differences of the canonical gravitational energy–momentum calculated in NC(P).

In Sec. IV we give conclusions and some remarks.

II. THE BEL–ROBINSON TENSOR AND BIANCHI IDENTITIES

The Bel–Robinson tensor is a special case of the so–called Maxwellian tensors. The Maxwellian tensors generalize the symmetric energy–momentum tensor of the electromagnetic field onto antisymmetric tensor fields. The general method of construction of such tensors was developed in [9]. In the following we apply this method to the Riemann tensor.

Let us consider Bianchi identities for the Riemann tensor

\[
R_{iklm} = R_{lmik} = -R_{kilm} = -R_{ikml} \quad (3)
\]

and their non–vanishing contractions

\[
\nabla d R_{bcde} \equiv \nabla a R_{bcde} + \nabla b R_{caled} + \nabla c R_{abde} \equiv 0 \quad (4)
\]

We realize that the identities (3)–(4) possess Maxwellian structure in the indices \((a, b, c)\).

\(^2\)The analogous expansions were obtained by using other energy–momentum pseudotensors of the gravitational field and also contain Bel–Robinson tensor. However, they are much more complicated; see e.g. [8].
Let us multiply (3) by $R^{bcd}f$. Then, after simple calculations we get the new identities

$$R^{bcd}f \nabla_b R_{acde} - \frac{1}{2} R^{bcd}f \nabla_a R_{bcde} \equiv 0. \quad (5)$$

Then let us transpose the indices $f$ and $e$ in (5)

$$R^{bcd}e \nabla_b R_{acdf} - \frac{1}{2} R^{bcd}e \nabla_a R_{bcdf} \equiv 0. \quad (6)$$

The sum of (5) and (6) gives

$$R^{bcd}f \nabla_b R_{acde} + R^{bcd}e \nabla_b R_{acdf}$$

$$- \frac{1}{2}(R^{bcd}f \nabla_a R_{bcde} + R^{bcd}e \nabla_a R_{bcdf}) \equiv 0. \quad (7)$$

Due to the identities (4) one can rewrite the identities (7) in the following form

$$\nabla_b(R^{bcd}f R_{acde} + R^{bcd}e R_{acdf} - \frac{1}{2}\delta^b_a R^{bcd}e R_{bcdf})$$

$$\equiv 2R_{ac}d e \nabla_{[d}R_{f]}^c + 2R_{ac}d f \nabla_{[d}R_{e]}^c. \quad (8)$$

The Bel–Robinson tensor $T_{aef}^b$ is easily indicated inside parenthesis on the left hand side of the identities (8) which determine the divergence of this tensor.

Thus, we see that the Bel–Robinson tensor and its divergence arise as a consequence of the Bianchi identities (3) and their contractions (4) only and they are neither connected with Einstein equations nor with the canonical formalism of the energy–momentum in GR.

However, by using of the Einstein equations

$$R^{i}_{k} = \beta(T^{i}_{k} - \frac{1}{2}\delta^{i}_{k}T) =: \beta E^{i}_{k}, \quad (9)$$

where $\beta = 8\pi G/c^4$, one can rewrite the identities (8) in the form

$$\nabla_b T_{aef}^b = 2\beta R_{ac}d e \nabla_{[d}E_{f]}^c + 2\beta R_{ac}d f \nabla_{[d}E_{e]}^c. \quad (10)$$

The equations (10) give the link between the divergence of the Bel–Robinson tensor and GR.

It follows from (10) that, in vacuum,
\[ \nabla_b T^b_{\ aef} = 0. \quad (11) \]

The dimensions of the components of the Bel–Robinson tensor are \((\text{length})^{(-4)}\); but it is a trivial fact that \(1/\beta^2 T^b_{\ afe}\) have dimensions of the energy–momentum square. This trivial fact was used in Ref. 3 with the aim of connecting the Bel–Robinson tensor with square of an energy–momentum tensor.


The problem of the energy–momentum in General Relativity (GR) was intensively studied by many authors (see e.g. [10—17]). The main results of these investigations are the following: 3

1. Owing to the non–tensorial character of the gravitational strengths \(\Gamma^i_{\ kl} = \{^i_{\ kl}\}\) the gravitational field in standard GR has no (and cannot have) any energy–momentum tensor. Any attempt to introduce such a tensor leads us beyound standard GR. Moreover, it is speculative and contradicts EEP.

From that it follows the non–localizability of the gravitational energy–momentum.

2. The best solution of the energy–momentum problem in standard GR seems to be given by the so–called canonical energy–momentum pseudotensor \(E^i_k\) proposed for

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3We do not consider here the so–called quasilocal quantities [18—22] because already for the Kerr spacetime differently defined quasilocal quantities give different results (see eg. [23]). Moreover, the term “quasilocal” is very obscure. We omit also Lorentz hypothesis (see eg. [24]) which is unsatisfactory from the physical point of view since it gives for gravitational field an energy–momentum tensor which vanishes in vacuum. But we think that Lorentz hypothesis is the best of the all trials to attribute an energy–momentum tensor to gravitational field.
gravitational field by Einstein and related to that pseudotensor, the canonical, double index, energy–momentum complex

$$E K_i^k := \sqrt{|g|} (T_i^k + E t_i^k),$$  \hfill (12)

for matter and gravitation which satisfies

$$\sqrt{|g|} (T_i^k + E t_i^k) = F U_i^{kl}. \hfill (13)$$

Here

$$F U_i^{kl} = (-) F U_i^{lk} = \alpha \frac{g_{ia} g^{jb}}{\sqrt{|g|}} \left[ (\Gamma_{lms}^k - \Gamma_{msl}^k) - \frac{1}{2} \left( \Gamma_{lmp}^k g_{iap} - \Gamma_{itk}^l g^{kb} \right) g_{ms} - \frac{1}{2} \left( \delta_{msl}^k + \delta_{msl}^k \right) \right].$$ \hfill (14)

are von Freud superpotentials, \( T_i^k \) are the components of a symmetric energy–momentum tensor of matter (the sources in the Einstein equations) and \( g \) is the determinant of the metric tensor; \( \partial_i \) denotes partial derivative. \( \alpha = 1/2 \beta = c^4/16\pi G \).

We have [25–27]

$$E t_i^k = \alpha \left\{ \delta_i^k \Gamma_{ms}^l \left( \Gamma_{mr}^i \Gamma_{sl}^r - \Gamma_{ms}^r \Gamma_{rl}^i \right) \right. \hfill (15)$$

$$+ g_{ms}^m \left[ \Gamma_{ms}^k \right] - \frac{1}{2} \left( \Gamma_{lmp}^k g_{iap} - \Gamma_{itk}^l g^{kb} \right) g_{ms}$$

$$- \frac{1}{2} \left( \delta_{msl}^k + \delta_{msl}^k \right) \left\}. \hfill (15)$$

The equations (13) can be obtained by rearranging of the Einstein equations having \( T_i^k \) as sources.

From (13) there follow the local or differential conservation laws

$$\left\{ [\sqrt{|g|} (T_i^k + E t_i^k)]_k \right. = 0, \hfill (16)$$

and, by using Stokes integral theorem, the integral conservation laws

$$\int_{\partial \Omega} \sqrt{|g|} (T_i^k + E t_i^k) d\sigma_k = 0. \hfill (17)$$
\( \partial \Omega \) is the boundary of a four-dimensional, compact domain \( \Omega \), and \( d\sigma_k \) is the three-dimensional integration element (see e.g. [17]).

The components \( E_t^k \) of the Einstein canonical energy–momentum pseudotensor of the gravitational field neither form a tensor nor other geometric object.

Any attempt of the physical interpretation of the Bel–Robinson tensor in the framework of GR should take into account the connection of the Bel–Robinson tensor and its divergence with Bianchi identities, the above two fundamental facts and the next, more important fact as follows: the Bel–Robinson tensor appears explicitly in the expansion of the differences

\[
E_t^k(y) - E_t^k(P)
\]

of the canonical energy–momentum calculated in normal coordinates [28–30] \( \text{NC}(P) \). (And in analogic expansions obtained when using other pseudotensors too [8]). This fact gives the most important connection between the Bel–Robinson tensor and GR as follows.

In the \( \text{NC}(P) \) \( \{y^i\} \) having the point \( P \) as their origin we have [38]

\[
E_t^k(y) - E_t^k(P) = \frac{1}{2} E_t^k \quad \text{shows}
\]

\[
\begin{align*}
E_t^k(y) &= \frac{1}{2} E_t^k \quad \text{shows}
\]

\[
R^k_{ilm}(P) + R^k_{ilm}(P) - \frac{1}{2} \delta^k_i R_{abc} \quad \text{shows}
\]

\[
2\delta^k_i R_{(i}|g(P)R^g_{|m}(P) - 3R_{i(l}(P)R^k_{|m)}(P) + R^k_{(i(l}(P)R^g_{|m)}(P)
\]

\[
+ R^k_{(i|g}(P)R^g_{|m)}(P)\} y^l y^m + R_3.
\]

In the formula (19) \( R_3 \) is the remainder of the third order, while

\[
T^k_{ilm} := R^{kab}_{i} R_{j}^{abm} + R^{kab}_{m} R_{i}^{abl} - \frac{1}{2} \delta^k_i R_{abc} \quad \text{shows}
\]

are the Bel–Robinson tensor components, and

\[\text{4Because } E_t^k(P) = E_t^k \quad \text{shows}\]

\[\text{4Because } E_t^k(P) = E_t^k \quad \text{shows}\]

the differences \( E_t^k(y) - E_t^k(P) \) have deeper physical meaning; for example, they admit introduction of superenergy and supermomentum tensors [31–37].
\[ P_{ilm}^k := R_{iabm}^{ kab} + R_{ibal}^{ kab} - \frac{1}{2} \delta_i^k R_{abm}^{ abc} \]  

(21)

are components of the tensor which is closely related to the Bel–Robinson tensor.

By using the Einstein equations in the form (9), one can rewrite (19) to the form

\[
E^t_i k(P) - E^t_i k(y) = \frac{\alpha}{9} [T_{ilm}^k(P) + P_{ilm}^k(P) \\
- \frac{1}{2} \delta_i^k R_{abc}^{ i}(P)(R_{abcm}(P) + R_{acbm}(P)) \\
+ 2 \delta_i^k \beta E_{i}(P) E_{m}^{g}(P) - 3 \beta E_{i}(P) E_{m}^{k}(P) + \beta R_{ig}(P) E_{m}^{g}(P) \\
+ \beta R_{g}(P) E_{m}^{g}(P)] y^l y^m + R_3.
\]  

(22)

In vacuum we have from (22)

\[
E^t_i k(P) - E^t_i k(y) = \frac{\alpha}{9} [T_{ilm}^k(P) + P_{ilm}^k(P) \\
- \frac{1}{2} \delta_i^k R_{abc}^{ i}(P)(R_{abcm}(P) + R_{acbm}(P))] y^l y^m + R_3 \\
= \frac{4\alpha}{9} [R^{k(ab)}_{(l)(P)} R_{ablm}(P) - \frac{1}{2} \delta_i^k R^{(bc)}_{i}(P) R_{abcm}(P)] y^l y^m + R_3.
\]  

(23)

We see from the above formulas (19)–(23) that the Bel–Robinson tensor really appears in the expansion of the differences (18).

Years ago, by using the expansion (23), we have showed [38] that the infinitesimal differences \( \Delta P_a \) of the free gravitational energy–momentum calculated in \( NC(P) \) are proportional to the components \( T_{a00}^0 \) of the Bel–Robinson tensor multiplied by \( \alpha = c^4/16\pi G \).

IV. CONCLUSION

The Bel–Robinson tensor follows from the Bianchi identities as the Maxwellian tensor for the Riemann tensor \( R_{iklm} \) and, within the framework of the standard GR, it can be connected with the differences of the canonical gravitational energy–momentum calculated in \( NC(P) \). It is easily seen from Sec. II, from the formulas (19)–(23) and from the Ref. 38. These facts give the most natural (and correct) physical interpretation of this tensor in the framework of the GR.
Although the gravitational field $\Gamma^i_{kl}$ has no energy–momentum tensor in standard GR, one can easily introduce there the so-called *canonical superenergy tensor* [31–37] for this field. The method of construction of the canonical superenergy tensor for gravitational field uses the expansion (19) and some kind of averaging. The Bel–Robinson tensor multiplied by $\alpha = c^4/16\pi G$ is the main, ”Maxwellian part” of such a tensor.

The canonical superenergy tensor for gravitational field does not vanish in vacuum; so, it gives us, for example, a very useful tool for local analysis of the gravitational radiation [34].

The idea of superenergy and its tensor is *universal* and easy to generalize onto matter field too; for example, one can easily introduce *the canonical superenergy tensor for matter* [31–35].

On the ”superenergy level” one can easily introduce *the canonical angular supermomentum tensors* for matter and gravitation [36,37] as well.

The *canonical superenergy tensors* of gravitation and matter and the *canonical angular supermomentum tensors* of gravitation and matter have much better geometrical and physical properties than the canonical objects from which they were obtained. Also, the integral superenergetic quantities have better properties than corresponding integral energetic quantities; especially, the superenergetic integrals have better convergence in asymptotically flat spacetimes (at spatial or null infinity).

Finally, the canonical superenergy and angular supermomentum tensors give a very good tool for local (and also to global) analysis of the gravitational and matter fields within the framework of the standard GR.

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REFERENCES


